Honors Brief Calculus – Lesson Notes: Unit 11 – Derivatives of Functions

4.2 – Derivative Notation, Simple Power Rule, Sum and Difference Formulas

Definition of derivative:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

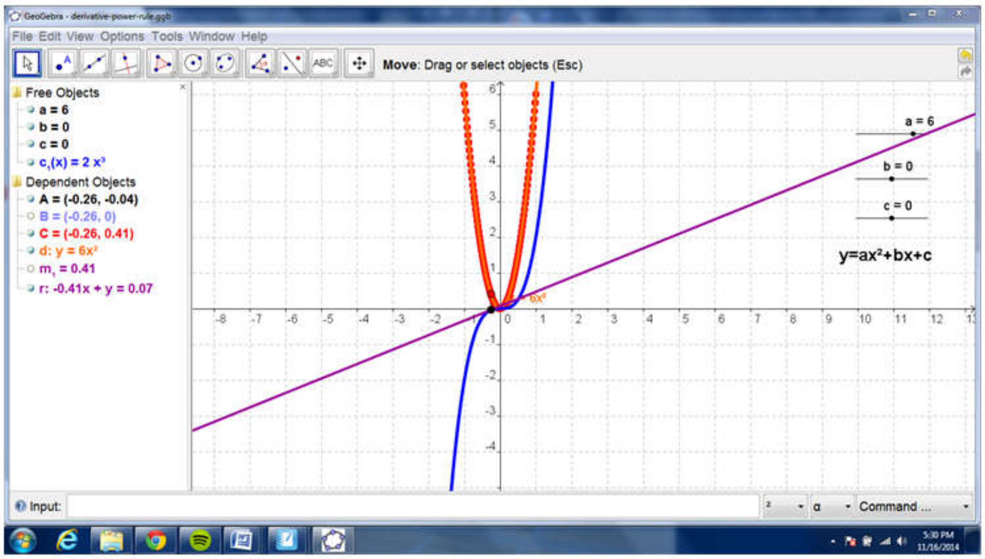
Notations for derivative:

$f'(x)$ $\quad y'$ $\quad \frac{dy}{dx}$ $\quad \frac{d}{dx}(y)$ $\quad \frac{d}{dx}f(x)$

prime notation (Lagrange) — Leibniz notation

$\frac{d}{dx}f(x)$ is read " compute the derivative of f with respect to x "

$\frac{d}{dt}s(t)$ is read " compute the derivative of s with respect to t "



Geogebra (derivative-power-rule)

$f(x)$	$f'(x)$	$f(x)$	$f'(x)$	$f(x)$	$f'(x)$	$f(x)$	$f'(x)$
1	0	x	1	x^2	$2x$	x^3	$3x^2$
2	0	$3x$	3	$2x^2$	$4x$	$2x^3$	$6x^2$
		$-2x$	-2	$-2x^2$	$-4x$		
				$3x^2$	$6x$		

The Simple Power Rule: $f(x) = x^n \qquad f'(x) = nx^{n-1}$

Derivative of a constant: $\frac{d}{dx}b = 0$

Derivative of a constant times a function: $\frac{d}{dx}\left[Cf(x)\right] = C\frac{d}{dx}f(x)$

Combining multiple terms:

$f(x) = 3x^2 - 2x \qquad f'(x) =$

$f(x) = 2x^3 - 2x^2 + 3x \qquad f'(x) =$

Sum and Difference Formulas:

$$\frac{d}{dx}\left[f(x) + g(x)\right] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

$$\frac{d}{dx}\left[f(x) - g(x)\right] = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$$

Find the derivative:

#2 $f(x) = -2$ #6 $f(x) = -8x^3$ #16 $f(x) = \frac{2}{3}x^6 - \frac{1}{2}x^4 + 2$

#22 $f(x) = \frac{6(x^3 - 2)}{5}$

#38 $f(x) = 8\sqrt[6]{x^3}$

#50 $f(x) = \frac{2}{x^5} - \frac{3}{x^3}$

#64 $f(x) = \frac{5}{\sqrt[4]{x}}$

#69 Find the indicated derivative:

$\frac{dA}{dR}$ if $A = \pi R^2$

#76 Find the value of the derivative at the indicated point.

