Precalculus – Lesson Notes: Chapter 6 Additional Trigonometry Topics

6.1 day 1: Law of Sines, Oblique Triangle Area Formula

We know how to do right triangle problems like this...

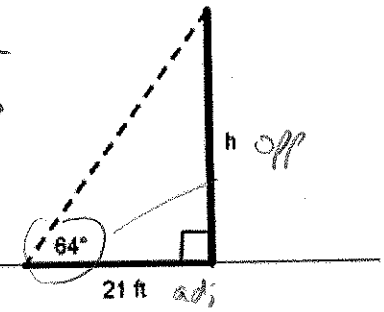
Find the height of the telephone pole:

$\tan\theta = \frac{opp}{adj}$

$\tan 64^\circ = \frac{h}{21}$

$h = 21\tan 64^\circ$

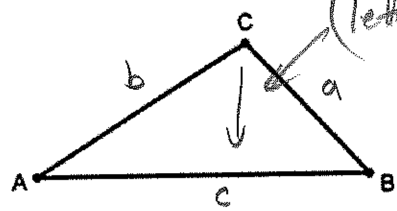
$\boxed{h = 43\text{ ft}}$



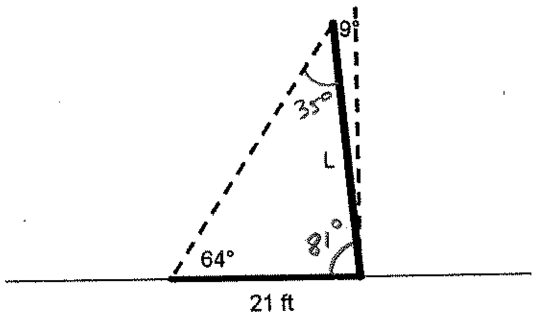
Law of Sines

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

(letters across match)



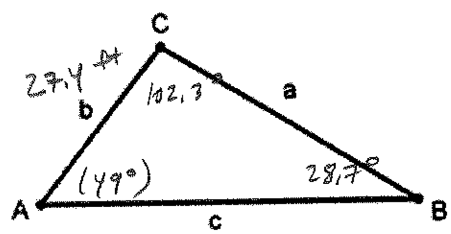
But what do we do if the pole is not vertical?



$\frac{L}{\sin 64^\circ} = \frac{21}{\sin 35^\circ}$

$(\sin 35^\circ)L = 21\sin 64^\circ$

$L = \frac{21\sin 64^\circ}{\sin 35^\circ}$ $\boxed{L = 32.9\text{ ft}}$



If C=102.3°, B=28.7°, and b=27.4 ft, find the remaining angle and sides.

$\angle A = 180^\circ - 102.3 - 28.7^\circ$

$\boxed{\angle A = 49^\circ}$

$\frac{c}{\sin 102.3^\circ} = \frac{27.4}{\sin 28.7^\circ}$

$(\sin 28.7^\circ)c = 27.4\sin 102.3^\circ$

$c = \frac{27.4\sin 102.3^\circ}{\sin 28.7^\circ}$

$\boxed{c = 55.7\text{ ft}}$

$\frac{a}{\sin 49^\circ} = \frac{27.4}{\sin 28.7^\circ}$

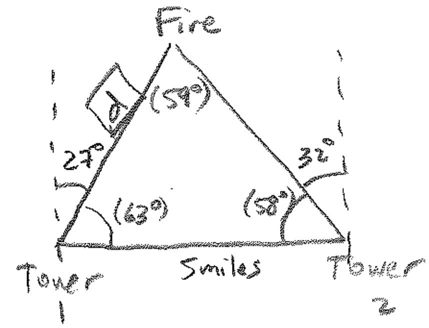
$(\sin 28.7^\circ)a = 27.4\sin 49^\circ$

$a = \frac{27.4\sin 49^\circ}{\sin 28.7^\circ}$

$\boxed{a = 43.1\text{ ft}}$

Example: Two fire ranger towers lie on the east-west line and are 5 miles apart. There is a fire with a bearing of N27°E from tower 1 and N32°W from tower 2. How far is the fire from tower 1?

