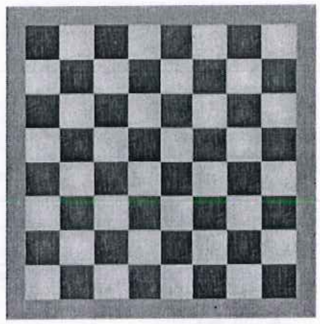
Precalculus – Lesson Notes: Chapter 3

3.1 day 1 – Exponential Functions

Consider this ancient riddle:

"A courtier once presented a Persian king with a beautiful, hand-made chessboard. The king asked what he would like in return for his gift, and the courtier surprised the king by asking for a single grain of rice to be placed on the first square, and then on each successive square, for double the amount of rice from the previous to be placed (e.g., two grains on the second square, four grains on the third, etc.) Thinking this a paltry sum, the king readily agreed and asked for the rice to be brought."



squares	rice	
start	1	$1 = 1(2)^0$
1	2	$1(2)$
2	4	$[1(2)]2 = 1(2)^2$
3	8	$[1(2)(2)]2 = 1(2)^3$
4	16	$= 1(2)^4$
5	32	$= 1(2)^5$
x	.	$= (1)(2)^x$
.	.	
63	?	$\approx$ 9,223,372,036,854,780,000

$>$9 quintillion

Did the king pay a fair price for his new chessboard?

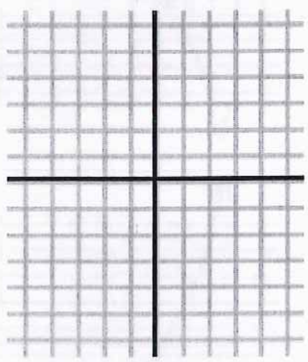
Exponential equation: $f(x) = a^x$

a = the 'base' of the exponential function

Graph each of these using a t-chart (do not use a calculator to graph):

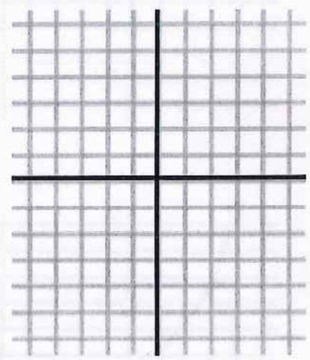
#1) $f(x) = 2^x$

x	f(x)
-2	
-1	
0	
1	
2	
3	



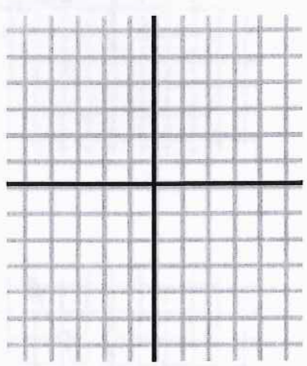
#2) $f(x) = 3^x$

x	f(x)
-2	
-1	
0	
1	
2	
3	



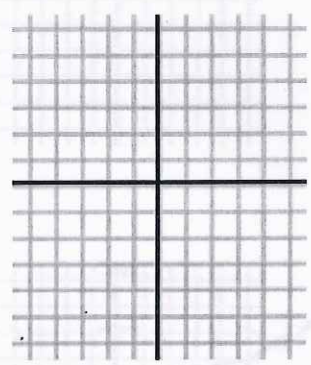
#3) $f(x) = -2^x$

x	f(x)
-2	
-1	
0	
1	
2	
3	



#4) $f(x) = 2^{-x}$

x	f(x)
-2	
-1	
0	
1	
2	
3	



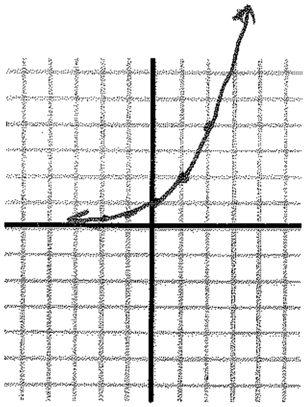
ecalculus – Exponential Functions 3.1 day 1 (groups finish)

these on board 1st

Graph each of these using a t-chart (do not use a calculator to graph):

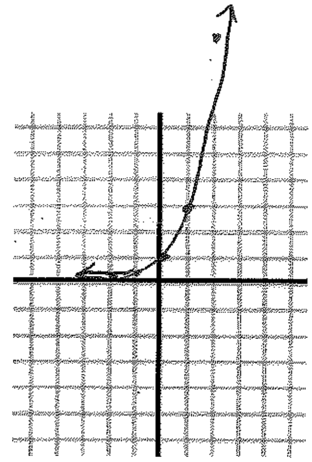
#1) $f(x)=2^x$

x	f(x)
-2	$2^{-2}=\frac{1}{2^2}=\frac{1}{4}$
-1	$2^{-1}=\frac{1}{2}$
0	$2^0=1$
1	$2^1=2$
2	$2^2=4$
3	$2^3=8$



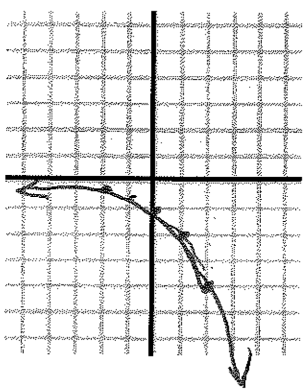
#2) $f(x)=3^x$

x	f(x)
-2	$3^{-2}=\frac{1}{3^2}=\frac{1}{9}$
-1	$3^{-1}=\frac{1}{3}$
0	$3^0=1$
1	$3^1=3$
2	$3^2=9$
3	



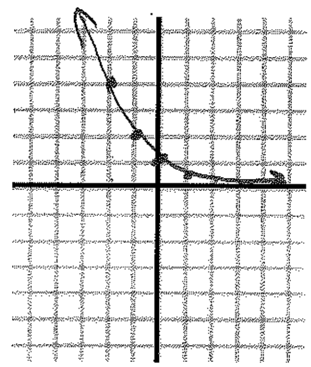
#3) $f(x)=-2^x$

x	f(x)
-2	$-2^{-2}=-\frac{1}{2^2}=-\frac{1}{4}$
-1	$-2^{-1}=-\frac{1}{2}$
0	$-2^0=-1$
1	$-2^1=-2$
2	$-2^2=-4$
3	$-2^3=-8$



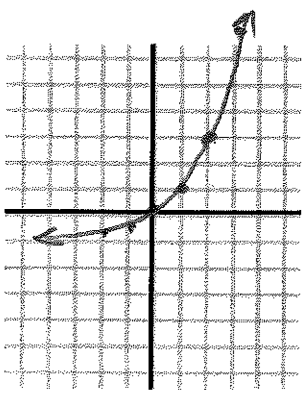
#4) $f(x)=2^{-x}$

x	f(x)
-2	$2^{-(-2)}=2^2=4$
-1	$2^{-(-1)}=2^1$
0	$2^0=1$
1	$2^{-1}=\frac{1}{2}$
2	$2^{-2}=\frac{1}{4}$
3	$2^{-3}=\frac{1}{8}$