$y = 1/x^2$ at (3, 1/9)

#102 Find any points at which the graph of *f* has a horizontal tangent line.

$f(x) = 3x^5 - 5x^3 - 1$

4.1 – Average vs. Instantaneous Rates of Change; Applications: Economics

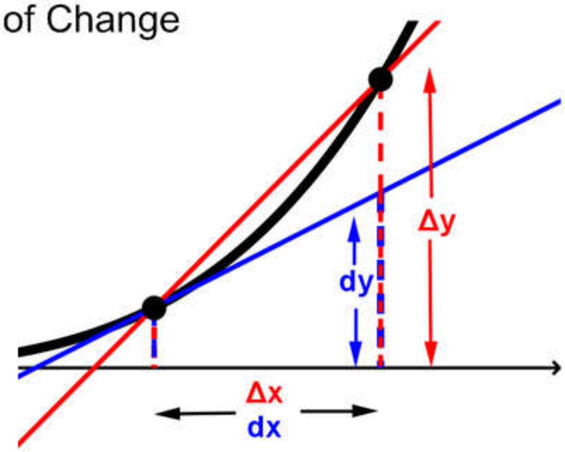
Instantaneous and Average Rate of Change

<u>Average rate of change</u> = slope
calculating using two data points

$$\frac{\Delta f}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

<u>Instantaneous rate of change</u>
= slope of tangent line at a point
= value of derivative at that point

$$\frac{dy}{dx} = f'(x)$$



Instantaneous and Average Rate of Change of: $f(x) = x^2 + 3x$

Average rate of change
<u>as x changes from 1 to 3:</u>

$f(1) =$

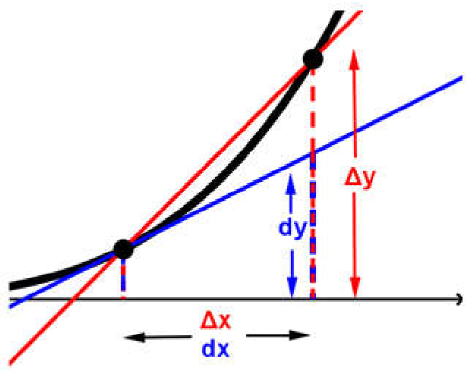
$f(3) =$

$\frac{\Delta f}{\Delta x} =$

<u>Instantaneous rate of change at x=1:</u>

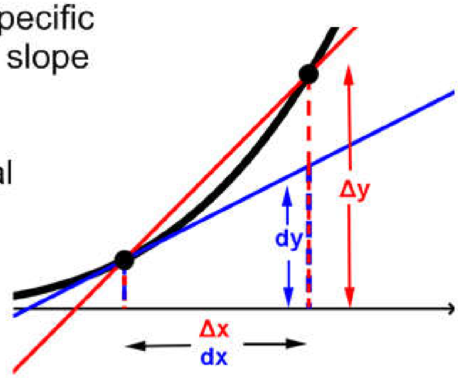
$f'(x) =$

$f'(1) =$



Average rate of change is calculated from specific data points, but is a number representing a slope which is constantly changing.

Instantaneous rate of change is a theoretical value calculated from a data model, but represents the exact rate of change at a specific x-value.



Both of these are variations of the idea of 'slope'.
Slope is a measure of how much y changes per unit change in x.

Application: Economics

In economics, Cost and Revenue are often modeled as functions of the number of units, x:

$C(x)$ = the cost of producing x units.

$R(x)$ = the money received from selling x units.

$C'(x)$ = the **marginal cost** = the increase in cost to produce 1 more unit.

$R'(x)$ = the **marginal revenue** = the increase in sales from selling 1 more unit.

$$C'(x) \approx C(x+1) - C(x)$$
$$R'(x) \approx R(x+1) - R(x)$$

#13 **Toy Truck Sales** At Dan's Toy Store the revenue, in dollars, derived from selling *x* electric trucks is $R = -0.005x^2 + 20x$

(a) What is the average rate of change in revenue due to selling 10 additional trucks after 1000 have been sold?

(b) What is the marginal revenue?

(c) What is the marginal revenue at x=1000?