$\angle Fire = 180° - 63° - 58° = 59°$



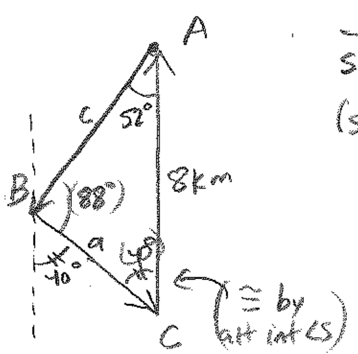
$$\frac{d}{\sin 58°} = \frac{5}{\sin 59°}$$

$$(\sin 59°)\, d = 5 \sin 58°$$

$$d = \frac{5 \sin 58°}{\sin 59°}$$

$$\boxed{d = 4.95 \text{ miles}}$$

Example: The course for a boat race starts at point A and proceeds in the direction S52°W to point B, then in the direction S40°E to point C, and finally back to point A. The point C lies 8 kilometers directly south of point A. Approximate the total distance of the race course.



$\angle B = 180° - 52° - 40°$

$\angle B = 88°$

$$\frac{a}{\sin 52°} = \frac{8}{\sin 88°}$$

$$(\sin 88°)a = 8 \sin 52°$$

$$a = \frac{8 \sin 52°}{\sin 88°}$$

$$a = 6.308 \text{ km}$$

$$\frac{c}{\sin 40°} = \frac{8}{\sin 88°}$$

$$(\sin 88°)c = 8 \sin 40°$$

$$c = \frac{8 \sin 40°}{\sin 88°}$$

$$c = 5.1454 \text{ km}$$

$$race = 8 + 6.308 + 5.1454$$

$$\boxed{race = 19.453 \text{ km}}$$

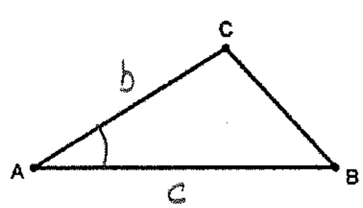
Area of Oblique Triangles Formula

(used when you know 2 sides, and the angle between these sides)

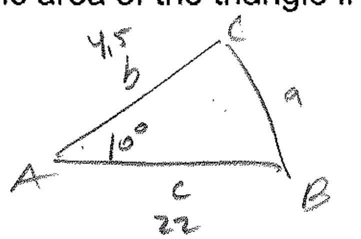
$$A_{triangle} = \frac{1}{2} bc \sin A$$

$$A_{triangle} = \frac{1}{2} ab \sin C$$

$$A_{triangle} = \frac{1}{2} ac \sin B$$



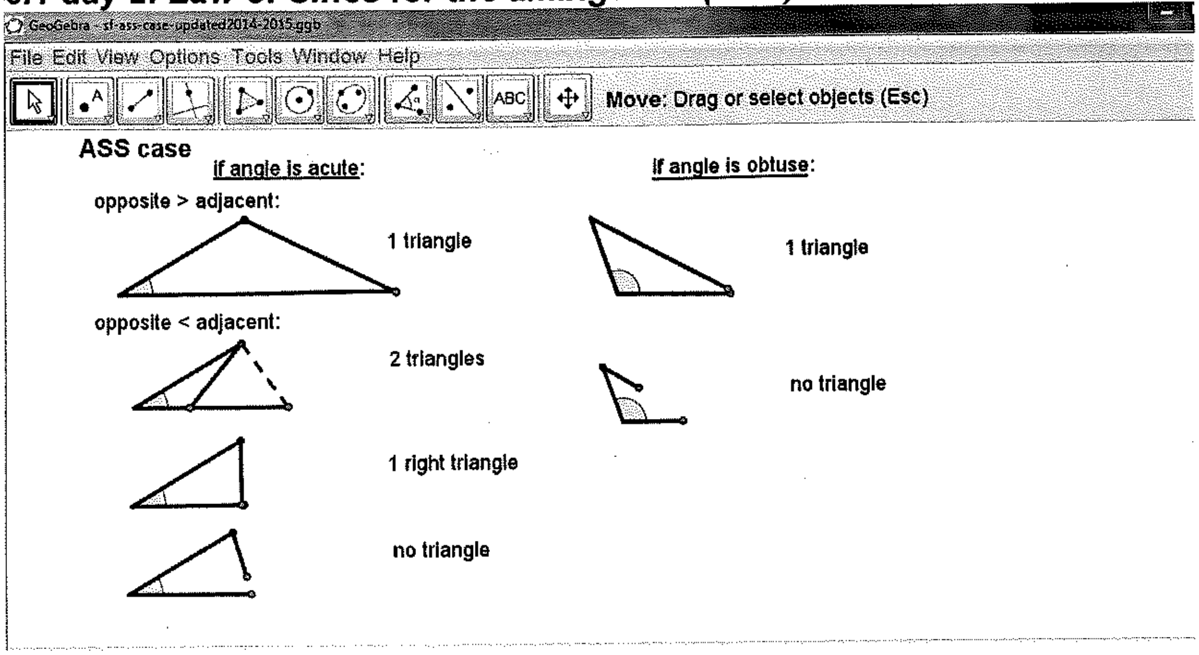
Find the area of the triangle if $A = 10°$, $b = 4.5$, $c = 22$