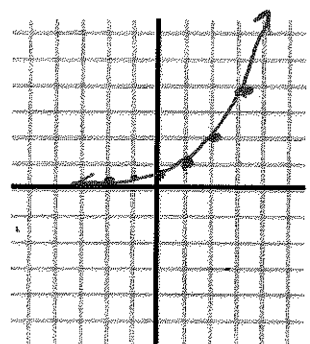
#5) $f(x)=2^{(x)}-1$

x	f(x)
-2	$2^{-2}-1=\frac{1}{2^2}-1=\frac{1}{4}-1$
-1	$2^{-1}-1=\frac{1}{2}-1=-\frac{1}{2}$
0	$2^0-1=1-1=0$
1	$2^1-1=1$
2	$2^2-1=3$
3	$2^3-1=7$



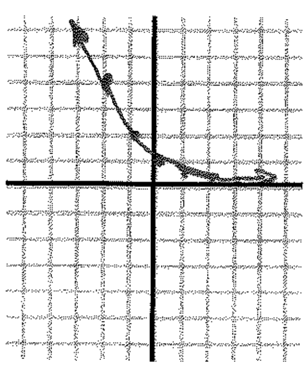
#6) $f(x)=2^{(x-1)}$

x	f(x)
-2	$2^{-2-1}=2^{-3}=\frac{1}{2^3}$
-1	$2^{-1-1}=2^{-2}=\frac{1}{4}$
0	$2^{0-1}=2^{-1}=\frac{1}{2}$
1	$2^{1-1}=2^0=1$
2	$2^{2-1}=2$
3	$2^{3-1}=2^2$



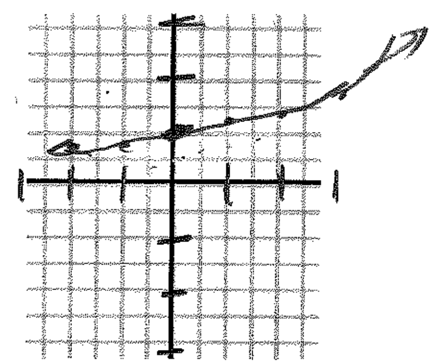
#7) $f(x)=\left(\frac{1}{2}\right)^x$

x	f(x)
-2	$\frac{1}{2}^{-2}=(2)^2=4$
-1	$\frac{1}{2}^{-1}=2$
0	$\frac{1}{2}^0=1$
1	$\frac{1}{2}^1=\frac{1}{2}$
2	$\left(\frac{1}{2}\right)^2=\frac{1}{4}$
3	

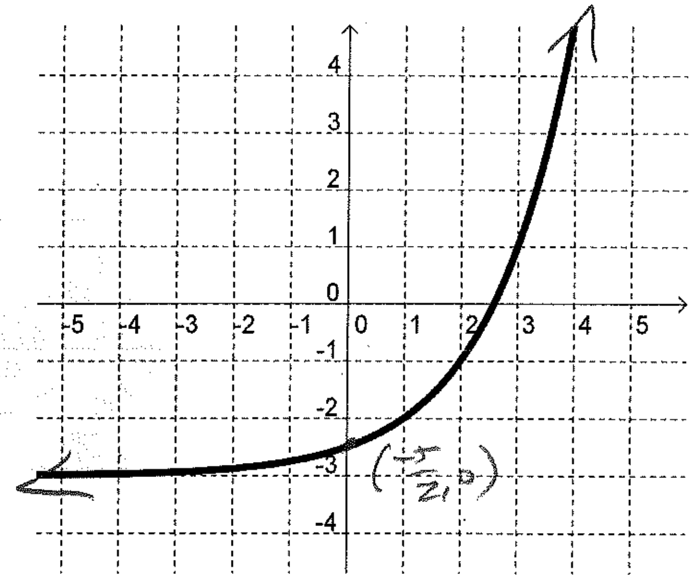


#8) $f(x)=\cancel{2^{(x-1)}}\ (1.2)^x$

x	f(x)
-2	$1.2^{-2}=0.694$
-1	$1.2^{-1}=.833$
0	1
1	1.2
2	1.44
3	1.728



$f(x) = 2^{(x-1)} - 3$ Graph looks like this:



1) What is the domain of this function?

$(-\infty, \infty)$

2) What is the range of this function?

$(-3, \infty)$

3) Does this function have an asymptote? If so, what is the equation of the asymptote?

$y = -3$

4) Can you find the y-intercept algebraically? (What does x have to be?)

$x = 0$

$f(0) = 2^{(0-1)} - 3$

$f(0) = 2^{-1} - 3$

$= \frac{1}{2} - 3$

$= \frac{1}{2} - \frac{6}{2} = \frac{-5}{2}$

so $\left(\frac{-5}{2}, 0\right)$

$-2\frac{1}{2}$

5) Can you find the x-intercept algebraically? (What does y have to be?)

$y = 0$

$0 = 2^{x-1} - 3$

$2^{x-1} = 3$

can't complete without a way to 'undo' 2^x

(next week)

3.1 day 2 – Exponential Functions: Interest Calculations

You put $100 in a bank account that earns 8% interest per year, but pays this interest out 4 times per year (2% every 3 months). Starting with your initial $100, figure out how much money is in your account every 3 months for ~~8~~ 2 years. (don't round)

months in account (start)	money in your account	amount of interest added $2b(.02)$	months in account (end)	money in your account
0	$100	$2	3	$102
3	$102	$2.04	6	$104.04
6	$104.04	$2.0808	9	$106.1208
9	$106.1208	$2.122416	12	$108.243216
12	$108.243216	$2.16486432	15	$110.4080803
15	$110.4080803	$2.208161606	18	$112.6162419
18	$112.6162419	$2.252324838	21	$114.8685667
21	$114.8685667	$2.297371335	24	$117.165938

What if I wanted to know how much money was in your account at the end of 12 years?

need a formula

take any line

$A_{end} = A_{start} + A_{start}(.02)$

$A_{end} = A_{start}(1 + .02)$

from beginning to any line:

$A_{end} = A_{start}(1+.02)^{\#\text{of lines}}$ # times paid per year * # years

$A_{end} = A_{start}\left(1 + \frac{\text{yearly interest}}{\#\text{ times paid per year}}\right)$

$$\boxed{A = P\left(1+\frac{r}{n}\right)^{nt}}$$

P = 'principal' starting amount

r = annual interest rate

n = number of times 'compounded' (paid out each year)

t = number of years in account

so: $P = 100$

$r = .08$

$n = 4$

$t = 12$

$A = 100\left(1+\frac{.08}{4}\right)^{4(12)}$

$\boxed{\$258.71}$

For compounding 'n' times per year:

$$A = P\left(1+\frac{r}{n}\right)^{nt}$$

compound annually:	n = 1
compound quarterly:	n = 4
compound monthly:	n=12
compound daily:	n = 365

Example: Invest \$5000 in an account with annual interest rate of 5% for 10 years. What is the amount in the account at the end of 10 years if the account compounds:

(a) annually $A = 5000\left(1+\frac{.05}{1}\right)^{(1)(10)} = \$8144.47$
$(n=1)$

(b) monthly $A = 5000\left(1+\frac{.05}{12}\right)^{(12)(10)} = \$8235.05$
$n=12$

(c) daily $A = 5000\left(1+\frac{.05}{365}\right)^{(365)(10)} = \$8243.32$
$n=365$

(d) every hour $A = 5000\left(1+\frac{.05}{8760}\right)^{8760(10)} = \$8243.59$
$n = 365 \times 24 = 8760$

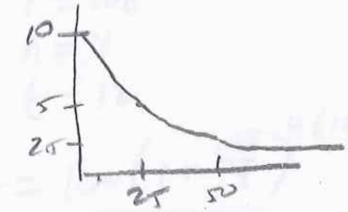
What if compounded every 'instant'...compounded 'continuously'?