(d) Intrepret $R'(1000)$

(e) For what value *of* x is $R'(x) = 0$?

#21 **Demand Function.** A certain item can be produced at a cost of $10 per unit. The demand equation for this item is $p = 90 - 0.02x$ where *p* is the price in dollars and *x* is the number of units.

Find:

(a) The revenue function

(b) The marginal revenue.

(c) The marginal cost.

(d) The break-even point(s).

(e) The number *x* for which marginal revenue equals marginal cost.

'Relative error' and 'percentage error'

The derivative of a function represents the change in a function per unit change in the input. This is a number, but is that change large or small? For example:

What are the instantaneous rates of change of these functions at x=3?

$$f(x) = 180x^2 \qquad g(x) = 10x^4$$

Which represents a 'larger change'?

Relative error (relative change) = $\dfrac{|\Delta f|}{f} \approx \dfrac{|f'(x)\Delta x|}{f(x)}$

$$f(x) = 180x^2 \qquad g(x) = 10x^4$$

4.3 – Product and Quotient Formulas for derivatives

The derivative of the sum of two functions is the sum of the derivatives of each function separately:

$$\frac{d}{dx}\left(x^3+x^2\right)=\frac{d}{dx}\left(x^3\right)+\frac{d}{dx}\left(x^2\right)$$

Is this also true for the derivative of the product of two functions?

$$\frac{d}{dx}\left(x^3\bullet x^2\right)\stackrel{?}{=}\frac{d}{dx}\left(x^3\right)\bullet\frac{d}{dx}\left(x^2\right)$$

More complicated for products - **<u>The product rule for derivatives</u>:** (proof on p.662 of textbook)

$$\frac{d}{dx}\left[f(x)\bullet g(x)\right]=f(x)\frac{d}{dx}\left[g(x)\right]+g(x)\frac{d}{dx}\left[f(x)\right]$$

$$\frac{d}{dx}\left[1st\bullet 2nd\right]=1st\frac{d}{dx}\left[2nd\right]+2nd\frac{d}{dx}\left[1st\right]$$

Using the product rule:

$$\frac{d}{dx}\left(x^3x^2\right)=$$

Similar for quotients - **<u>The quotient rule for derivatives</u>:**

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]=\frac{g(x)\frac{d}{dx}\left[f(x)\right]-f(x)\frac{d}{dx}\left[g(x)\right]}{\left[g(x)\right]^2}$$

$$\frac{d}{dx}\left[\frac{high}{low}\right]=\frac{low\frac{d}{dx}\left[high\right]-high\frac{d}{dx}\left[low\right]}{\left[low\right]^2}$$

"low d-high minus high d-low over low squared"

p.663

Find the derivative of the function by using the formula for the derivative of a product.

2. $f(x) = (3x-4)(2x+5)$

14. $y = \frac{5}{3}(\sqrt{u}-2)(3u+2)$

Find the derivative.

22. $f(x) = \dfrac{2x^2-1}{5x+2}$

26. $f(x) = 1 - \dfrac{1}{x} + \dfrac{1}{x^2}$

29. $f(x) = \dfrac{3x^2-2x+1}{\sqrt{x}}$

48. $f(x) = \dfrac{(2-3x)(1-x)}{(x+2)(3x+1)}$

51. The value v of a luxury car after t years is:

$$v(t) = \frac{10{,}000}{t} + 6000 \qquad 1 \le t \le 6$$

(a) What is the average rate of change in value from $t = 2$ to $t = 5$?
(b) What is the instantaneous rate of change in value?
(c) What is the instantaneous rate of change in value after 2 years?
(d) What is the instantaneous rate of change in value after 5 years?

4.4/4.5 – The Chain Rule, Extended Power Rule

Intuitive 'proof' of the Chain Rule...

Derivative = Slope = Rate of Change

In a sense, ratios are also rates of change: There are 12 inches per foot: $\frac{\text{12 inches}}{\text{1 foot}}$

There are 5280 feet per mile: $\frac{\text{5280 feet}}{\text{1 mile}}$

How many inches are there per mile?

$$\frac{\text{12 inches}}{\text{1 foot}} * \frac{\text{5280 feet}}{\text{1 mile}} = \frac{\text{63360 inches}}{\text{1 mile}}$$

$$\frac{di}{df} \cdot \frac{df}{dm} = \frac{di}{dm} \quad \text{so:} \quad \frac{di}{dm} = \frac{di}{df} \cdot \frac{df}{dm}$$

The Chain Rule:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Why is this helpful?

