Honors Brief Calculus – Lesson Notes: Unit 14 (Ch6) – Integral Calculus

6.1 – Antiderivatives; The Indefinite Integral

If we are given a function, we can find the derivative:

$$f(x) = 3x^4 \longrightarrow f'(x) = 12x^3$$

If we are given a derivative function, could we find the function from which it came:

$$f'(x) = 12x^3 \longrightarrow$$

'Reversing' the process of finding the derivative is called finding the **antiderivative**.

Symbols for antiderivatives:

If $f(x) = 2x$

$F(x) = x^2$ is an antiderivative of $f(x)$

But the following are all also antiderivatives of $f(x)$:

$$F(x) = x^2 + 1$$
$$F(x) = x^2 - 22$$
$$F(x) = x^2 + 15{,}432{,}167$$
$$F(x) = x^2 + \frac{2}{7}$$

All the antiderivatives of $f(x)$ are of the form:

$$F(x) = x^2 + K$$

The process of taking an antiderivative of $f(x)$ is represented with the **integral sign** like this:

$$\int f(x)\,dx = F(x) + K$$

Basic Integration Formulas

$$\int c\,dx = cx + K$$

$$\int x^r\,dx = \frac{x^{r+1}}{r+1} + K \qquad (r \neq 1)$$

2. $\int -4\,dx$

6. $\int x^{4/3}\,dx$

$$\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx + K$$
$$\int [f(x) - g(x)]\,dx = \int f(x)\,dx - \int g(x)\,dx + K$$
$$\int c\,f(x)\,dx = c\int f(x)\,dx$$

$$\int e^x\,dx = e^x + K$$

$$\int \frac{1}{x}\,dx = \ln x + K$$

26. $\int x(x+2)\,dx$

18. $\int \left(\frac{x+1}{x}\right)dx$

$$\int e^{ax}\,dx = \frac{e^{ax}}{a} + K$$

20. $\int \left(\frac{8}{x} - e^{-x}\right)dx$

28. $\int \frac{x^6 + x^2 + 1}{x^3}\,dx$

44. Population Growth

There are currently 20,000 citizens of voting age in a small town. Demographics indicate that the voting population will ***change*** at the ***rate of*** $2.2t - 0.8t^2$ (in thousands of voting citizens) where t denotes time in years.

How many citizens of voting age will there be 3 years from now?

#41 **Cost Function** A company determines that the marginal cost of producing x units of a particular commodity during 1 day of operation is

$$C'(x) = 6x - 141$$

where the production cost is in dollars. The selling price of the commodity is fixed at \$9 per unit, and the fixed cost is \$1800 per day.

a. Find the cost function.

b. Find the revenue function.

c. Find the profit function.

d. What is the maximum profit that can be obtained in one day of operation?

e. Graph the revenue, cost, and profit functions.

6.2 – Integration by Substitution

Some integrals cannot be evaluated by using the basic integration formulas, so we need other integration techniques. One of these is **integration by substitution** which is based on the Chain Rule.

Evaluating the derivative of this function by the Chain Rule:

$$y=\left(x^2+5\right)^4$$

$u=x^2+5 \qquad y=u^4$

$\frac{du}{dx}=2x \qquad \frac{dy}{du}=4u^3$

$$\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}$$

$$\frac{dy}{dx}=4u^3(2x)$$

$$\frac{dy}{dx}=4\left(x^2+5\right)^3(2x)$$

Substitution method of integration:

$$\int 4\left(x^2+5\right)^3 2x\,dx$$

$u=x^2+5$

$\frac{du}{dx}=2x$

$du=2x\,dx$

$4\int u^3\,du$

$4\left(\frac{1}{4}u^4\right)+K$

u^4+K

$\left(x^2+5\right)^4+K$

Procedure:

1) Select a complicated portion of the integral that is 'inside' (the inside function of a composite function) and make this function 'u'.

2) Find $\frac{du}{dx}$ and solve for du.

3) Rewrite the integral using only u and du. Multiply by a constant outside if needed to make all the pieces of du.

4) Find the integral.

5) Substitute the x expression for u.