For continuous compounding: $A = Pe^{rt}$

(e) continuously $A = 5000e^{.05(10)} = \$8243.61$

Other applications of exponential functions

<u>Radioactive Decay</u>: Let y represent the mass of a quantity of a radioactive element whose half-life is 25 years. After t years, the mass (in grams) is $y = 10\left(\frac{1}{2}\right)^{\frac{t}{25}}$



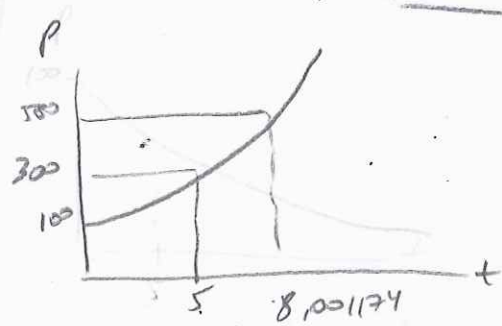
(a) What is the initial mass (when t=0)?

$y = 10\left(\frac{1}{2}\right)^{\frac{0}{25}} = \boxed{10 \text{ grams}}$

(b) How much of the initial mass is present after 80 years? $y = 10\left(\frac{1}{2}\right)^{\left(\frac{80}{25}\right)} = \boxed{1.088 \text{ grams}}$

<u>Bacteria population</u>: Given by $P(t) = 100e^{0.2197t}$

(a) What is the initial bacteria population? $\boxed{100}$
(b) Graph (calculator)
(c) What is the bacteria population at 5 hours?
(d) When does the bacteria population reach 580?



(c) $P(5) = 100e^{.2197(5)} = 299.966$
$\approx \boxed{300}$

(d) (use calculator)
(add $y=580$ line and intersect) $\approx \boxed{8 \text{ hours}}$

3.2 – Logarithms

Solve: $2^x - 3 = 0$

$\quad\;\; +3 \quad +3$

$2^x = 3$

need to 'undo' 2^x

exponent $\longrightarrow \boxed{2^x} \longrightarrow$ number

raises base 2
to input x

number $\longrightarrow \boxed{\log_2(number)} \longrightarrow$ exponent

gives exponent base 2
needs to be raised to
to get the number

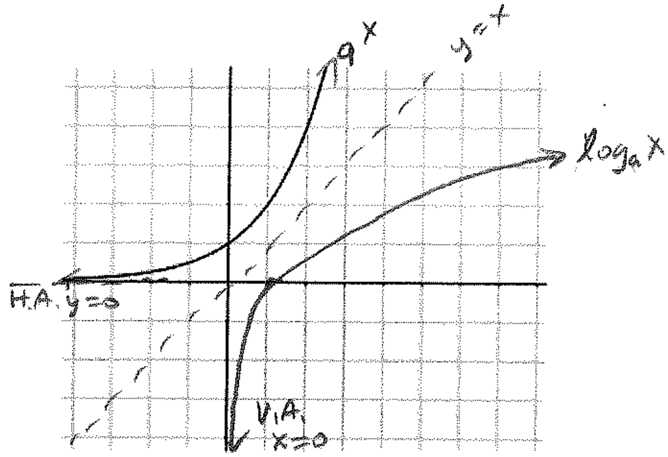
Exponential and logarithmic functions (of same base) are inverses of each other

$f(x) = a^x \Leftrightarrow f^{-1}(x) = \log_a(x)$

exponential function

logarithmic function

"log base a of x"



Logarithm and exponential functions (of same base) are inverses - they 'undo' each other:

$3^x = 8$

$\log_3(3^x) = \log_3(8)$

$x = \log_3(8)$

"a logarithm is an exponent"...

$\log_4 x = 2$

$4^{(\log_4 x)} = 4^{(2)}$

$x = 4^2$

$x = 16$

$$a^y = x \qquad \leftrightarrow \qquad y = \log_a(x)$$

"Exponential form" "Logarithmic form"

if a=10 (base is 10):

$10^2 = 100 \Leftrightarrow \log_{10}(100) = 2$

$10^3 = 1000 \Leftrightarrow \log_{10}(1000) = 3$

if a=2 (base is 2):

$2^2 = 4 \Leftrightarrow \log_2(4) = 2$

$2^3 = 8 \Leftrightarrow \log_2(8) = 3$

raise base to an exponent, get a number $\leftrightarrow$ log of that number, gives the exponent

1) set log = x
2) write in exponential form

$\log_2(32) = x$ $\boxed{5}$

$2^x = 32$
$2^5 = 32$
so $x = 5$

$\log_3(27) = x$ $\boxed{3}$

$3^x = 27$
$3^3 = 27$

$\log_{10}\left(\frac{1}{100}\right) = x$ $\boxed{-2}$

$10^x = \frac{1}{100}$
$10^{-2} = \frac{1}{100}$

$\log_4(2) = x$ $\boxed{\frac{1}{2}}$

$4^x = 2$
$4^{1/2} = \sqrt{4} = 2$

$f(x) = \log_{10}(x)$ 'common logarithmic function'

(calculators have buttons for common log, and natural log)

$f(x) = \log_e(x)$ 'natural logarithmic function'

$\log_e(x) = \ln(x)$

if base isn't marked assumed to be base 10

$\log(x)$ means $\log_{10}(x)$

A few properties of logarithmic functions:

$\log_a 1 = 0$ $\ln 1 = 0$

$\log_a a = 1$ $\ln e = 1$

$\log_a\left(a^x\right) = x$ $\ln\left(e^x\right) = x$

$a^{(\log_a x)} = x$ $e^{(\ln x)} = x$

if $\log_a x = \log_a y$, *then* $x = y$
if $\ln x = \ln y$, *then* $x = y$
} 1:1 property

Examples: Find x...

