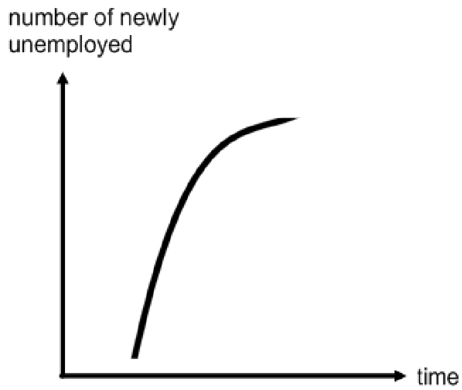
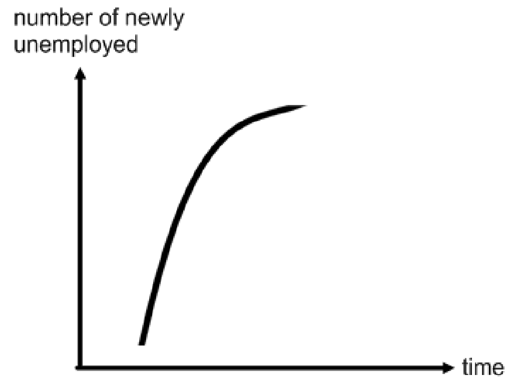
Hrs Brief Calculus – Lesson Notes: Unit 10 (Ch3) - Limits; Derivative of a Function

3.1 – The idea of a Limit; Finding Limits using tables and graphs

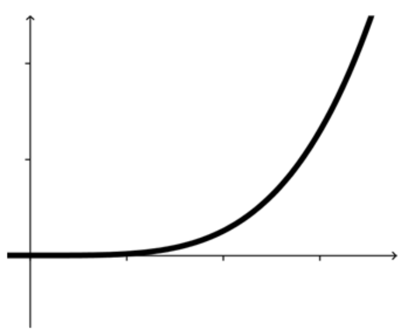
Algebra/Functions:
Focus is on functions that can model the data values of real-world scenarios.



Calculus:
Focus is on rate of change of functions and on how values 'accumulate'.



Preview: rate of change



Limits:

The concept of the limit of a function is what bridges the gap between the mathematics of algebra and the mathematics of calculus.

$$\lim_{x \to c} f(x) = N$$

Read:
"The limit of f of x as x approaches c equals the number N."

This means:
For all x approximately equal to c, but not equal to c, the value f(x) is approximately equal to N.

What is the domain of this function? $f(x) = \frac{x^2 - 1}{x - 1}$

Graph the function in your calculator.

Specific values of f(x) when x is close to 1:

x	f(x)
0	
0.5	
0.75	
0.9	
0.99	
0.999	

x	f(x)
2	
1.5	
1.25	
1.1	
1.01	
1.001	

Criterion for $\lim_{x \to c} f(x)$ to exist: $\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x)$

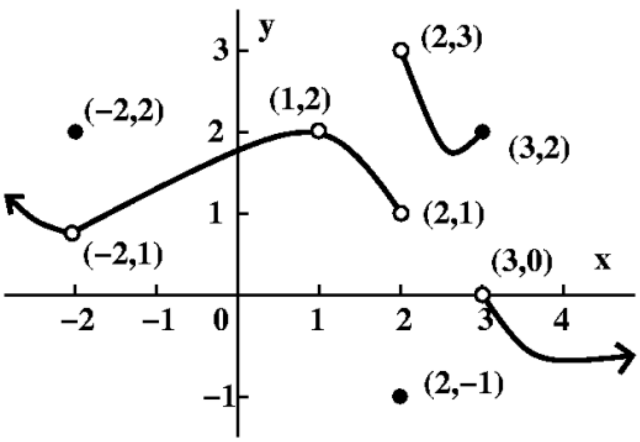
The limit N of a function $y = f(x)$ as x approaches the number c does not depend on the value of f at c.

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1}$$

5. Use a calculator to complete each table and evaluate the indicated limit $\lim_{x \to 2} f(x) =$

x	1.9	1.99	1.999
$f(x) = \frac{x^2 - 4}{x - 2}$			

x	2.1	2.01	2.001
$f(x) = \frac{x^2 - 4}{x - 2}$			



Given the graphical definition of the function, find the following:

$f(2)$ $\lim_{x \to -2} f(x)$

$\lim_{x \to 2^-} f(x)$ $\lim_{x \to 1} f(x)$

$\lim_{x \to 2^+} f(x)$ $\lim_{x \to 2} f(x)$

$\lim_{x \to 2} f(x)$ $\lim_{x \to 3} f(x)$

Use the graph to determine whether $\lim_{x \to c} f(x)$ exists.