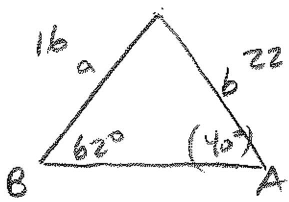
$$A_{triangle} = \frac{1}{2}(4.5)(22)\sin 10°$$

$$\boxed{A_{triangle} = 8.6 \ u^2}$$

6.1 day 2: Law of Sines for the ambiguous (ASS) case



B=62°, b=22, a=16
Find A.



$$\frac{16}{\sin A} = \frac{22}{\sin 62^\circ}$$

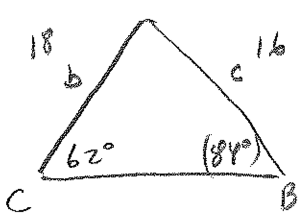
$$22\sin A = 16\sin 62^\circ$$

$$\sin A = \frac{16\sin 62^\circ}{22}$$

$$\sin A = .6421\ldots$$

$$A = \sin^{-1}(.6421\ldots) = \boxed{40^\circ}$$

C=62°, c=16,b=18
Find B.



$$\frac{18}{\sin B} = \frac{16}{\sin 62^\circ}$$

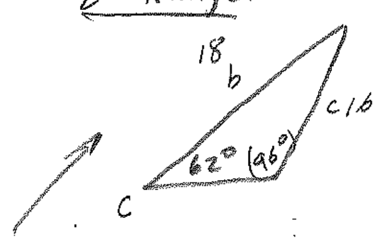
$$16\sin B = 18\sin 62^\circ$$

$$\sin B = \frac{18\sin 62^\circ}{16}$$

$$\sin B = .9933\ldots$$

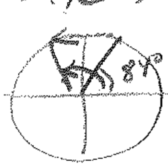
$$B = \sin^{-1}(.9933\ldots) = \boxed{84^\circ}$$

2nd triangle:



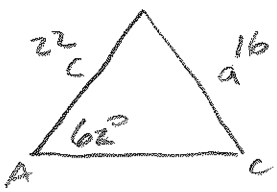
but opp < adj so 2nd triangle:

the 'other' B angle is across y-axis:



'other' $B = 180^\circ - 84^\circ = \boxed{96^\circ}$

A=62°, a=16, c=22
Find C.



$$\frac{22}{\sin C} = \frac{16}{\sin 62^\circ}$$

$$16 \sin C = 22 \sin 62^\circ$$

$$\sin C = \frac{22 \sin 62^\circ}{16}$$

$$\sin C = 1.214..$$

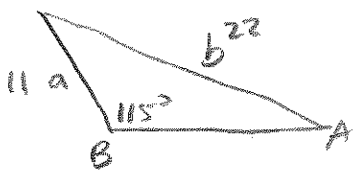
$$C = \sin^{-1}(1.214..)$$

error (because sin can't be > 1)

So angle C possible

so $\boxed{\text{no triangle}}$

B=115°, b=22, a=11
Find A.



$$\frac{11}{\sin A} = \frac{22}{\sin 115^\circ}$$

$$22 \sin A = 11 \sin 115^\circ$$

$$\sin A = \frac{11 \sin 115^\circ}{22}$$

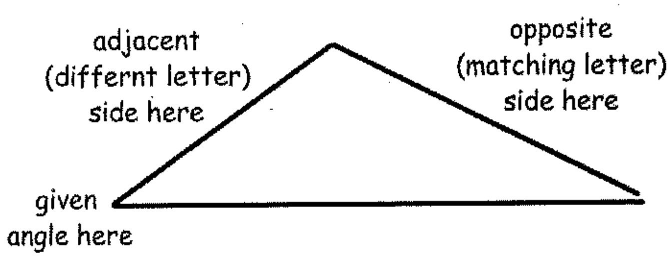
$$\sin A = .45315...$$

$$A = \sin^{-1}(.45315...) = \boxed{27^\circ}$$

(always either 1 or no triangle with obtuse given angle)