$\log_5 1 = x$

$5^x = 1$
$5^0 = 1$
$\boxed{x = 0}$

$\log_3 3 = x$

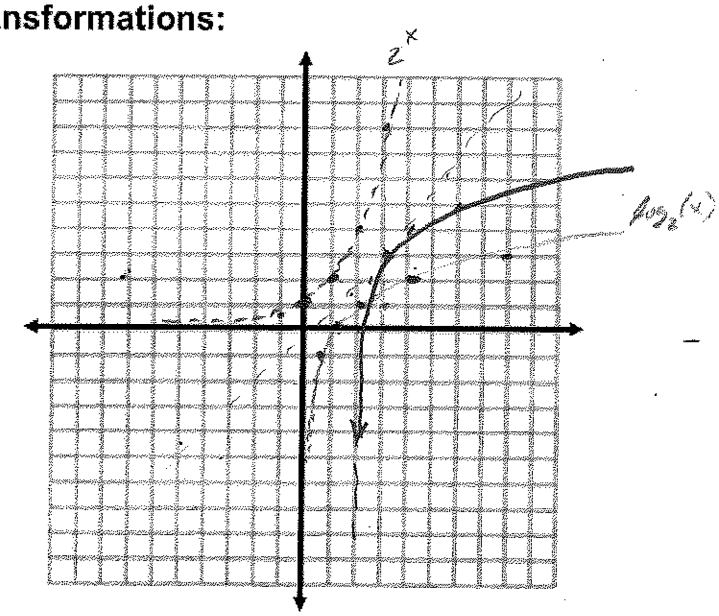
$3^x = 3$
$3^1 = 3$
$\boxed{x = 1}$

$\log_5 x = \log_5 8$

$\boxed{x = 8}$

To graph a logarithmic function by hand, graph the inverse exponential function and reflect over y=x line, then apply transformations:

$y = \log_2(x-2)+3$ inverse of $y = 2^x$



Domain
$(2, \infty)$

Range
$(-\infty, \infty)$

Asymptote
V.A. $x = 2$

x-intercept
$(y=0)$
$0 = \log_2(x-2)+3$
$\log_2(x-2) = -3$
$2^{-3} = x-2$
$\frac{1}{8} = x-2$
$2 + \frac{1}{8} = x = \frac{17}{8}$ $\boxed{\left(\frac{17}{8}, 0\right)}$

y-intercept
$(x=0)$
$y = \log_2(0-2)+3$
$y = \log_2(-2)+3$
not in domain
$\boxed{\text{no y-int}}$

increasing/
decreasing
incr: $(2, \infty)$
decr: nowhere

HAlg3-4, 3.3 day 1 Notes – Properties of Logarithms

The logarithmic function works the same way as other functions, accepts an input, provides an output:

$$f(x) = \log_{10}(x)$$
$$f(1000) = \log_{10}(1000)$$
$$f(1000) = 3$$

$1000 \xrightarrow{x} \boxed{\log(x)} \longrightarrow 3$

$10^{?}$

2 ways to evaluate this function...

By hand

$\log_{10}(1000) \stackrel{\text{S}}{=} x$

$10^x = 1000$

$x = 3$

Calculator

LOG(1000)

If our calculators can only find log base 10 or base e, how do we find $\log_2 8$?

Change of base formula: $\log_a x = \dfrac{\log_b x}{\log_b a}$ changes from base a to base b.

Base b can be anything, but we usually use b=10 or b=e because we have calculator keys for logarithms in those bases.

Examples:	using 'log' key:	using 'ln' key:
$\log_2 8 =$	$\dfrac{\log_{10} 8}{\log_{10} 2} = \dfrac{.90308...}{.3010...} = 3$	$\dfrac{\ln 8}{\ln 2} = \dfrac{2.079...}{.6931...} = 3$
$\log_4 30 =$	$\dfrac{\log_{10} 30}{\log_{10} 4} = 2.4534...$	$\dfrac{\ln 30}{\ln 4} = 2.4534...$
$\log_5 18 =$	$\dfrac{\log_{10} 18}{\log_{10} 5} = 1.7958...$	$\dfrac{\ln 18}{\ln 5} = 1.7958...$

More properties of logarithms:

$$\log_a(uv) = \log_a(u) + \log_a(v) \qquad \ln(uv) = \ln(u) + \ln(v)$$
$$\log_a\left(\frac{u}{v}\right) = \log_a(u) - \log_a(v) \qquad \ln\left(\frac{u}{v}\right) = \ln(u) - \ln(v)$$
$$\log_a u^n = n\log_a u \qquad \ln u^n = n\ln u$$

(Note: 'similar' to exponent rules: $x^u x^v = x^{(u+v)}$ and $\dfrac{x^u}{x^v} = x^{(u-v)}$)

Examples using properties:

$\log_3(2xy) = \log_3 2 + \log_3 x + \log_3 y$

$\log_5(2x^3y^2) = \log_5 2 + \log_5 x^3 + \log_5 y^2$
$= \log_5 2 + 3\log_5 x + 2\log_5 y$

$\log_4(4y) = \log_4 4 + \log_4 y$
$= 1 + \log_4 y$

$\ln\left(\dfrac{14x^3}{2h^2w^4}\right) = \ln(14x^3) - \ln(2h^2w^4)$
$= \ln 14 + \ln x^3 - [\ln 2 + \ln h^2 + \ln w^4]$
$= \ln 14 + 3\ln x - \ln 2 - 2\ln h - 4\ln w$

$\log_5(h) + \log_5(w) = \log_5(hw)$

$\log_5(2x) - 2\log_5(y) = \log_5(2x) - \log_5(y^2)$
$= \log_5\left(\dfrac{2x}{y^2}\right)$

$2\log_3 x - 3\log_3 y + \dfrac{1}{2}\log_3 z =$
$\log_3(x^2) - \log_3(y^3) + \log_3(z^{1/2})$
$\log_3\left(\dfrac{x^2 z^{1/2}}{y^3}\right)$
$\log_3\left(\dfrac{x^2\sqrt{z}}{y^3}\right)$

$\dfrac{1}{3}(2\ln x - 4\ln y - \ln(z+2)) =$
$\dfrac{1}{3}[\ln x^2 - \ln y^4 - \ln(z+2)]$
$\dfrac{1}{3}\ln\left(\dfrac{x^2}{y^4(z+2)}\right)$
$\ln\left(\dfrac{x^2}{y^4(z+2)}\right)^{1/3}$
$\ln\left(\sqrt[3]{\dfrac{x^2}{y^4(z+2)}}\right)$

HAlg3-4, 3.3 day 2 Notes – Rewriting log expressions, Applications of logarithms

Sometimes, you can find the exact values of logarithmic expressions without a calculator or rewrite logarithmic express using log properties:

teacher $\log_6 \sqrt[3]{6}$ $\quad \log_6 6^{1/3}$ $\quad \frac{1}{3}\log_6 6$ $\quad (6^x = 6, x=1)$ $\quad \frac{1}{3}(1)$ $\quad \boxed{\frac{1}{3}}$

students $\log_5 \frac{1}{125}$ $\quad \log_5 1 - \log_5 125$ $\quad 5^x = 1$, 0 $\quad 5^x = 125$, -3 $\quad \boxed{-3}$ $\quad$ or $\log_4 (64)$ $\quad \log_4 4^3 = 3$

students $\log_4(-16)$ $\quad 4^x = -16$ (not possible)

teacher $\log_4 2 + \log_4 32$ $\quad 4^x = 2$ $\quad 4^x = 32 \to$ but $2^5 = 32$ rewrite 4^x as $(2^2)^x$, $2^{2x} = 32 = 2^5$, $2x = 5$, $x = \frac{5}{2}$