Some functions are difficult to differentiate directly, but can be expressed as a 'composite' function and differentiated using the chain rule:

$$f(x) = \left(x^3 - 2x - 1\right)^{100}$$

$$f\left(\left[g(x)\right]\right) = \left(\left[x^3 - 2x - 1\right]\right)^{100}$$

'Inside function': $g(x) = x^3 - 2x - 1$

$u = x^3 - 2x - 1$

'Outside function': $f(u) = u^{100}$

$y = u^{100}$

Then, you can find the derivative using the Chain Rule:

$$y = \left(x^3 - 2x - 1\right)^{100}$$
$$y = u^{100}$$
$$u = x^3 - 2x - 1$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$\frac{dy}{dx} = \left(100\left(x^3 - 2x - 1\right)^{99}\right) \cdot \left(3x^2 - 2\right)$$

<u>The Chain Rule</u>: $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

$$\frac{d}{dx} f\left(g(x)\right) = f'\left(g(x)\right) \cdot g'(x)$$

'derivative of the outside time derivative of the inside'

<u>The Extended Power Rule</u>: $\frac{d}{dx}\left[f(x)\right]^r = r\left[f(x)\right]^{r-1} \cdot f'(x)$

Find dy/dx using the Chain Rule.

$$y = (2x+5)^3$$

$$y = u^3\ ,\quad u = 2x+5$$

$$y = \left[\left(\frac{1}{x+2}\right)^2 - 1\right]^3$$

$$y = \left(u^2 - 1\right)^3\ ,\quad u = \frac{1}{x+2}$$

Find the derivative using the Extended Power Rule.

$$f(x) = \left(x^2 - 1\right)^4$$

$$f(x) = 3x^2\left(x^2+1\right)^3$$

$$f(x) = \left[x(x+4)\right]^4$$

$$f(x) = x^2\sqrt{4x-1}$$

$$f(x) = \left(\frac{x^2}{x+5} \right)^4$$

$$f(x) = \frac{2x^3}{(x^2 - 4)^2}$$

62. The weekly revenue R in dollars resulting from the sale of x typewriters is

$$R(x) = \frac{100x^5}{(x^2 + 1)^2} \qquad 0 \le x \le 100$$

Find

(a) The marginal revenue

(b) The marginal revenue at $x = 40$

(c) The marginal revenue at $x = 60$

(d) Interpret the answers to b and c.

4.5 – Derivatives of Exponential and Logarithmic Functions

Derivative of an exponential function:

(Must go back to definition of derivative)

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$f(x) = a^x$$

$$f'(x) = \lim_{h\to 0}\frac{\left[a^{x+h}\right]-\left[a^x\right]}{h}$$

$$f'(x) = \lim_{h\to 0}\frac{a^{x+h}-a^x}{h}$$

$$f'(x) = \lim_{h\to 0}\frac{a^x a^h-a^x}{h}$$

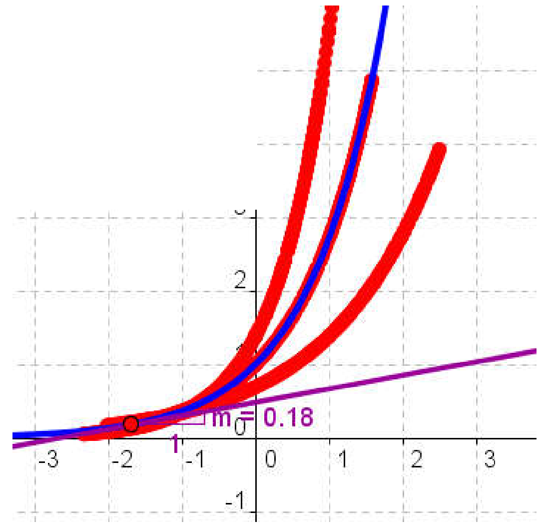
$$f'(x) = \lim_{h\to 0}\frac{a^x\left(a^h-1\right)}{h}$$

