Precalculus – Lesson Notes: 9.5-9.7 Binomial Theorem, Combinatorics, Probability

9.5 Binomial Theorem

- ones on outside
- Symmetry (left/right
- numbers in middle are ↖↗ added
- 1st exponent for x = original exponent
- x exponents drop by 1 →, y exponents increase by 1 →
- exponents in each term add to original exponent.

$(x+y)^0 = 1$

$(x+y)^1 = x + y$

$(x+y)^2 = x^2 + 2xy + y^2$

$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$

$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$

$(x+y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$

Find all patterns

1 6 15 20 15 6 1

What would the expansion be for $(x+y)^6$?

$x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6$

Pascal's triangle:

```
row 0 →          1
row 1 →        1   1
              1   2   1
            1   3   3   1
          1   4   6   4   1
row 5 →  1   5  (10)  10   5   1
         ↑   ↑    ↑
       col 0 col 1 col 2 - - -
```

$_5C_2$ row col

$_nC_r = \dfrac{n!}{(n-r)!r!}$ row col

☆ Calculator ☆

The Binomial Theorem:

$$(x+y)^n = {}_nC_0x^n + {}_nC_1x^{n-1}y + {}_nC_2x^{n-2}y^2 + \ldots + {}_nC_rx^{n-r}y^r + \ldots + {}_nC_ny^n$$

where $_nC_r = \dfrac{n!}{(n-r)!r!}$ or a row from Pascal's triangle

'x' and 'y' can be more complex...

$(2x-3)^4 = {}_4C_0(2x)^4(-3)^0 + {}_4C_1(2x)^3(-3)^1 + {}_4C_2(2x)^2(-3)^2 + {}_4C_3(2x)^1(-3)^3 + {}_4C_4(2x)^0(-3)^4$

$= 1(16x^4)(1) + 4(8x^3)(-3) + 6(4x^2)(9) + 4(2x)(-27) + 1(1)(81)$

$= \boxed{16x^4 - 96x^3 + 216x^2 - 216x + 81}$

-96 is the 'coefficient' for the x^3 term

A. Use the binomial theorem to expand the expression $(2x - y)^4$

${}_4C_0(2x)^4(-y)^0 + {}_4C_1(2x)^3(-y)^1 + {}_4C_2(2x)^2(-y)^2 + {}_4C_3(2x)^1(-y)^3 + {}_4C_4(2x)^0(-y)^4$

$1(16x^4)(1) + 4(8x^3)(-y) + 6(4x^2)(y^2) + 4(2x)(-y^3) + 1(1)(y^4)$

$\boxed{16x^4 - 32x^3y + 24x^2y^2 - 8xy^3 + y^4}$

B. Expand $(4x - 1)^5$

${}_5C_0(4x)^5(-1)^0 + {}_5C_1(4x)^4(-1)^1 + {}_5C_2(4x)^3(-1)^2 + {}_5C_3(4x)^2(-1)^3 + {}_5C_4(4x)^1(-1)^4 + {}_5C_5(4x)^0(-1)^5$

$1(1024x^5)(1) + 5(256x^4)(-1) + 10(64x^3)(1) + 10(16x^2)(-1) + 5(4x)(1) + 1(1)(-1)$

$\boxed{1024x^5 - 1280x^4 + 640x^3 - 160x^2 + 20x - 1}$

C. Find the coefficient of $x^{12}y^3$ in the expansion of $(4x - 5y)^{15}$.

${}_{15}C_0(4x)^{15}(-5y)^0$

${}_{15}C_3(4x)^{12}(-5y)^3$

$455(16777216x^{12})(-125y^3)$

$\boxed{-9.5421\,10^{11}}\,x^{12}y^3$

coefficient

D. Find the coefficient of $x^{10}y^8$ in the expansion of $(-3x^2 + 2y^4)^7$.

${}_7C_0(-3x^2)^7(2y^4)^0$

x^{14} x^{12}

${}_7C_2(-3x^2)^5(2y^4)^2$

$21(-243x^{10})(4y^8)$

$\boxed{-20412}\,x^{10}y^8$

coefficient

9.6 Counting problems (Combinatorics)

Simple counting problems

List all possibilities

A computer generates integers randomly between 1 and 12. 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12