$\frac{1}{2} + \frac{5}{2} = \frac{6}{2}$ $\boxed{3}$

Practice:

teacher $\log_5 \frac{1}{15}$ $\quad \log_5 1 - \log_5 15$
$0 - \log_5(3 \cdot 5)$
$0 - \log_5 5 - \log_5 3$
$0 - 1 - \log_5 3$
$\boxed{-1 - \log_5 3}$

students $\log_2(4^2 \cdot 3^4)$ $\quad \log_2 4^2 + \log_2 3^4$
$2\log_2 4 + 4\log_2 3$
$2(2) + 4\log_2 3$
$\boxed{4 + 4\log_2 3}$

students $\log_{10}\left(\frac{\cancel{39}\,3}{300}\right) \to \log_{10} 3 - \log_{10} 100$ $\quad 10^x = 100$
$\boxed{\log_{10} 3 - 2}$

students $\ln \frac{6}{e^2}$ $\quad \ln 6 - \ln e^2$
$\ln 6 - 2\ln e$
$\ln 6 - 2$

Applications of logarithms: Situations where a variable changes rapidly at first, then varies more slowly. Example:

Students participating in a psychological experiment attended several lectures. After the last lecture, and every month for the next year, the students were tested to see how much of the material they remembered. The average scores for the group were given by the memory model:

$f(t) = 90 - 15\log_{10}(t+1) \quad 0 \le t \le 12$ where t is time in months.

(a) What was the average score on the original exam (t=0)? $\quad 90 - 15\log_{10}(1)$ $\quad 10^x = 1$, $x = 0$ $\quad 90 - 0$ $\quad \boxed{90}$

(b) What was the average score after 6 months? $\quad 90 - 15\log_{10}(7) = \boxed{77.3}$

(c) What was the average score after 12 months? $\quad 90 - 15\log_{10}(13) = \boxed{73.3}$

(d) How long did it take for the average score to decrease to 75?

$90 - 15\log_{10}(t+1) = 75$
$15\log_{10}(t+1) = 15$
$\log_{10}(t+1) = 1$
$10^1 = t + 1$
$10 = t + 1$
$-1 \quad -1$
$9 = t$

$\boxed{9 \text{ months}}$

HAlg3-4, 3.4 day 1 Notes – Solving Exponential and Logarithmic Equations

Reminder: How do we solve for x in these cases?

$2^x = 3$ Compare: $\frac{2x}{2}=\frac{3}{2}$ $x^2=3$ $\sqrt{x^2}=\sqrt{3}$

need to 'undo' 2^x

$\log_2(2^x)=\log_2(3)$

$x=\log_2(3)$

$\log_2 x = 3$

$2^{(\log_2 x)}=2^3$

$x=2^3$

$x=8$

easier: write in exp. form:

$\log_2 x = 3$

$2^3=x$

$8=x$

Strategies for solving exponential and logarithmic equations:

Use inverse functions:

- First, isolate the term with x.
- Use log properties to combine to one term.
- Substitute a variable to get a quadratic.

Use the 1:1 property for logarithms or exponents:

- Find a common base on both sides.

#1. Solve for x: $\ln x = -1$ (use inverse function)

$e^{\ln x}=e^{-1}$

$x=e^{-1}$

$\boxed{x=0.3678\ldots}$

**(check answer)

#2. Solve for x: $\log_{10} x = -\frac{1}{2}$ (use inverse function)

$10^{-1/2}=x$

$x=\frac{1}{10^{1/2}}$

$\boxed{x=\frac{1}{\sqrt{10}}}$

or $10^{(\log_{10} x)}=10^{-1/2}$

$x=10^{-1/2}$

** check answer

#3. Solve for x: $5e^{x+2}-8=14$ (isolate term w/x, then use inverse)

$5e^{x+2}=22$

$e^{x+2}=\frac{22}{5}$

$\ln(e^{x+2})=\ln\left(\frac{22}{5}\right)$

$x+2=\ln\left(\frac{22}{5}\right)$

$\boxed{x=-2+\ln\left(\frac{22}{5}\right)}$

$x=-0.51839\ldots$

** check answer

#4. Solve for x: $\log_{10} x - \log_{10}(x-3) = 1$ (log properties, then inverse)

$\log_{10}\left(\frac{x}{x-3}\right) = 1$

$10^1 = \frac{x}{x-3}$

$\frac{x}{x-3} = \frac{10}{1}$

$x = 10(x-3)$

$x = 10x - 30$

$-9x = -30$

$x = \frac{30}{9} = \boxed{\frac{10}{3}}$

**check answer

#5. Solve for x: $\ln\sqrt{x+2} = \ln x$ (use 1:1 property)

$\sqrt{x+2} = x$

$(\sqrt{x+2})^2 = x^2$

$x+2 = x^2$

$x^2 - x - 2 = 0$

$(x-2)(x+1) = 0$

$\boxed{x = 2}$, $x = -1$ extraneous

**check answer

#6. Solve for x: $\left(\frac{3}{4}\right)^x = \frac{27}{64}$ (get same base, use 1:1 property)

$\left(\frac{3}{4}\right)^x = \left(\frac{3^3}{4^3}\right)$

$\left(\frac{3}{4}\right)^x = \left(\frac{3}{4}\right)^3$

$\boxed{x = 3}$

**check answer

#7. Solve for x: $e^{2x} - e^x - 20 = 0$ (substitute a variable to get a quadratic)

$u = e^x$

$u^2 - u - 20 = 0$

$(u-5)(u+4) = 0$

$(e^x - 5)(e^x + 4) = 0$

**check answer

$e^x - 5 = 0$ $\quad$ $e^x + 4 = 0$

$e^x = 5$ $\quad$ $e^x = -4$

$\ln(e^x) = \ln(5)$ $\quad$ $\ln(e^x) = \ln(-4)$

$\boxed{x = \ln 5}$ $\quad$ $x = \ln(-4)$ -4 not in domain of ln

#8. Solve for x: $\ln(x-2) + \ln(2x-3) = 2\ln x$ (log properties, then 1:1)

$\ln[(x-2)(2x-3)] = \ln(x^2)$

**check answer

$(x-2)(2x-3) = x^2$

$2x^2 - 7x + 6 = x^2$

$x^2 - 7x + 6 = 0$

$(x-6)(x-1) = 0$

$\boxed{x = 6}$, $x = 1$ ext.