$$f'(x) = a^x\lim_{h\to 0}\frac{\left(a^h-1\right)}{h}$$

$$f'(x) = a^x\lim_{h\to 0}\frac{\left(a^h-1\right)}{h}$$

$$f'(x) = a^x\ln(a)$$

$$\boxed{\frac{d}{dx}\left[a^x\right] = a^x\ln(a)}$$



$$\frac{d}{dx}\left[a^x\right] = a^x\ln(a)$$

$$\frac{d}{dx}\left[2^x\right] = 2^x\ln(2) = (0.693..)2^x$$

$$\frac{d}{dx}\left[3^x\right] = 3^x\ln(3) = (1.0986..)3^x$$

$$\frac{d}{dx}\left[e^x\right] = e^x\ln(e) = (1)e^x$$

$$\boxed{\frac{d}{dx}\left[e^x\right] = e^x}$$

e^x is the only function whose derivative is itself.

$$\boxed{\frac{d}{dx}\left[e^{g(x)}\right] = e^{g(x)}\frac{d}{dx}\left[g(x)\right]}$$

(by the chain rule)

What about the derivative of a logarithmic function?

$$y = \ln(x)$$
$$y = \log_e(x)$$
$$e^y = x$$
$$e^{\ln x} = x$$

$$\frac{d}{dx}\left[e^{\ln x}\right] = \frac{d}{dx}[x]$$
$$e^{\ln x}\frac{d}{dx}[\ln x] = 1$$
$$\frac{d}{dx}[\ln x] = \frac{1}{e^{\ln x}}$$
$$\frac{d}{dx}[\ln x] = \frac{1}{x}$$

$$\boxed{\frac{d}{dx}[\ln x] = \frac{1}{x}}$$

$$\boxed{\frac{d}{dx}\left[\ln\left(g(x)\right)\right] = \frac{1}{g(x)}\frac{d}{dx}\left[g(x)\right]}$$

(by the chain rule)

$$\frac{d}{dx}[\log_a x] = \frac{d}{dx}\left[\frac{\ln x}{\ln a}\right]$$
$$\frac{d}{dx}[\log_a x] = \frac{1}{\ln a}\frac{d}{dx}[\ln x]$$
$$\frac{d}{dx}[\log_a x] = \frac{1}{\ln a}\frac{1}{x}$$
$$\boxed{\frac{d}{dx}[\log_a x] = \frac{1}{x\ln a}}$$

Summary of exponential and logarithmic derivative rules:

$$\frac{d}{dx}\left[e^x\right] = e^x \qquad \frac{d}{dx}[\ln x] = \frac{1}{x}$$

$$\frac{d}{dx}\left[a^x\right] = a^x \ln(a) \qquad \frac{d}{dx}[\log_a x] = \frac{1}{x \ln a}$$

$$\frac{d}{dx}\left[e^{g(x)}\right] = e^{g(x)} \frac{d}{dx}\left[g(x)\right] \qquad \frac{d}{dx}\left[\ln(g(x))\right] = \frac{1}{g(x)} \frac{d}{dx}\left[g(x)\right]$$

Find the derivative of each function.

2. $f(x) = 2e^x$

6. $f(x) = \dfrac{1}{e^{-4x}}$

7. $f(x) = e^{4x^2}$

9. $f(x) = \ln\left(6x^2\right)$

8. $f(x) = x^2 e^x$

28. $f(x) = e^{x+(1/x)}$

36. $f(x) = \dfrac{1}{2}\ln 4x$

42. $f(x) = \ln \sqrt[3]{x}$

Use logarithmic differentiation to find the derivative.