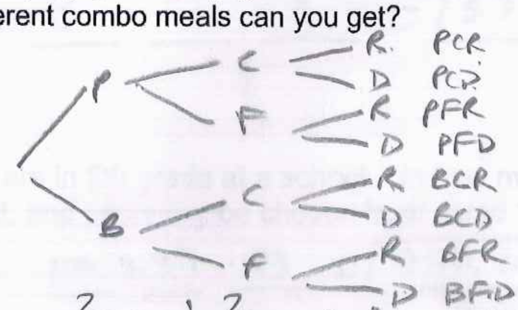
In how many ways can the number be an even integer?

$\boxed{\text{6 ways}}$

Pairings - tree diagram

At a snack bar a combo meal consists of: a main item of pizza (P) or a burger (B), a side item of chips (C) or fries (F), and a drink which is regular (R) or diet (D).

How many different combo meals can you get?



$\boxed{\text{8 meals}}$

1 'box' per 'choice' or event → $\underset{\text{main}}{\underline{2}} \cdot \underset{\text{side}}{\underline{2}} \cdot \underset{\text{drink}}{\underline{2}} = \boxed{8}$

Fundamental Counting Principle (multiplication principle)

"Let E_1 and E_2 be two events. The first event E_1 can occur in m_1 different ways. After E_1 has occurred, E_2 can occur in m_2 different ways. The number of ways that the two events can occur is: m_1m_2."

Practice: Students must select 1 of 2 math courses, 1 of 3 science courses, and 1 of 5 social studies courses. How many different class groupings are possible?

$\underset{\text{math}}{\underline{2}} \cdot \underset{\text{science}}{\underline{3}} \cdot \underset{\text{social studies}}{\underline{5}} = \boxed{30}$

Example: 8 pieces of paper on which are written the numbers 1 through 8 are put in a box. One piece of paper is drawn out and the number recorded. The paper is replaced in the box. Another piece of paper is drawn out and recorded, replaced and a paper is drawn out and recorded a third time, forming a 3 digit number.

How many different 3 digit numbers are possible?

$\underline{8} \cdot \underline{8} \cdot \underline{8} = \boxed{512}$

Example: Same as above, except that once drawn out, a piece of paper is not replaced in the box.

$\underline{8} \cdot \underline{7} \cdot \underline{6} = \boxed{336}$

Practice:

How many different 7-digit telephone numbers are possible if the 1st digit cannot be a zero or a one?

(2-9) (0-9) (0-9)

8 10 10 10 10 10 10 = $\boxed{8000000}$

8 horses run in a race. In how many different ways can these horses come in 1st, 2nd and 3rd place?

(1-8)

$8 \cdot 7 \cdot 6 = \boxed{336}$

something different (choosing out of a set)

100 students are in 8th grade at a school. In how many ways can a student body president, vice president, and secretary be chosen from these 100 students?

(1-100)

$100 \cdot 99 \cdot 98 = \boxed{970,200}$

Permutation - A permutation of n different elements is an ordering of elements with one element first, another second, etc. ORDER MATTERS.

Can compute permutations using 'boxes' (as above) or using the permutation formula:

Number of permutations of n elements taken r at a time is:

$$_nP_r = \frac{n!}{(n-r)!}$$

Horses example: Select 3 horses from 8, order matters.

$$_8P_3 = \frac{8!}{(8-3)!} = \frac{8!}{5!} = \frac{8 \cdot 7 \cdot 6 \cdot \cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}}{\cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}} = 336$$ (or calculator)

What if some elements are identical?