$\ln(1-2)\ldots$

$\ln(-1)$

not in domain

HAlg3-4, 3.4 day 2 Notes – Solving Exponential and Logarithmic Equations

Summary of solving strategies:

1) Combine all exponential or logarithmic expressions into 1 term on 1 side, then use inverse.

$$\log_4 x - \log_4 (x-1) = \frac{1}{2}$$

$\log_4\left(\frac{x}{x-1}\right) = \frac{1}{2}$

$4^{\log_4\left(\frac{x}{x-1}\right)} = 4^{1/2} = \sqrt{4}$

$\frac{x}{x-1} = 2$

$\frac{x}{x-1} = 2$

$x = 2(x-1)$

$x = 2x - 2$

$\boxed{2 = x}$

2) Combine all exponential or logarithmic expressions into 1 term on each side, then use 1:1.

$$\ln x - \ln 5 = 0$$

$\ln x = \ln 5$

$\boxed{x = 5}$

How would we handle this one? $\ln x = x^2 - 2$

$e^{\ln x} = e^{(x^2-2)}$

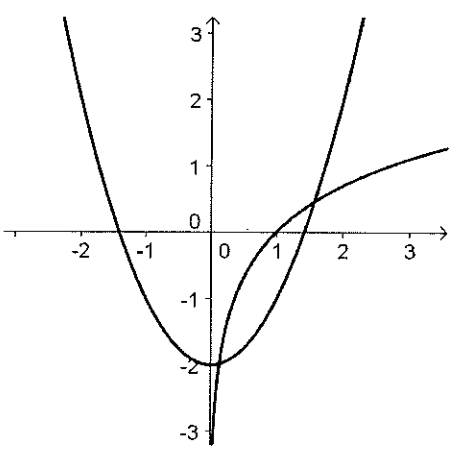
$x = e^{(x^2-2)}$??

$x = e^{x^2} \cdot e^{-2}$

Regular strategies don't work in cases like this, so we resort to graphing calculator. Two ways to do this:

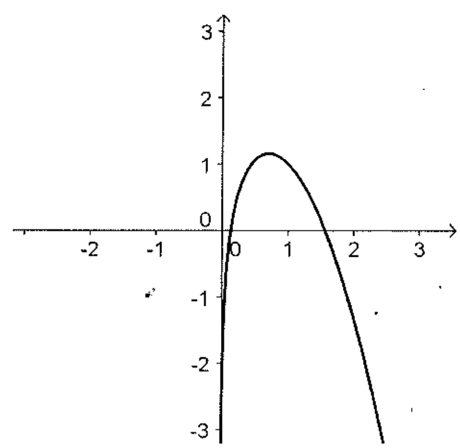
1) Enter each side of the equation in as a separate equation and find the intersection points.

(Use calculator 'intersect' feature)
2nd (trace) for CALC menu, intersect
--try window x: -5 to 5, y: -5 to 5



2) Move everything to one side to make a single equation = 0, then find zeros.

(Use calculator 'zero' feature)
2nd (trace) for CALC menu, zero
--try window x: -1 to 3, y: -2 to 2



Can also use graphing to quickly check for extraneous solutions:
Solve for x: $\log_{10} x + \log_{10}\left(x^2 - 8\right) = \log_{10} 8x$

drop for precalc

$\log_{10} x(x^2-8) = \log_{10} 8x$

$x(x^2-8) = 8x$

$x^3 - 8x = 8x$

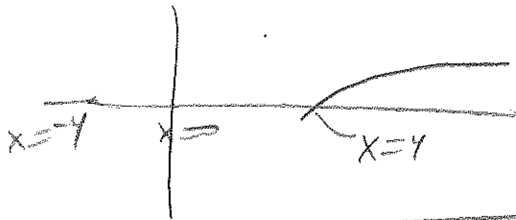
$x^3 - 16x = 0$

$x(x^2-16) = 0$

$x(x+4)(x-4) = 0$

$\underbrace{\cancel{x=0}, \cancel{x=-4}}_{\text{extraneous}}, \ x=4$

graph $\log x + \log(x^2-8) - \log 8x = 0$



Modelling

replace w/ Geogebra car example

$$C = C_0 e^{kt}$$

t	C
0	18000
2	13000
6	6781.32
7.87	5000

$$C = 18000 e^{kt}$$

$$13000 = 18000 e^{k(2)}$$

$$\frac{13}{18} = e^{2k}$$

$$\ln\left(\frac{13}{18}\right) = 2k$$

$$k = \frac{\ln\left(\frac{13}{18}\right)}{2} = -.1627$$

$$C = 18000 e^{-.1627t}$$

C @ 6yrs

$$C = 18000 e^{-.1627(6)}$$

$$= 6781.32$$

5000 @ t = ?

$$5000 = 18000 e^{-.1627t}$$

$$\frac{5}{18} = e^{-.1627t}$$

$$\ln\left(\frac{5}{18}\right) = -.1627t$$

$$t = \frac{\ln\left(\frac{5}{18}\right)}{-.1627} = 7.87 \text{ yrs}$$

Example: How long would it take for an investment to double if the interest was compounded continuously at 8%? $r = .08$

$$A = Pe^{rt}$$

Starting amount = P
at time t, amount doubles to 2P

@ t=0

$A = Pe^{.08(0)}$
$A = Pe^{0}$
$A = P(1)$
$A = P$

t	A
0	P
t	2P

$$A = Pe^{rt}$$
$$2P = Pe^{.08t}$$
$$2 = e^{.08t}$$
$$e^{.08t} = 2$$
$$\ln(e^{.08t}) = \ln 2$$
$$.08t = \ln 2$$
$$t = \frac{\ln 2}{.08}$$
$$t = \boxed{8.66 \text{ years}}$$

Example: You have \$50,000 to invest. You need to have \$350,000 to retire in 30 years. At what continuously compounded interest rate would you need to invest to reach your goal?

$P = \$50,000$
$A = \$350,000$ when $t = 30$
$r = ?$

$$A = Pe^{rt}$$
$$350000 = 50000\, e^{r(30)}$$
$$35 = 5e^{30r}$$
$$7 = e^{30r}$$
$$e^{30r} = 7$$
$$\ln(e^{30r}) = \ln 7$$
$$30r = \ln 7$$
$$r = \frac{\ln 7}{30}$$
$$r = .0649$$

$\boxed{\approx 6.5\%}$

HAlg3-4, 3.5 day 1 Notes – Exponential and Logarithmic Models

Five most common math models using exponential or logarithmic functions:

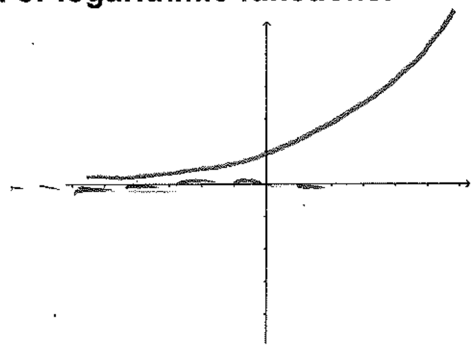
1) Exponential growth model: $y = ae^{bx} \quad (b > 0)$