2. $\int (3x-5)^4\,dx$

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6. $\int \left(x^2-2\right)^3 x\,dx$

12. $\int \frac{x}{\sqrt[3]{1+x^2}}dx$

14. $\int x\sqrt{x+3}\ dx$

18. $\int e^{x^3+1}x^2 dx$

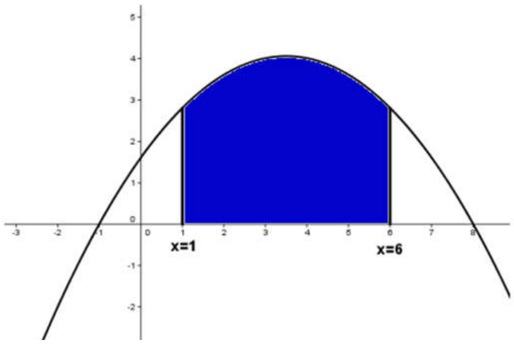
24. $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\ dx$

28. $\int \frac{(x+1)}{\left(x^2+2x+3\right)^2} dx$

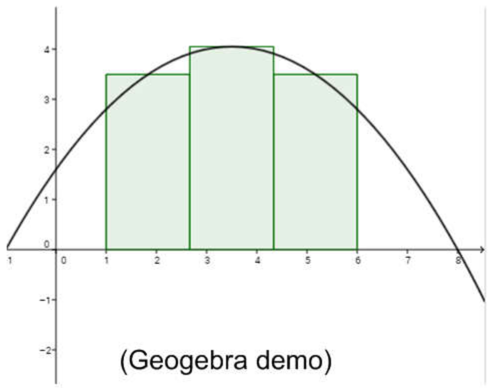
6.6 – Definite Integrals; Riemann Sums

Integral means more than 'antiderivative'...

Could we find the area 'under' a function's curve between two x-values?

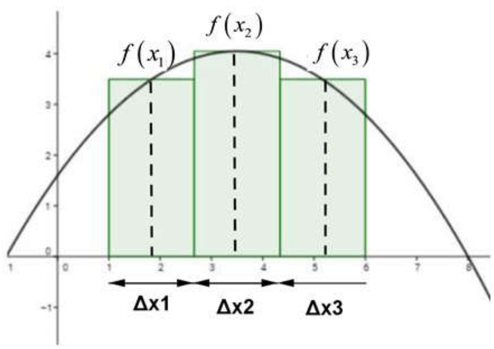


We could approximate this area by filling it with rectangles and adding the areas of the rectangles:



This form of calculation is called a **Riemann Sum**:

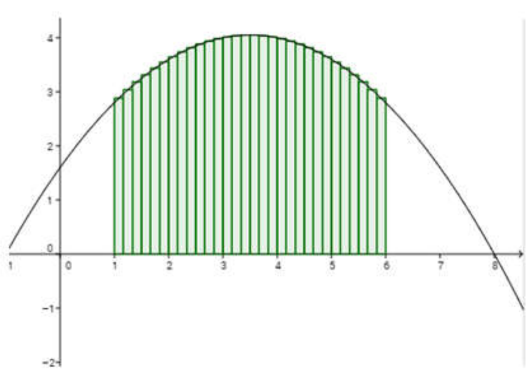
$$Area = \sum_i f(x_i)\,\Delta x_i$$



If the number of rectangles is increased (Δx approaches zero), the area of the Riemann sum approaches the true area under the curve.

$$\int_a^b f(x)dx = \lim_{\Delta x \to 0} \sum_i f(x_i) \bullet \Delta x_i$$

The result is called the **definite integral** from a to b (from x=1 to x=6 in this case).



Today, we're going to approximate the definite integral area by computing Riemann sums, using this procedure:

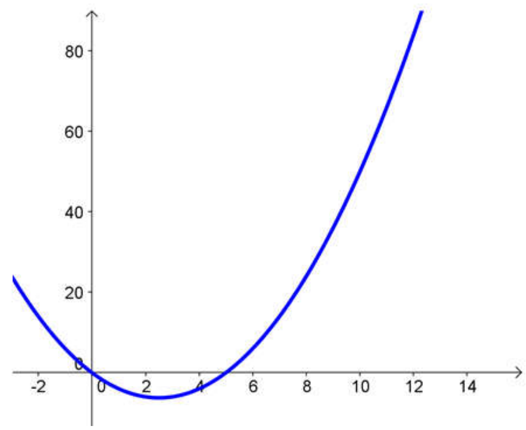
1. Divide interval into subintervals of equal length.