Example: In how many distinguishable ways can the letters BANANA be written?

out of 6 letters, select all 6 (order matters)

$$_6P_6 = \frac{6!}{(6-6)!} = \frac{6!}{0!} = 6!$$

$B A_3 N A_1 N A_2$

$$\frac{6!}{3!\,2!\,1!} = \boxed{60}$$

$A_1 A_2 A_3$

$A_1 A_3 A_2$

$A_2 A_1 A_3$

$A_2 A_3 A_1$

$A_3 A_1 A_2$

$A_3 A_2 A_1$

$6 = 3!$

$\underline{3} \cdot \underline{2} \cdot \underline{1}$

Number of distinguishable permutations of n objects is:

$$\frac{n!}{n_1! \cdot n_2! \cdot n_3! \cdot \ldots \cdot n_k!}$$

try MISSIPPIPPI

What if order does not matter?

Example: 100 students are in 8th grade in a school.
In how many ways can 3 students be chosen to form a student council?

$$\frac{\underline{100} \cdot \underline{99} \cdot \underline{98}}{6} = \frac{970{,}200}{6} = \boxed{161{,}700}$$

Jill	Bob	Jane
Jill	Jane	Bob
Bob	Jill	Jane
Bob	Jane	Jill
Jane	Bob	Jill
Jane	Jill	Bob

$3! =$

6

Combination - A combintation is a subset of n elements taken r at a time, where ORDER DOESN'T MATTER.

Number of combinations of n elements taken r at a time is:

$$_nC_r = \frac{n!}{(n-r)!r!}$$

Example: 100 students are in 8th grade in a school. In how many ways can 3 students be chosen to form a student council?

Combination (order doesn't matter)

$$_{100}C_3 = 161{,}700$$

Practice: In how many ways can 3 letters be chosen from the letters A, B, C, D, E if order of the letters does not matter?

$$_5C_3 = 10$$

Day 2

More complex examples:

A shipment of 25 television sets contains 3 defective units.
In how many ways can a vending company purchase 4 of these units and receive
(a) all good units,
(b) 2 good units,
(c) at least 2 good units?

choosing 4

25 TVs

3 bad 22 good

(a) all good

4 good, 0 bad

$$\underbrace{_{22}C_4}_{\text{ways to choose good}} \cdot \underbrace{_3C_0}_{\text{ways to choose bad}}$$

$7315 \cdot 1$

$\boxed{7315}$

(b) 2 good (2 bad)

$$\underbrace{_{22}C_2}_{\text{good}} \cdot \underbrace{_3C_2}_{\text{bad}}$$

$231 \cdot 3$

$\boxed{693}$

(c) at least 2 good

2 good or 3 good or 4 good

$_{22}C_2\ _3C_2 \qquad _{22}C_3\ _3C_1 \qquad _{22}C_4\ _3C_0$

$231 \cdot 3 \qquad 1540 \cdot 3 \qquad 7315 \cdot 1$

$(693) + (4620) + (7315)$

$\boxed{12628}$

2 bad AND 2 good

Multiply

this OR this

ADD

'AND' = multiply
'OR' = add

In how many ways can 5 girls and 3 boys walk through a doorway single file?

$\underline{8} \cdot \underline{7} \cdot \underline{6} \cdot \underline{5} \cdot \underline{4} \cdot \underline{3} \cdot \underline{2} \cdot \underline{1} = 8! = \boxed{40,320}$

What if girls must enter before boys?

$\underline{5} \cdot \underline{4} \cdot \underline{3} \cdot \underline{2} \cdot \underline{1} \cdot \underline{3} \cdot \underline{2} \cdot \underline{1} = \boxed{720}$

Try this

Three couples have reserved seats in one row at a concert. In how many ways can they be seated?

(6 people)

$\underline{6} \cdot \underline{5} \cdot \underline{4} \cdot \underline{3} \cdot \underline{2} \cdot \underline{1} = \boxed{720}$

What if couples wish to sit together?