- populations (unrestrained)
- interest

$$A = Pe^{rt}$$

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

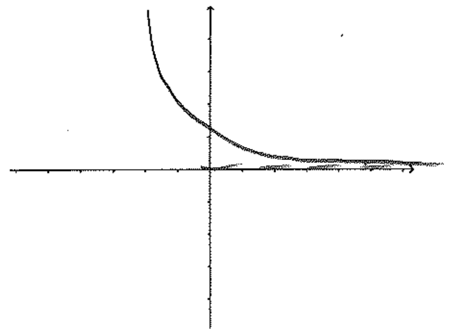
growth $= (>1)^x$
decay $= (<1)^x$



2) Exponential decay model: $y = ae^{-bx} \quad (b > 0)$

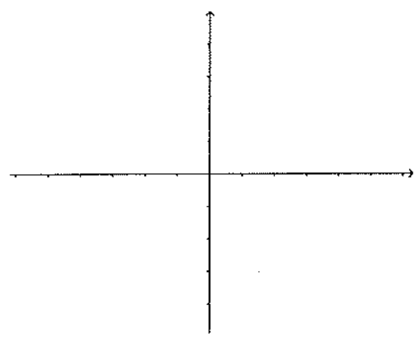
- radioactive decay

$$Q(t) = Q_0\left(\frac{1}{2}\right)^{\frac{t}{halflife}}$$



3) Gaussian model: $y = ae^{-(x-b)^2/c}$ 'bellcurve'

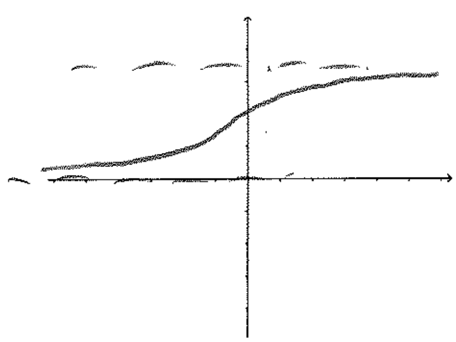
- IQ
- test scores



4) Logistic growth model: $y = \dfrac{a}{1 + be^{-rx}}$

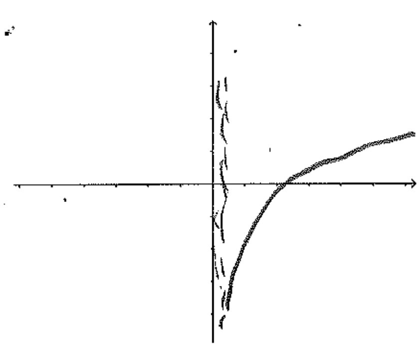
(sigmoidal curve)

- spread of virus
- populations in constrained environment



5) Logarithmic models: $y = a + b\ln x \quad y = a + b\log_{10} x$

- sound intensity
- earthquake magnitude



To create a model, pick the general equation most closely matching data and solve (using data points) to find constants for specific model.

Similar to 'find an equation of a line with a given slope through a point':
Find the equation of a line with slope of -2 through point (1,4). What is the y value of the point on the line if x=5?

$y = mx + b$
$y = -2x + b$
$(4) = -2(1) + b$
$4 = -2 + b$
$6 = b$

x	y
1	4
5	-4

complete model:
$y = -2x + 6$
use the model
$y = -2(5) + 6$
$y = -10 + 6$
$y = -4$

Compound Interest (example of exponential growth)

Separate models for compounding n times per year, and compounding continuously:

n times per year: $A = P\left(1 + \frac{r}{n}\right)^{nt}$ continuous compounding: $A = Pe^{rt}$

Example, solving to find constants: Find a model equation for the amount in an investment account for time t in years. The investment is \$5000, interest is compounded continuously, and investments in this account double every 8 years.

$A = Pe^{rt}$
$5000 = Pe^{r(0)}$
$5000 = P(1)$
$P = 5000$

t	A
0	5000
8	10000

$A = 5000e^{rt}$
$10000 = 5000e^{r(8)}$
$2 = e^{r8}$
$\ln 2 = \ln(e^{r8})$
$\ln 2 = r \cdot 8$
$r = \frac{\ln 2}{8} \approx .086643$

$A = 5000e^{.086643t}$

New definition: Effective Yield (EY) (or 'effective interest rate')
Effective yield = equivalent annual interest rate (compounded annually) that would give the same interest as the actual investment returns.

Effective Yield = (total interest earned in a year) / (amount invested)
For continuous compounding, Effective Yield = $e^r - 1$

Example: If \$100 is invested in an account with an annual interest rate of 6% compounded monthly, what is the effective interest rate (effective yield)?

$A = P\left(1 + \frac{r}{n}\right)^{nt}$
$A = 100\left(1 + \frac{.06}{12}\right)^{12t}$
at 1yr, t=1
$A = 100\left(1 + \frac{.06}{12}\right)^{12(1)}$
$A = 106.16778$

6% annual interest rate
but actually earned \$6.16778 interest
so effective interest rate is:
$\frac{6.16778}{100} = .0616778$
$\approx \boxed{6.17\%}$

groups *

Example: Find all missing values to complete the table for an investment which compounds continuously:

** do all teacher example 1st (both pages)

	Initial Investment	Annual % rate	Time to double	Amount after 10 years
1	\$1000	12%	5.77 yrs	\$ 3320.12
2	\$750	9.2%	7.5 years	\$ 1889.88
3	\$600	9.2%	7.5 yrs	\$1505
4	8986.58	8%	8.7 yrs	\$20,000

1

$A=Pe^{rt}$

$A=1000e^{.12t}$

$2000=1000e^{.12t}$

$2=e^{.12t}$

$\ln(2)=.12t$

$t=\frac{\ln 2}{.12}=5.77$

$A=1000e^{.12(10)}$

$=3320.12$

t	A
0	1000
?	2000

2 example

$A=Pe^{rt}$

$A=750e^{rt}$

$1500=750e^{r(7.5)}$

$2=e^{r(7.5)}$

$\ln 2=r(7.5)$

$r=\frac{\ln 2}{7.5}=.0924$

$A=750e^{.0924(10)}$

$=1889.88$

t	A
0	750
7.5	1500

3

$A=Pe^{rt}$

$A=600e^{rt}$

$1505=600e^{r(10)}$

$\frac{1505}{600}=e^{r(10)}$

$\ln\left(\frac{1505}{600}\right)=r(10)$

$r=\frac{\ln\left(\frac{1505}{600}\right)}{10}$

$r=.092$

$1200=600e^{.092t}$

$2=e^{.092t}$

$\ln 2=.092t$

$t=\frac{\ln 2}{.092}\approx 7.5$

t	A
0	600
10	1505
?	1200

4

$A=Pe^{rt}$

$A=Pe^{.08t}$

$\frac{20000}{e^{.08(10)}}=\frac{Pe^{.08(10)}}{e^{.08(10)}}$

$P=\frac{20000}{e^{.08(10)}}=8986.58$

$2P=Pe^{.08t}$

$2=e^{.08t}$

$\ln 2=.08t$

$t=\frac{\ln 2}{.08}=8.66$

t	A
0	P
10	20000
?	2P

Find all missing values to complete the table for an investment which compounds continuously:

$A = Pe^{rt}$

	Initial Investment	Annual % rate	Time to double	Amount after 10 years
1	\$1000	12%	______	______
(example)	\$750	______	7.5 years	______
2	\$600	______	______	\$1505
3	______	8%	______	\$20,000

Find all missing values to complete the table for the decay of a radioactive substance: $Q = Q_0\left(\frac{1}{2}\right)^{\frac{t}{halflife}}$

	Half-life (years)	Initial Quantity	Amount After 1000 years	Equation model
4	1620	2.3g	1.5 g	$Q = 2.3\left(\frac{1}{2}\right)^{\frac{t}{1620}}$
5	5730	3 g	2.658g	$Q = 3\left(\frac{1}{2}\right)^{t/5730}$
6	1943	10 g	7 g	$Q = 10\left(\frac{1}{2}\right)^{t/1943}$

(4)

$Q = Q_0\left(\frac{1}{2}\right)^{\frac{t}{hl}}$

t	Q
1000	1.5

$1.5 = Q_0\left(\frac{1}{2}\right)^{\frac{1000}{1620}}$

$\frac{1.5}{.6518...} = \frac{Q_0(.6518947...)}{.6518...}$

$2.3 = Q_0$

(5)

$Q = Q_0\left(\frac{1}{2}\right)^{\frac{t}{hl}}$

t	Q
0	3

$Q = 3\left(\frac{1}{2}\right)^{\frac{t}{5730}}$

$Q = 3\left(\frac{1}{2}\right)^{\frac{1000}{5730}}$

$Q = 2.658$

(6)

$Q = Q_0\left(\frac{1}{2}\right)^{\frac{t}{hl}}$

t	Q
0	10
1000	7

$Q = 10\left(\frac{1}{2}\right)^{\frac{t}{hl}}$

$7 = 10\left(\frac{1}{2}\right)^{\frac{1000}{hl}}$

$\frac{7}{10} = \frac{10\left(\frac{1}{2}\right)^{1000/hl}}{10}$

$\frac{7}{10} = \left(\frac{1}{2}\right)^{1000/hl}$

$\log_{1/2}(7/10) = \frac{1000}{hl}$

$hl\left(\log_{1/2}(7/10)\right) = 1000$

$hl = \frac{1000}{\log_{1/2}(7/10)}$

$hl = \frac{1000}{\left(\frac{\log(7/10)}{\log(1/2)}\right)}$

$hl = 1000 \cdot \frac{\log(1/2)}{\log(7/10)} = 1943.358$

HAlg3-4, 3.5 day 2 Notes – Exponential and Logarithmic Models

partners pair do these

More examples of exponential and logarithmic models:

d1 row1

#1. **Population** The population of a city is given by the model: $P = 240{,}360e^{.012t}$ where t=0 represents the starting year, 2000. According to this model, when will the population reach 275,000?

$275{,}000 = 240{,}360e^{.012t}$

$e^{.012t} = \frac{275{,}000}{240{,}360}$

$.012t = \ln \frac{275000}{240{,}360}$

$t = \frac{\ln \frac{275000}{240{,}360}}{.012} = 11.219$

$\boxed{2011}$

t	P
0	240,360
t	275000

d2 row2

#2. **Bacteria Growth**. The number of bacteria, N, in a culture is given by the model: $N = 250e^{kt}$ where t is the time (in hours.) If N=280 when t=10, estimate the time required for the population to double in size.

$280 = 250e^{k(10)}$

$\frac{280}{250} = e^{10k}$

$10k = \ln \frac{280}{250}$

$k = \frac{\ln \frac{280}{250}}{10} = .0113328...$

$2 = e^{.0113328t}$

$\ln 2 = .011332..t$

$t_{double} = \frac{\ln 2}{k} = 61.16 \text{ hrs}.$

$\boxed{\approx 61.2 \text{ hrs}}$

t	N
0	250
10	280
t	500

d3 row3

#3. **Depreciation**. A computer that costs \$4600 new has a 'book value' of \$3000 after 2 years.

(a) Find the linear model for the value of the computer over time: $V = mt + b$

$3000 = m(2) + 4600$

$-1600 = m2$

$-800 = m$

$\boxed{V = -800t + 4600}$

t	V
0	4600
2	3000

(b) Find the exponential model for the value over time: $V = ae^{kt}$

$3000 = 4600e^{k(2)}$

$\frac{3000}{4600} = e^{k(2)}$

$\ln\left(\frac{30}{46}\right) = k(2)$

$k = \frac{\ln(30/46)}{2}$

$k = -.2137$

$\boxed{V(t) = 4600e^{-.2137t}}$

row6

(c) Plot both using a graphing calculator. Which model depreciates faster in the first year?

$\boxed{\text{exponential model}}$

(d) Use each model to find the book values of the computer at 1 year and at 3 years.

lin: $V(1) = -800(1) + 4600 = \$3800$

$V(3) = -800(3) + 4600 = \$2200$

(use calc, table feature)

exp: $V(1) = 4600e^{(-.2137)(1)} = \$3714.90$

$V(3) = 4600e^{(-.2137)(3)} = \$2422.9$

#4. **Sales and Advertising**. The sales, S (in thousands of units), of a product after x hundred dollars is spent on advertising is: $S = 10(1-e^{kx})$. When \$500 is spent on advertising, 2500 units are sold.

(a) Complete the model by solving for k.

$2.5 = 10(1-e^{k5})$ $\quad e^{k5} = .75$

$.25 = 1-e^{k5}$ $\quad k5 = \ln .75$

$-.75 = -e^{k5}$ $\quad k = \frac{\ln .75}{5} = -0.0575364$

d2

(b) Estimate the number of units that will be sold if \$700 is spent on advertising.

row 4

$S = 10(1-e^{(-.0575364)7})$

$S = 3.315$ thousand units

so $\boxed{3315}$

#5. **Intensity of Sound**. The level of sound, β (in decibels), with an intensity I is: $\beta(I) = 10\log\frac{I}{I_0}$ where I_0 is an intensity of 10^{-12} watt per square meter (faintest sound that can be heard by a human.)

Determine the level of sound, β, if:

(a) $I = 10^{-3.5}$ watt per square meter (jet 4 miles from takeoff)

$\beta = 10\log\frac{10^{-3.5}}{10^{-12}} = 10\log 10^{(-3.5+12)} = 10\log 10^{8.5} = 10(8.5) = \boxed{85\ dB}$

(b) $I = 10^{-3}$ watt per square meter (diesel truck at 25 feet)

d2

$\beta = 10\log\frac{10^{-3}}{10^{-12}} = 10\log\frac{10^{12}}{10^{3}} = 10\log 10^{9} = 10(9) = \boxed{90\ dB}$

row 5

(c) $I = 10^{-1.5}$ watt per square meter (auto horn at 3 feet)

$\beta = 10\log\frac{10^{-1.5}}{10^{-12}} = 10\log\frac{10^{12}}{10^{1.5}} = 10\log 10^{10.5} = 10(10.5) = \boxed{105\ dB}$