2. Pick a number x in each subinterval and evaluate $f(x)$.

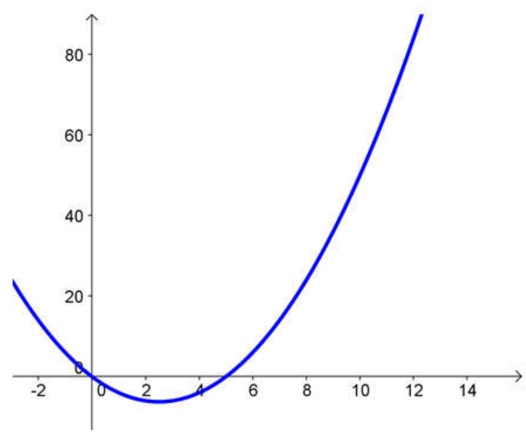
3. Find the sum of the areas of each rectangle formed
   (A = base x height)

#2 By dividing [0, 10] into **two** subintervals of equal length; always pick x_i as the **right** endpoint of each subinterval.

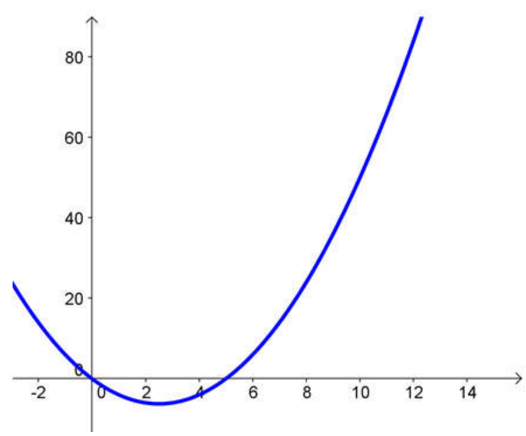
$$\int_0^{10} \left(x^2 - 5x\right) dx$$



#4 By dividing [0, 10] into **five** subintervals of equal length; always pick x_i as the **left** endpoint of each subinterval.



#6 By dividing [0, 10] into **five** subintervals of equal length; always pick x_i as the **midpoint** of each subinterval.



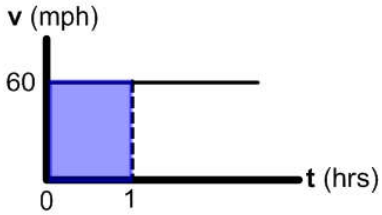
6.4 – Evaluating Definite Integrals; The Fundamental Theorem of Calculus

Start with two seemingly different ideas (that use the same notation)...

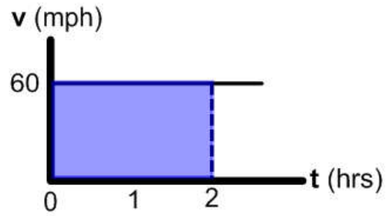
$$\int f(x)\,dx \qquad\qquad \int_a^b f(x)\,dx$$

Antiderivative of $f(x)$, $F(x)$ Area under $f(x)$ curve from x=a to x=b

Consider a car traveling at a constant 60 mph:



$$\begin{aligned} area\ of\ rect &= v(t)\cdot \Delta t \\ &= 60\,\frac{miles}{hr}\cdot 1\ hr \\ &= 60\ miles \\ &= distance\ traveled \\ &\ \ from\ 0-1\ hrs \end{aligned}$$



$$\begin{aligned} area\ of\ rect &= v(t)\cdot \Delta t \\ &= 60\,\frac{miles}{hr}\cdot 2\ hr \\ &= 120\ miles \\ &= distance\ traveled \\ &\ \ from\ 0-2\ hrs \end{aligned}$$

Area under the velocity curve = the total distance traveled

But we also know that the velocity function is the derivative of the distance (displacement) function...

$$v(t) = s'(t)$$

...and therefore the distance function is the antiderivative of the velocity function.

$$s(t) = \int v(t)\,dt$$

$$s(t) = V(t)$$ ← antiderivative of the velocity function

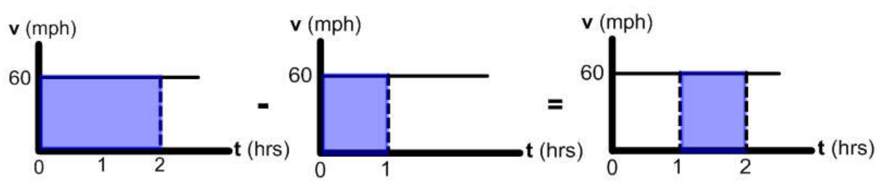
the total distance traveled = antiderivative of velocity