$\underline{6}\ \underline{1} \cdot \underline{4} \cdot \underline{1} \cdot \underline{2} \cdot \underline{1} = \boxed{48}$

5 cards are selected from an ordinary deck of 52 playing cards.
In how many ways can you get a full house?
(3 of a kind and two of another, e.g. 8-8-8-5-5).

$_{13}C_1$	$_4C_3$	$_{12}C_1$	$_4C_2$
# ways to choose card to have 3 of	# ways to get 3 of this card	# ways to choose card to have 2 of	# ways to get 2 of this card

$13 \cdot 4 \cdot 12 \cdot 6 = 3,744$

ways to draw any 5 cards: $_{52}C_5 = 2,598,960$

$P = \frac{3744}{2598960} = 0.00144$

(0.144%)

9.7 day 1 Probability

If you roll a 6-sided, fair die, what is the probability that you will roll a 4?

Possible outcomes = $\{1, 2, 3, 4, 5, 6\}$

Desired outcomes ______ 1

Probability = $\frac{1}{6}$

What is the probability that you roll an even number? $\{1, 2, 3, 4, 5, 6\}$ $\frac{3}{6} = \boxed{\frac{1}{2}}$

Terms:

Any happening whose result is uncertain is called an **experiment.**
Possible results of the experiment are **outcomes.**
The set of all possible outcomes is called the **sample space.**
Any subcollection of a sample space is called an **event.**

Probability of an Event

$$P(E) = \frac{n(E)}{n(S)} = \frac{\textit{number of desired outcomes}}{\textit{total number of outcomes}}$$

Probability is a number between 0 and 1 (usually expressed as a fraction or decimal):

impossible certain

$$0 \le P(E) \le 1$$

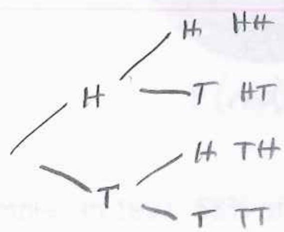
Probability of the complement of an Event (Probability of an event <u>not</u> occurring):

$$P(E') = 1 - P(E)$$

$P(\text{rain}) = 0.6$ (60%) then $P(\text{not rain}) = 1 - 0.6 = 0.4$ (40%)

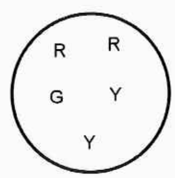
Examples: Find the sample space

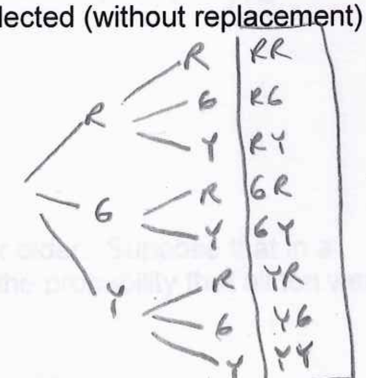
Two coins are tossed



$\{HH, HT, TH, HH\}$

2 marbles are selected (without replacement)





If two coins are tossed, what is the probability that both land heads up?

$\{HH, HT, TH, TT\}$ $P(HH) = \boxed{\frac{1}{4}}$

If a card is drawn from a standard deck of cards, what is the probability that it is an ace?

$P(\text{ace}) = \frac{4}{52} = \boxed{\frac{1}{13}}$

(4 aces in a deck of 52 cards)

Two 6-sided dice are tossed. What is the probability that the total of the two dice is 7?

$\underset{\text{ways 1st die can land}}{6} \cdot \underset{\text{ways 2nd die can land}}{6} = 36$

desired sum = 7, how many ways?