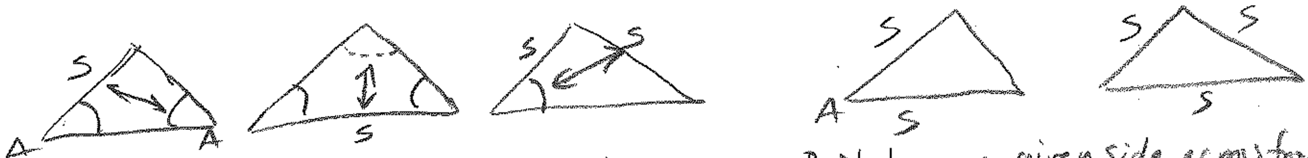
It is helpful if you:

- Always draw the angle on the left and bottom side horizontal.
- Always add the 'adjacent' side connecting to angle above:



6.2: Law of Cosines; Heron's Triangle Area Formula

Law of Sines works for AAS, ASA, and ASS cases. What about SAS and SSS?



Given a side across from an angle
LAW OF SINES

Don't have a given side across from an angle.
NEED SOMETHING ELSE (LAW OF COSINES)

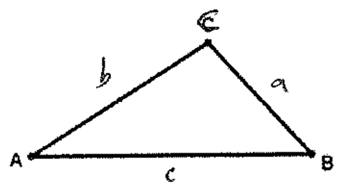
Law of Cosines:

this side — across from — the angle

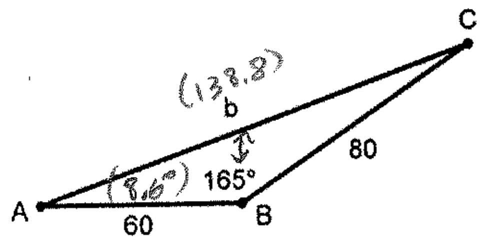
$$c^2 = a^2 + b^2 - 2ab\cos C \qquad \cos C = \frac{a^2+b^2-c^2}{2ab}$$

Pythagorean Theorem with a correction term

Same equation, solved for the cos



Example (SAS case):
Find remaining sides and angles.



$b^2 = a^2 + c^2 - 2ac\cos B$

$b^2 = 80^2 + 60^2 - 2(80)(60)\cos 165°$

$b^2 = 19272.88\ldots$

$\boxed{b = 138.8}$

Then, can use law sines:

$\frac{80}{\sin A} = \frac{138.8}{\sin 165°}$

$(138.8)\sin A = 80\sin 165°$

$\sin A = \frac{80\sin 165°}{138.8}$

$\sin A = .149175\ldots$

$A = \sin^{-1}(.149175\ldots)$

$A = 8.579$

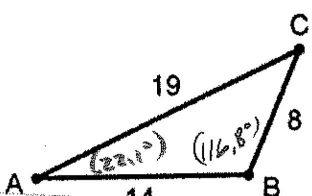
$\boxed{A = 8.6}$

Then, sum angles to 180°

$\angle C = 180° - 165° - 8.6°$

$\boxed{\angle C = 6.4°}$

Example (SSS case):
Find the angles.



Find the largest angle 1st

(to avoid ambiguous case problems)

largest angle is across from largest side

Find B 1st:

$19^2 = 14^2 + 8^2 - 2(14)(8)\cos B$

$361 = 196 + 64 - 224\cos B$

$\frac{101}{-224} = \frac{-224\cos B}{-224}$

$\cos B = -.45089\ldots$

$\boxed{B = \cos^{-1}(-.45089\ldots) = 116.8°}$

Law of sines for 2nd angle:

$\frac{8}{\sin A} = \frac{19}{\sin 116.8°}$

$(19)\sin A = 8\sin(116.8°)$

$\sin A = \frac{8\sin(116.8°)}{19}$

$\sin A = .3758\ldots$

$\boxed{A = \sin^{-1}(.3758\ldots) = 22.1°}$

Sum to 180° for last angle

$C = 180° - 116.8° - 22.1°$

$\boxed{C = 41.1°}$

Heron's Area Formula:

$$\textit{area of triangle} = \sqrt{s(s-a)(s-b)(s-c)}, \quad \textit{where } s = \frac{(a+b+c)}{2}$$

Example: Find the area of a triangle with sides of 3.5, 10.2, and 9.