Since the total distance traveled = area under the velocity curve
and the total distance traveled = antiderivative of velocity

Area under the velocity curve = Antiderivative of velocity

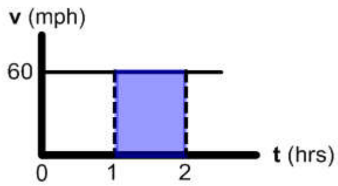
$$\int_0^t v(t)\,dt = V(t)$$

And if we wanted to find the distance traveled between time t=1 and t=2 we could subtract one area from the other:

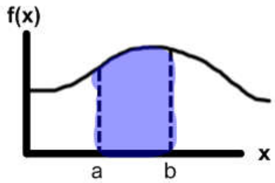


$$\int_0^2 v(t)\,dt = V(2) \qquad \int_0^1 v(t)\,dt = V(1) \qquad \int_1^2 v(t)\,dt = V(2)-V(1)$$

It turns out that this idea, that the area under a function curve over an x-interval is equal to the antiderivative evaluated at the endpoints, is generalizable to all functions, not just constant functions...



$$\int_1^2 v(t)\,dt = V(2) - V(1)$$

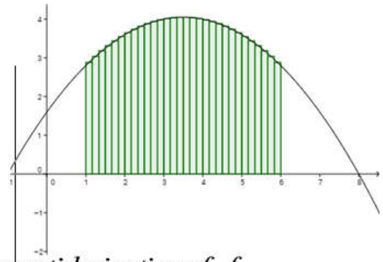


$$\int_a^b f(x)\,dx = F(b) - F(a)$$

...and is called the ***Fundamental Theorem of Calculus***.

Definition of Definite Integral: Area under a curve:

$$\int_a^b f(x)\,dx = \lim_{\Delta x \to 0} \sum_i f(u_i) \bullet \Delta x_i$$



Fundamental Theorem of Calculus states:

$$\int_a^b f(x)\,dx = F(b) - F(a) \quad \textit{where } F \textit{ is an antiderivative of } f.$$

Very surprising! To evaluate an integral (find area under a curve) we only need to find the antiderivative function, and evaluate it at the end points.

The definite integral is the change in the antiderivative.

$$\int_0^{10} \left(x^2 - 5x\right) dx$$

Using Riemann Sums

#6 By dividing [0, 10] into **five** subintervals of equal length; always pick u_i as the **midpoint** of each subinterval.

interval	u	f(u)		Δx		area
(0,2)	1	-4	*	2	=	-8
(2,4)	3	-6	*	2	=	-12
(4,6)	5	0	*	2	=	0
(6,8)	7	14	*	2	=	28
(8,10)	9	36	*	2	=	72
						80

Using Fundamental Theorem of Calculus

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Properties of Definite Integrals

- $\int_a^b f(x)dx = -\int_b^a f(x)dx$
- $\int_a^a f(x)dx = 0$
- $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$
  where c is between a and b
- $\int_a^b cf(x)dx = c\int_a^b f(x)dx$
- $\int_a^b [f(x)dx \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$

2. $\int_1^2 (2x+1)dx$

4. $\int_{-2}^0 \left(e^x + x^2\right)dx$

24. $\int_0^1 x^2 e^{x^3} dx$

31. $\int_0^2 \frac{\left(e^{3x} + e^{-x}\right)}{e^{2x}} dx$

Other Applications of the Definite Integral

For the learning curve $f(x) = cx^k$, the total number (N) of labor-hours required to produce units numbered *a* through *b* is

Learning Curves $$N = \int_a^b f(x)dx = \int_a^b cx^k dx$$

When the rate of sales of a product is a known function, say *f(t),* where *t* is the time, the total sales of this product over a time period T are

Total sales over Time $$T = \int_o^T f(t)dt$$

Amount of Annuity with Continuous Compounding

$$A = \int_0^N Pe^{rt}dt$$

P = annual payments *r* % = interest rate

A = amount of annuity after *N* payments

#42 **Annuity** If $1200 is deposited each year in a savings account paying 5% per annum compounded continuously, how much is in the account after 3 years?