1 6, 2 5, 3 4, 4 3, 5 2, 6 1

6 ways

$P(\text{sum is } 7) = \frac{6}{36} = \boxed{\frac{1}{6}} = .16666$

$(= 16.7\%)$

In a state lottery, a player chooses 6 different numbers from 1 to 40. If these 6 numbers match the 6 winning numbers (order does not matter), the player wins. What is the probability of winning if a single ticket is purchased?

total possible numbers ${}_{40}C_6 = 3{,}838{,}380$

so $P(1 \text{ ticket wins}) = \frac{1}{3{,}838{,}380} = 0.00000026$

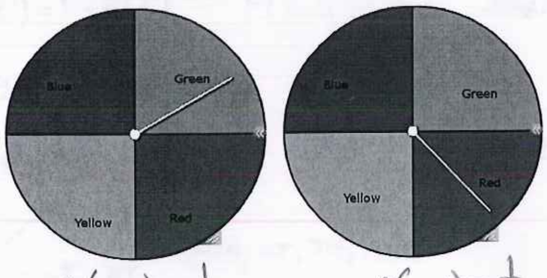
$(= 0.000026\%)$

Independent Events – Two events are independent if the occurrence of one has no effect on the occurrence of the other.

"and" = multiply probabilities

Probability of <u>both</u> independent events occurring: $P(A \text{ and } B) = P(A) \cdot P(B)$

Example: If two 4-color spinners below are spun, what is the probability that the both spinners will land on red?



$P(\text{red}) = \frac{1}{4}$ $P(\text{red}) = \frac{1}{4}$

$P(\text{both red}) = \frac{1}{4} \cdot \frac{1}{4} = \boxed{\frac{1}{16}}$

Example: In 1997, 58% of the population of the U.S. were 30 years old or older. Suppose that in a survey, ten people were chosen at random from the population. What is the probability that all ten were 30 years or older?

$P(\text{a person is } 30+) = 0.58$

$P(10 \text{ people } 30+) = (.58)(.58)(.58)(.58)(.58)(.58)(.58)(.58)(.58)(.58) = (0.58)^{10} = 0.004308$

$(\approx 0.43\%)$

Example: What is the probability of tossing two 6-sided dice and getting a sum of at least 8?

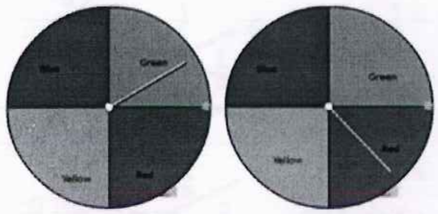
	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

$P(\text{at least } 8) = \frac{15}{36} = \boxed{\frac{5}{12}} \approx \boxed{.417}$

9.7 day 2 Probability

Probability of both independent events occurring: $P(A \text{ and } B) = P(A) \bullet P(B)$

Example: If two 4-color spinners below are spun,
what is the probability that the both spinners will land on red?



$P(\text{Red and Red}) = P(\text{red}) \bullet P(\text{red})$

$= \frac{1}{4} \bullet \frac{1}{4}$

$\Rightarrow \boxed{\frac{1}{16}}$

Mutually Exclusive Events
Two events from the same sample space, A and B, are mutually exclusive if they have no outcomes in common.

Probability of either of mutually exclusive events occurring: $P(A \text{ or } B) = P(A) + P(B)$

Example: The personnel department of a company has compiled data on employee's number of years of service, shown in the table. In an employee is chosen at random, what is the probability that the employee has 9 or fewer years of service?