$s = \frac{3.5+10.2+9}{2}$

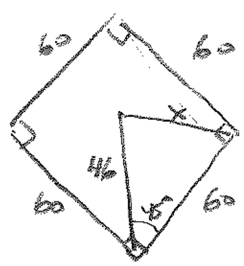
$s = 11.35$

$A_{triangle} = \sqrt{11.35(11.35-3.5)(11.35-10.2)(11.35-9)}$

$= \sqrt{240.7859\ldots}$

$\boxed{\approx 15.5\ u^2}$

Example: A baseball diamond is a square with sides of 60 ft. The pitcher's mound is not halfway between home plate and 2nd base, it is 46 ft from home plate.
How far is the pitcher's mound from 1st base?



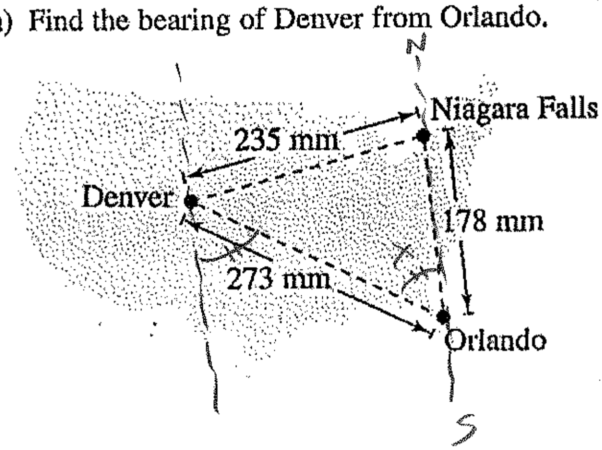
$x^2 = 46^2 + 60^2 - 2(46)(60)\cos 45°$

$x^2 = 1812.77\ldots$

$\boxed{x = 42.6\ ft}$

29. ***Navigation*** On a map, Orlando is 178 millimeters due south of Niagara Falls, Denver is 273 millimeters from Orlando, and Denver is 235 millimeters from Niagara Falls.

(a) Find the bearing of Denver from Orlando.



$235^2 = 178^2 + 273^2 - 2(178)(273)\cos X$

$55225 = 31684 + 74529 - 97188\cos X$

$\frac{-50988}{-97188} = \frac{-97188\cos X}{-97188}$

$\cos X = .5246\ldots$

$X = \cos^{-1}(.5246\ldots)$

$X = 58.356°$

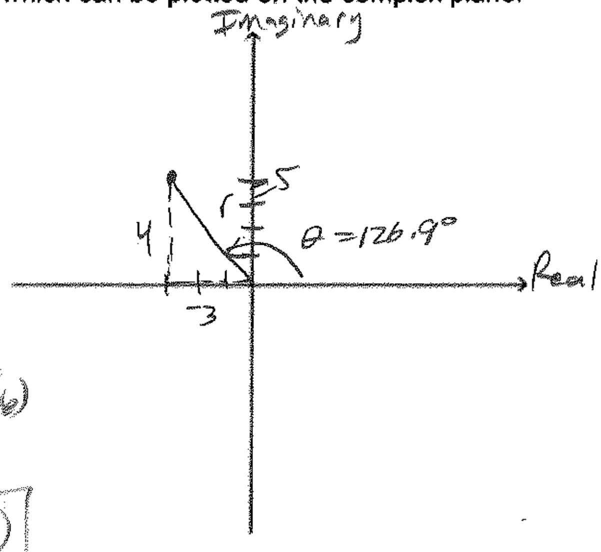
Denver to Orlando

$\boxed{S\ 58.356°\ E}$

6.5 (2 days): Trigonometric Form of Complex Numbers; DeMoivre's Theorem

We've defined a complex number, having a real and imaginary part which can be plotted on the complex plane:

$$z=-3+4i$$



Two ways to specify a complex number:

$r^2=3^2+4^2$ $\quad \tan\theta=\frac{y}{x}$

$r=\sqrt{9+16}$ $\quad \tan\theta=\frac{4}{-3}$

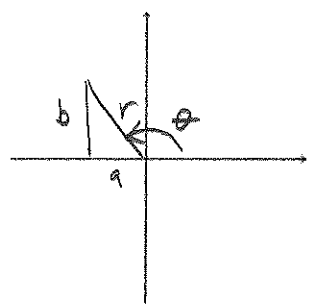
$r=\sqrt{25}$ $\quad \theta=\tan^{-1}(\frac{-4}{3})=-53.1°$ (wrong quadrant, other side of circle) $+180°$

$r=5$

$\theta=126.9°$

$\boxed{-3+4i}$ or $\boxed{5(\cos 126.9° + i\sin 126.9°)}$

More generally:



Standard (or rectangular) Form

$$z=a+bi$$

$r=\sqrt{a^2+b^2}$

$\tan\theta=\frac{b}{a}$ (careful about quadrant)

a – 'real component'

b – 'imaginary component'

Trigonometric (or polar) Form

$$z=(r\cos\theta)+i(r\sin\theta)$$

$$z=r(\cos\theta+i\sin\theta)$$

$a=r\cos\theta$

$b=r\sin\theta$

r – 'modulus'

θ – 'argument'

Converting between forms:

To convert from standard to trigonometric form:

$z=a+bi \longrightarrow z=r(\cos\theta+i\sin\theta)$

1) Find the radius: $r=\sqrt{a^2+b^2}$

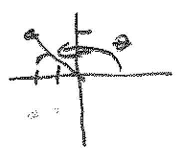
2) Find the angle: $\tan\theta=\dfrac{b}{a}$

(unit circle or $\tan^{-1}$ in calculator)

3) Write in trigonometric form: $z=r(\cos\theta+i\sin\theta)$

(always sketch)

Example: Write $-2+2i$ in trig form



1) $r=\sqrt{(-2)^2+(2)^2}$

$r=\sqrt{4+4}$

$\boxed{r=\sqrt{8}}$

2) $\tan\theta=\frac{b}{a}=\frac{2}{-2}=-1$

(special angle value)

$\theta=180°-45°=\boxed{135°}$

3) $\boxed{\sqrt{8}(\cos 135° + i\sin 135°)}$

To convert from trigonometric to standard form:

$z=r(\cos\theta+i\sin\theta) \longrightarrow z=a+bi$

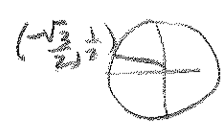
1) Distribute the radius: $z=(r\cos\theta)+(r\sin\theta)i$

2) Use unit circle or calculator for cos and sin, multiply by r to find a and b: $z=(r\cos\theta)+(r\sin\theta)i$

$z=\quad(a)\quad+\quad(b)i$

Write $4\left(\cos\dfrac{5\pi}{6}+i\sin\dfrac{5\pi}{6}\right)$ in a+bi form

1) $4\cos\frac{5\pi}{6}+i\,4\sin\frac{5\pi}{6}$



2) $\frac{5\pi}{6}$ is special angle:

$4(-\frac{\sqrt{3}}{2})+i\,4(\frac{1}{2})$

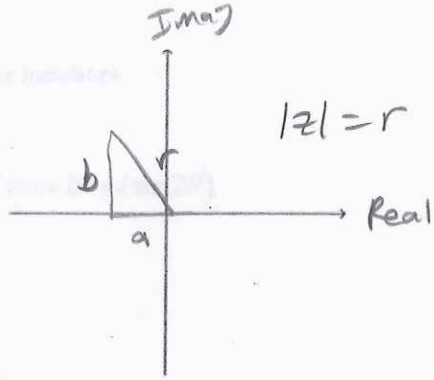
3) $\boxed{-2\sqrt{3}+2i}$

Absolute Value of a complex number:

defined to be distance from origin

$$|z| = |a + bi|$$

$$|z| = \sqrt{a^2 + b^2}$$



$|z| = r$ (always positive and real)

Example: Find the absolute values:

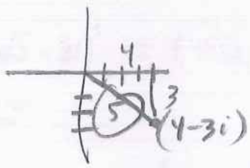
(a) $z = 4 - 3i$

$|z| = \sqrt{(4)^2 + (-3)^2}$

$|z| = \sqrt{16 + 9}$

$|z| = \sqrt{25}$

$\boxed{|z| = 5}$

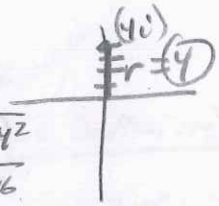


(b) $z = 4i$

$|z| = \sqrt{0^2 + 4^2}$

$|z| = \sqrt{0 + 16}$

$\boxed{|z| = 4}$

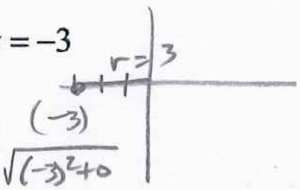


(c) $z = -3$

$|z| = \sqrt{(-3)^2 + 0}$

$|z| = \sqrt{9}$

$\boxed{|z| = 3}$



Multiplication and Division of Complex Numbers

Standard form: Like we learned last semester... Example: $(-2 + 2i)(3 - i)$
(divide by multiplying top and bottom by complex conjugate of denominator)