6.5 – Area Under a Graph

Properties of Area

I. Area $\geq$ 0

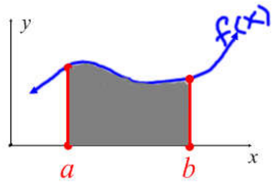
II. If A and B are two nonoverlapping regions with areas that are known, then

Total area of A and B = Area of A + Area of B

Area under a Graph

Suppose $y = f(x)$ is a continuous function defined on a closed interval I and $f(x) \geq 0$ for all points x in I. Then, for $a < b$ in I, the definite integral

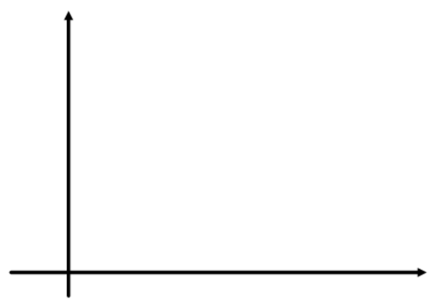
$$\int_a^b f(x)dx$$



is the area **under** the graph of $y = f(x)$ and **above** the x-axis between the lines $x = a$ and $x = b$.

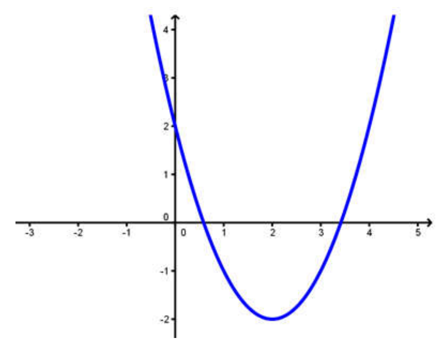
6. Find the area enclosed by

$f(x) = x^2 - 4$, the x-axis, and the lines $x = 2$ and $x = 4$

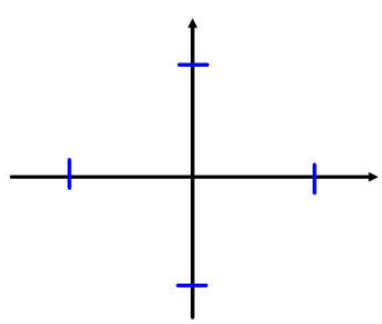


If $f(x) \leq 0$ and $a \leq x \leq c$, then $-f(x) \geq 0$ and by symmetry, the area A equals:

$$A = \int_a^c [-f(x)]\,dx = -\int_a^c f(x)\,dx$$



Find the area enclosed by $f(x) = x^3$, the x-axis, $x = -1$, and $x = 1/2$

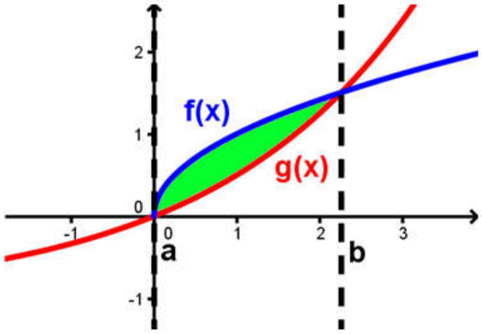


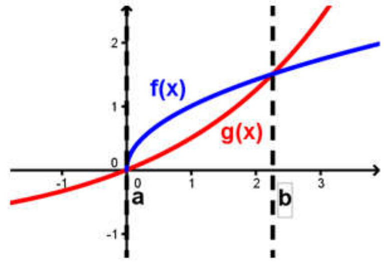
Area of a Region Bounded by Two Graphs

If f and g are continuous on $[a,b]$ and $g(x) \le f(x)$ for all x in the interval, then the area of the region bounded by the graphs of f, g, $x = a$, and $x = b$ is given by:

$$A = \int_a^b \left[f(x) - g(x) \right] dx$$

(why?)