54. $f(x)=\left(3x^2+4\right)^3\left(x^2+1\right)^4$

48. $f(x)=\ln(\ln x)$

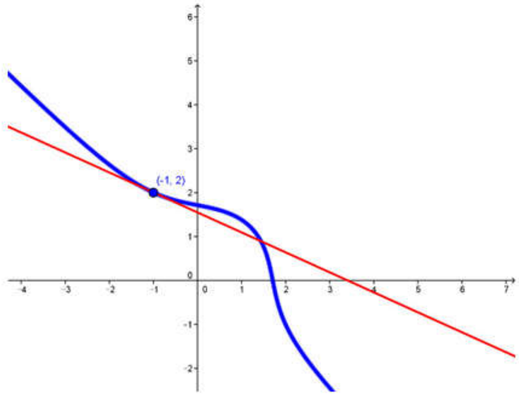
52. $f(x)=x\ln\sqrt[3]{3x+1}$

4.7 – Implicit Differentiation

Given this equation:

$$x^3 + xy + y^3 = 5$$

How could you find the slope of the line tangent to this curve at the point (-1, 2)?



Some equations can be solved for y.
These are called 'explicit equations'.

But some equations are either difficult or impossible to solve for y.
These are called 'implicit equations'.

We can still differentiate (find the derivative) of an implicit equation, but we need a procedure which doesn't require solving for y first.
This procedure is called **implicit differentiation**.

Steps to differentiate implicity

1. Differentiate both sides with respect to x. For terms with y, use Chain Rule (differentiate wrt y, and multiply by $\frac{dy}{dx}$)
2. Collect all terms with $\frac{dy}{dx}$ on one side and all other terms on the other side.
3. Factor out $\frac{dy}{dx}$
4. Solve for $\frac{dy}{dx}$

Example:

$$3x^2 - 2y^2 = 6$$

Given this equation:

$$x^3 + xy + y^3 = 5$$

How could you find the slope of the line tangent to this curve at the point (-1, 2)?

Practice (groups): Find the derivative

4. $x^3 y = 5$

6. $x^2 y + xy^2 = x + 1$

14. $\frac{1}{x^2} + \frac{1}{y^2} = 6$

18. $x^2 + y^2 = \frac{2y}{x}$

22. $x^2 + y^2 = (3x - 4y)^2$

4.6 – Higher-Order Derivatives; Velocity and Acceleration

The derivative of a function is, itself, a function. We could therefore take the derivative of this function. This is called the 2nd derivative of the original function. You can do this repeatedly:

First derivative: $y' = f'(x) = \frac{dy}{dx} = \frac{d}{dx} f(x)$

Second derivative: $y'' = f''(x) = \frac{d^2 y}{dx^2} = \frac{d^2}{dx^2} f(x)$

Third derivative: $y''' = f'''(x) = \frac{d^3 y}{dx^3} = \frac{d^3}{dx^3} f(x)$

Fourth derivative: $y^{(4)} = f^{(4)}(x) = \frac{d^4 y}{dx^4} = \frac{d^4}{dx^4} f(x)$

nth derivative: $y^{(n)} = f^{(n)}(x) = \frac{d^n y}{dx^n} = \frac{d^n}{dx^n} f(x)$

Find f' and f'' .

6. $f(x) = -4x^3 + x^2 - 1$

14. $f(x) = \frac{x+1}{x^2}$

22. Find the indicated derivative.

$f^{(5)}(x)$ if $f(x) = 4x^3 + x^2 - 1$

In physics, an equation that gives the position of an object as a function of time is called a 'displacement function' and is usually denoted by $s(t)$.

The <u>average velocity</u> between two points is:

$$average\ velocity = \frac{change\ in\ position}{change\ in\ time}$$

$$average\ velocity = \frac{\Delta s}{\Delta t}$$

The <u>instantaneous velocity</u> and <u>acceleration</u> at a point are:

$$instantaneous\ velocity = v = s'(t)$$

$$acceleration = a = v'(t) = s''(t)$$

Find the velocity v and acceleration a of an object whose position s at time t is given:

#32 $s = 16t^2 + 10t + 1$

44. Falling Body an object is propelled vertically upward with an initial velocity of 39.2 meters per second. The distance s (in meters) of the object from the ground after t seconds is

$$s = -4.9t^2 + 39.2t$$

a. What is the velocity of the object at any time t ?

b. When will the object reach its highest point?

c. What is the maximum height?

d. What is the acceleration of the object at any time t ?

e. How long is the object in the air?

f. What is the velocity of the object upon impact?

g. What is the total distance traveled by the object?