Trigonometric form: $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

$$z_1 z_2 = r_1(\cos\theta_1 + i\sin\theta_1) \cdot r_2(\cos\theta_2 + i\sin\theta_2)$$

$$z_1 z_2 = r_1 r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2))$$

$$\frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)}$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2))$$

Example: $z_1 = 8(\cos 120° + i\sin 120°)$, $z_2 = 6(\cos 150° + i\sin 150°)$

Find $z_1 z_2$

$8 \cdot 6(\cos(120° + 150°) + i\sin(120° + 150°))$

$\boxed{48(\cos(270°) + i\sin(270°))}$

Find $\frac{z_1}{z_2} = \frac{8}{6}(\cos(120° - 150°) + i\sin(120° - 150°))$

$= \boxed{\frac{4}{3}(\cos(-30°) + i\sin(-30°))}$

DeMoivre's Theorem: used to find powers of complex numbers

$$z = r(\cos\theta + i\sin\theta)$$
$$z^2 = rr(\cos(\theta+\theta) + i\sin(\theta+\theta)) = r^2(\cos 2\theta + i\sin 2\theta)$$
$$z^3 = r^3(\cos 3\theta + i\sin 3\theta)$$

$$z^n = [r(\cos\theta + i\sin\theta)]^n$$
$$z^n = r^n(\cos n\theta + i\sin n\theta)$$

Example: Evaluate $(-2\sqrt{3}-2i)^5 = (4(\cos 210° + i\sin 210°))^5$

1) convert to trig form

$-2\sqrt{3}-2i$

$r = \sqrt{a^2+b^2}$
$r = \sqrt{(2\sqrt{3})^2+(2)^2}$
$r = \sqrt{4 \cdot 3 + 4}$
$r = \sqrt{16}$
$r = 4$

$\tan\theta = \frac{b}{a} = \frac{-2}{-2\sqrt{3}}$
$\tan\theta = \frac{1}{\sqrt{3}}$
(divide by 2 trick)
$\tan\theta = \frac{1/2}{\sqrt{3}/2}$ y x

$\theta = 210°$

2) Use DeMoivre's Theorem

$[4(\cos 210° + i\sin 210°)]^5$
$= 4^5(\cos(5 \cdot 210°) + i\sin(5 \cdot 210°))$
$= 1024(\cos 1050° + i\sin 1050°)$
$= \boxed{1024(\cos 330° + i\sin 330°)}$

find easier to locate coterminal angle:
1050°
−360°
690°
−360
330°

Complex Roots:

What are the square roots of 4?

$(2)^2 = 4 \quad (-2)^2 = 4$

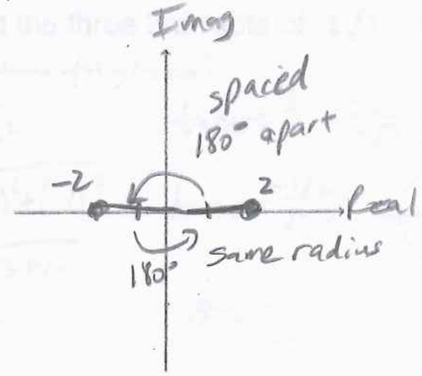
2 and −2

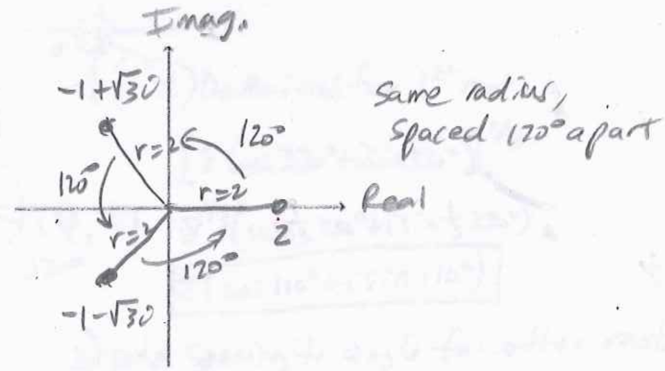
What are the cube roots of 8?

$z^3 = 8$ also $-1+\sqrt{3}i$ and $-1-\sqrt{3}i$

$(-1+\sqrt{3}i)(-1+\sqrt{3}i)(-1+\sqrt{3}i)$
$(-1+\sqrt{3}i)(1-\sqrt{3}i-\sqrt{3}i+3i^2)$
$(-1+\sqrt{3}i)(1-2\sqrt{3}i-3)$
$(-1+\sqrt{3}i)(-2-2\sqrt{3}i)$
$2+2\sqrt{3}i-2\sqrt{3}i-2 \cdot 3i^2$
$2+6$
8 ✓

Let's plot the roots found above on the complex plane:





For a complex number, z, there will be n complex roots.