Yrs of Service	Number Employees
0-4	8
5-9	23
10-14	12
15-19	15
20-24	18
25-29	14
30-34	4
35-39	4
40-44	2
	100

$P(0\text{-}4) = \frac{8}{100}$ (8%)

$P(5\text{-}9) = \frac{23}{100}$ (23%)

$P(9 \text{ or less}) = P(0\text{-}4 \text{ 'or' } 5\text{-}9) = P(0\text{-}4) + P(5\text{-}9)$

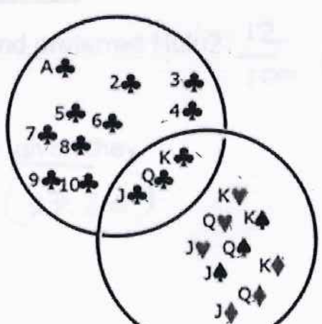
$= \frac{8}{100} + \frac{23}{100}$

$= \boxed{\frac{31}{100} \ (31\%)}$

Probability of a Union

Example: One card is selected from a standard deck. What is the probability that the card is either a club or a face card?

There are 13 clubs, and there are 3 face cards in each suit (12 face cards in all), but some cards are both clubs and face cards. To solve, use a **Venn diagram**:



Probability of a Union with overlap:

club or face   club   face   club and face

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

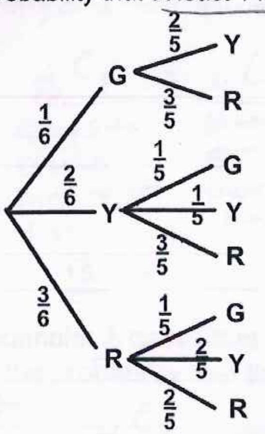
$$\frac{22}{52} = \frac{13}{52} + \frac{12}{52} - \frac{3}{52}$$

Probability of two independent events both occurring ('and'): $P(A \text{ and } B) = P(A) \bullet P(B)$

Probability of either of two mutually exclusive events occurring ('or', no overlap): $P(A \text{ or } B) = P(A) + P(B)$

Probability of either of two events occurring ('or', with overlap): $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$

Example: A bag contains: 1 green marble, 2 yellow marbles and 3 red marbles.
If three marbles are drawn from the bag one-at-a-time (without replacement) what is the probability that at least 1 red marble is drawn?



GY	$\frac{1}{6}\frac{2}{5}$	$= \frac{2}{30}$
GR	$\frac{1}{6}\frac{3}{5}$	$= \frac{3}{30}$
YG	$\frac{2}{6}\frac{1}{5}$	$= \frac{2}{30}$
YY	$\frac{2}{6}\frac{1}{5}$	$= \frac{2}{30}$
YR	$\frac{2}{6}\frac{3}{5}$	$= \frac{6}{30}$
RG	$\frac{3}{6}\frac{1}{5}$	$= \frac{3}{30}$
RY	$\frac{3}{6}\frac{2}{5}$	$= \frac{6}{30}$
RR	$\frac{3}{6}\frac{2}{5}$	$= \frac{6}{30}$

GYYRRR

GR or YR or RG or RY or RR

$\frac{3}{30} + \frac{6}{30} + \frac{3}{30} + \frac{6}{30} + \frac{6}{30}$

$\frac{24}{30} = \frac{6 \cdot 4}{6 \cdot 5} = \boxed{\frac{4}{5}}$

Quicker way:

$P(\text{at least 1 red}) = 1 - P(\text{no red})$

How many ways to get no red?

GY YG YY

$\frac{1}{6}\frac{2}{5}$ $\frac{2}{6}\frac{1}{5}$ $\frac{2}{6}\frac{1}{5}$

$\frac{2}{30} + \frac{2}{30} + \frac{2}{30} = \frac{6}{30} = \frac{6 \cdot 1}{6 \cdot 5} = \frac{1}{5} = P(\text{no red})$

so $P(\text{at least 1 red}) = 1 - \frac{1}{5} = \boxed{\frac{4}{5}}$

Example: One hundred people were asked which video streaming service they preferred. The results are shown in the table:

	Netflix	Hulu	No opinion	**Total**
Women	17	13	12	**42**
Men	28	12	18	**58**
Total	**45**	**25**	**30**	***100***

a) What is the probability that a random person selected is male and preferred Hulu? $\frac{12}{100}$ (12%)

b) What is the probability that a random person selected is female, given they preferred Netflix?