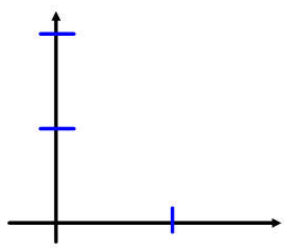




15. Find the area enclosed by $f(x) = x^2 + 1$ and $g(x) = x + 1$

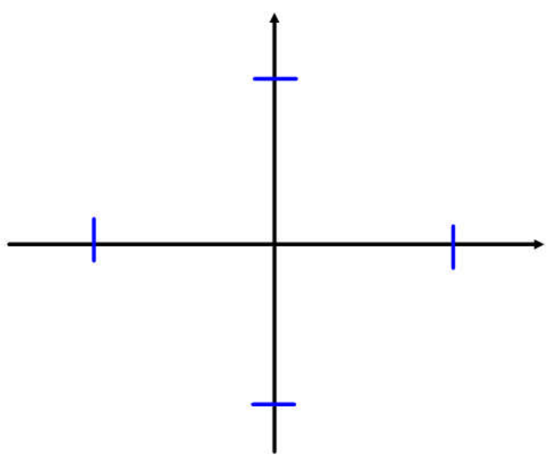
*find points of intersection

*sketch graph



*set up definite integral and solve

27. Find the area enclosed by $y = x^2$, $y = x$, $y = -x$



6.3 – Integration by Parts

- Basic integration formulas (powers, exponents, logarithms) work for some integrals.
- Integration by substitution (based on chain rule) works for some integrals.
- Other integrals can be evaluated by the **integration by parts** procedure,

which is based on the product rule:

$$\frac{d}{dx}[uv] = u\frac{d}{dx}[v] + v\frac{d}{dx}[u]$$

$$\frac{d}{dx}[uv] = u\frac{dv}{dx} + v\frac{du}{dx}$$

Integrating...

$$uv = \int u\frac{dv}{dx}dx + \int v\frac{du}{dx}dx$$

$$uv = \int u\,dv + \int v\,du$$

Solving for 1st term on right...

$$\boxed{\int u\,dv = uv - \int v\,du}$$

Integration by parts

Steps to Integrate by Parts

step 1: dx is always a part of dv

step 2: It must be possible to integrate dv

step 3: u and dv are chosen so that $\int v\,du$ is easier to evaluate than the original integral $\int u\,dv$; this often happens when u is simplified by differentiation.

Use integration by parts to evaluate the integrals:

$\int xe^{-3x}\,dx$

$\int x^2 e^{2x}\,dx$

$$\int x^2 \ln 5x \, dx$$

$$\int x^3 (\ln x)^2 \, dx$$

$$\int_1^2 x \ln x \, dx$$

6.7 – Differential Equations

$\frac{dy}{dx} = f(x)$ is called a ***differential equation***.

A function $y = F(x)$ for which $dy/dx = f(x)$ is a ***solution*** of the differentiable equation

The ***general solution*** of $dy/dx = f(x)$ consists of all the antiderivatives of f.

We've been finding solutions of differential equations when we find antiderivatives:

$\frac{dy}{dx} = 5x^2 + 2$ ⟵ (differential equation)

What is the antiderivative of the right side?

$y = \frac{5}{3}x^3 + 2x + K$ ⟵ (general solution of the differential equation)

What is the equation if y=5 when x=3? ⟵ (boundary condition)

$$(5) = \frac{5}{3}(3)^3 + 2(3) + K$$

$$-46 = K$$

$y = \frac{5}{3}x^3 + 2x - 46$ ⟵ (particular solution)

This form of differential equation is called a **separable differential equation** because the variables can be separated and gathered on separate sides:

$$\frac{dy}{dx} = 5x^2 + 2$$

$$dy = (5x^2 + 2)dx$$

$$\int dy = \int (5x^2 + 2)dx$$

$$y = \frac{5}{3}x^3 + 2x + K$$

$$\frac{dy}{dx} = \frac{4x}{3y^2}$$

One very important application of differential equations is developing equations where the rate of change is proportional to the amount.

Uninhibited Population Growth

If a colony of rabbits contains a small number of rabbits, the *rate of increase* in population is small, because there are not many rabbits available to breed. If the population is large the *rate of increase* in population is also large, because more rabbits can pair up to breed. The *rate* of population increase is proportional to the amount of rabbits:

$$\frac{dP}{dt} = kP$$

22. **Radioactive Decay** $A = A_0 e^{kt}$
If 25% of a radioactive substance disappears in 10 years, what is the half-life of the substance?

Find the general solution of the differential equation.

2. $\frac{dy}{dx} = 5x^2 - 4x + 2$

6. $\frac{dy}{dx} = y^2$

Find the particular solution.

10. $\frac{dy}{dx} = x^2 + 4$
$y = 1$ when $x = 0$

12. $\frac{dy}{dx} = x^2 + x$
$y = 5$ when $x = 3$

18. $\frac{dy}{dx} = x + e^x$
$y = 4$ when $x = 0$