- The first root is found using DeMoivre's theorem:

$$\sqrt[n]{z} = \sqrt[n]{r(\cos\theta + i\sin\theta)}$$

$$z^{\frac{1}{n}} = \left[r(\cos\theta + i\sin\theta)\right]^{\frac{1}{n}} = r^{\frac{1}{n}}\left(\cos\frac{1}{n}\theta + i\sin\frac{1}{n}\theta\right)$$

$$\text{First root} = \sqrt[n]{r}\left(\cos\frac{\theta}{n} + i\sin\frac{\theta}{n}\right)$$

- The other roots will be evenly spaced around a circle, so keep radius the same and add the 'root spacing' to the angles:

$$\text{'root spacing'} = \frac{360°}{n}$$

What are the cube roots of 8?

$8 = 8(\cos 0° + i\sin 0°)$

$\sqrt[3]{8} = 8^{1/3} = [8(\cos 0° + i\sin 0°)]^{1/3}$

$= 8^{1/3}[\cos\frac{1}{3}0° + i\sin\frac{1}{3}0°]$

1st root $= \boxed{2(\cos 0° + i\sin 0°)} = \boxed{2}$

spacing $= \frac{360°}{3} = 120°$

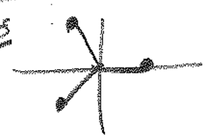
2nd root $= \boxed{2(\cos 120° + i\sin 120°)} = 2(-\frac{1}{2}) + i(2)(\frac{\sqrt{3}}{2})$

$= \boxed{-1 + \sqrt{3}i}$

3rd root $= \boxed{2(\cos 240° + i\sin 240°)} = 2(-\frac{1}{2}) + i(2)(-\frac{\sqrt{3}}{2})$

$= \boxed{-1 - \sqrt{3}i}$

roots



Find the four 4th roots of 16: $= 16(\cos 0° + i\sin 0°)$

$\sqrt[4]{16} = [16(\cos 0° + i\sin 0°)]^{1/4}$

1st root $= 16^{1/4}(\cos\frac{1}{4}0° + i\sin\frac{1}{4}0°)$

$= \boxed{2(\cos 0° + i\sin 0°)} = \boxed{2}$

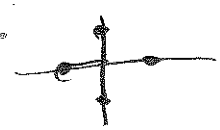
spacing $= \frac{360°}{4} = 90°$

2nd root $= \boxed{2(\cos 90° + i\sin 90°)} = 2(0 + i(1)) = \boxed{2i}$

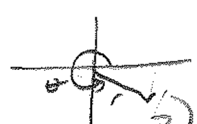
3rd root $= \boxed{2(\cos 180° + i\sin 180°)} = 2(-1 + i(0)) = \boxed{-2}$

4th root $= 2(\cos 270° + i\sin 270°) = 2(0 + i(-1)) = \boxed{-2i}$

roots



Find the three 3rd roots of $4\sqrt{3} - 4i$



1) convert to trig form:

$r = \sqrt{a^2 + b^2}$

$r = \sqrt{(4\sqrt{3})^2 + (-4)^2}$

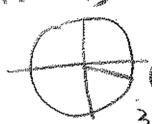
$r = \sqrt{16 \cdot 3 + 16}$

$r = \sqrt{64}$

$r = 8$

$\tan\theta = \frac{b}{a} = \frac{-4}{4\sqrt{3}} = \frac{-1}{\sqrt{3}}$

$\tan\theta = \frac{-1/2}{\sqrt{3}/2}$



$\theta = 330°$

$4\sqrt{3} - 4i = 8(\cos 330° + i\sin 330°)$

2) DeMoivre's for 1st root:

$[8(\cos 330° + i\sin 330°)]^{1/3}$

$8^{1/3}(\cos\frac{1}{3}330° + i\sin\frac{1}{3}330°)$

$\boxed{2(\cos 110° + i\sin 110°)}$

3) add spacing to angle for other roots:

spacing $= \frac{360°}{3} = 120°$

2nd root $= 2(\cos(110° + 120°) + i\sin(110° + 120°))$

$= \boxed{2(\cos 230° + i\sin 230°)}$

3rd root $= \boxed{2(\cos 350° + i\sin 350°)}$