$\frac{17}{45} = .378$ (37.8%)

Example: A shipment of 50 peaches contains 8 rotten peaches. If you randomly pick 12 peaches from the shipment what is the probability that exactly 6 of them will be rotten? (so 2 are good

50 peaches

8 rotten → (choose 6)

42 good → (choose 6)

$_8C_6 \cdot {}_{42}C_6$

ways to pick 6 rotten ones · # ways to pick 2 good ones

$28 \cdot 5245786$

146882008 ways to get exactly 6 rotten

total ways to pick 12 peaches $= {}_{50}C_{12} = 15890700$

$P(\text{exactly 6 rotten}) = \frac{146882008}{1{,}213996511 \cdot 10^{11}}$

$= 0.0012$

(~0.12%)

Example: If 5 cards are drawn from a standard deck of 52 playing cards, what is the probability that the 5 cards make a full house?

ways to get a full house:

$_{13}C_1 \cdot {_4C_3} \cdot {_{12}C_1} \cdot {_{13}C_2}$

$_{13}C_1$	$_4C_3$	$_{12}C_1$	$_{13}C_2$
# ways to choose card to get 3 of	# ways to get 3 of those cards	# ways to choose card to get 2 of	# ways to get 2 of those cards

$13 \cdot 4 \cdot 12 \cdot 6 = 3744$ ways to get a full house

ways to draw 5 cards

$_{52}C_5 = 2598960$

$P(\text{full house}) = \frac{3744}{2598960}$

$= 0.00144$

$(\approx 0.14\%)$

Example: A committee of 3 is to be selected at random from a group of 4 boys and 5 girls. What is the probability that the committee selected will consist entirely of boys?

ways to choose all boys

$_4C_3$ choose 3 boys $\cdot$ $_5C_0$ choose no girls

$4 \cdot 1 = 4$ ways

ways to choose a committee of 3

$_9C_3 = 84$

$P(\text{all boys}) = \frac{4}{84}$

$= 0.0476$

$(\approx 4.8\%)$

RRRR YY BBB

Example: A bag contains: 4 red, 2 yellow and 3 blue marbles. A marble is taken out and its color recorded, then, without replacement, another marble is taken out and its color recorded. What is the probability that at least 1 blue marble was drawn out of the bag?

opposite is no blue

ways to get no blue:

RR RY YR YY

$\frac{4}{9}\cdot\frac{3}{8} + \frac{4}{9}\cdot\frac{2}{8} + \frac{2}{9}\cdot\frac{4}{8} + \frac{2}{9}\cdot\frac{1}{8}$

$\frac{12}{72} + \frac{8}{72} + \frac{8}{72} + \frac{2}{72} = \frac{30}{72} = \frac{5}{12}$

so $P(\text{at least 1 Blue}) = 1 - \frac{5}{12} = \boxed{\frac{7}{12}}$

$= 0.583$

(58.3%)

Example: A shipment of 20 CD players contains 4 defective units. A retail outlet has ordered 5 of these units, and will receive 5 at random from the shipment. What is the probability that:
(a) exactly 4 units are good?
(b) at least one unit is good?

20 → 4 bad, 16 good

(a) exactly 4 good (and 1 bad)

$_{16}C_4$ 4 good $\cdot$ $_4C_1$ 1 bad

$1820 \cdot 4$

7280

total ways to get 5 units out of 20

$_{20}C_5 = 15504$

$P(\text{exactly 4 good}) = \frac{7280}{15504}$

$= 0.46955$

$(\approx 47\%)$

(b) at least one good (opposite is no good)

0 good 5 bad ← impossible to get all bad (only 4 bad units)

so $P(\text{all bad}) = 0$

and $P(\text{at least 1 good}) = 1 - P(\text{all bad})$

$= 1 - 0$

$= 1$

(100% chance at least one good unit)