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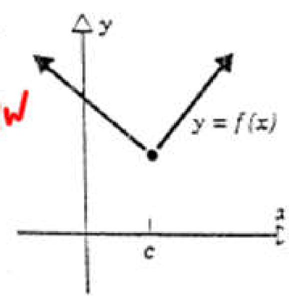

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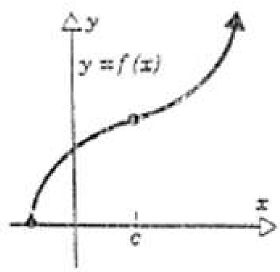

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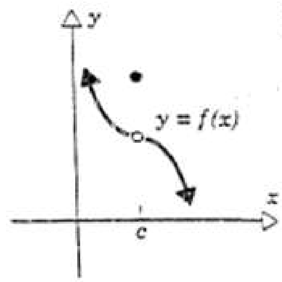

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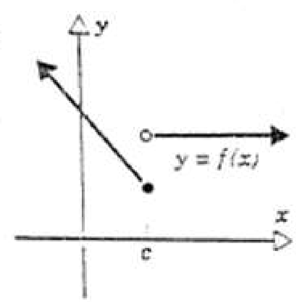

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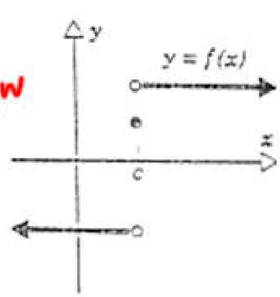

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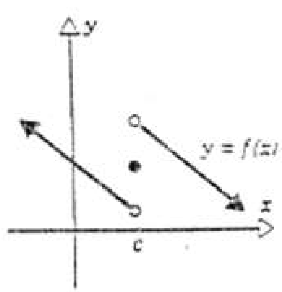

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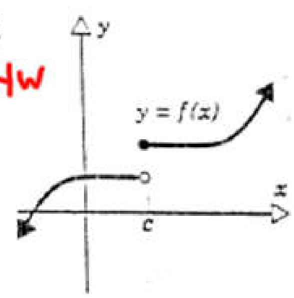

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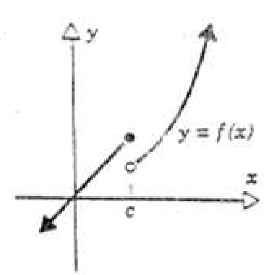


Determine whether $\lim_{x \to c} f(x)$ exists by graphing.

If it exists, find $\lim_{x \to c} f(x)$

32. $f(x) = \begin{cases} 2x-1 & \text{if } x \neq 1 \\ 2 & \text{if } x = 1 \end{cases}$

30. $f(x) = \begin{cases} 2x+1 & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$

Find the limit.

39. $\lim_{x \to 3^+} (x-5)$

42. $\lim_{x \to 0^+} \frac{3x}{x^3}$

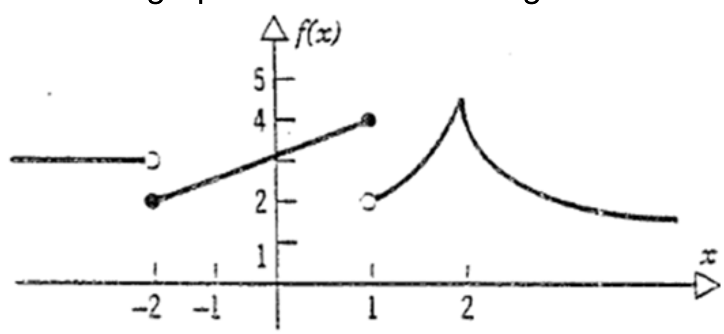
44. $\lim_{x \to 3^+} \frac{6}{x-3}$

50. Find the limit $\lim_{x \to 1^-} f(x)$ and $\lim_{x \to 1^+} f(x)$ for the function

$$f(x) = \begin{cases} 3x-1 & \text{if } x < 1 \\ 2 & \text{if } x = 1 \\ 3x & \text{if } x > 1 \end{cases}$$

Does $\lim_{x \to 1} f(x)$ exist?

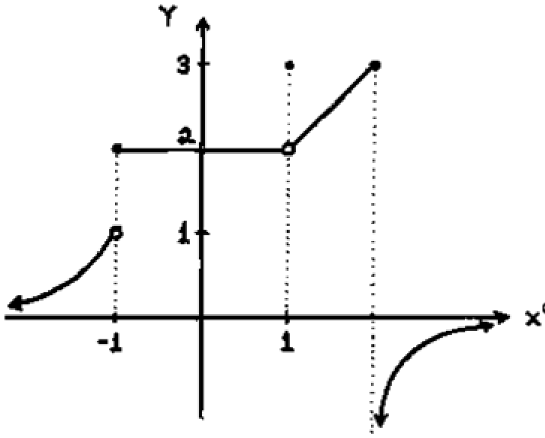
Use the graph to find the following limits:



(a) $\lim_{x \to -2^+} f(x)$ (b) $\lim_{x \to -2^-} f(x)$ (c) $\lim_{x \to -2} f(x)$

(d) $\lim_{x \to 1^+} f(x)$ (e) $\lim_{x \to 1^-} f(x)$ (f) $\lim_{x \to 1} f(x)$

(g) $\lim_{x \to 2^+} f(x)$ (h) $\lim_{x \to 2^-} f(x)$ (i) $\lim_{x \to 2} f(x)$



$\lim_{x \to -1^-} f(x)$ $\lim_{x \to 1^-} f(x)$ $\lim_{x \to 2^-} f(x)$

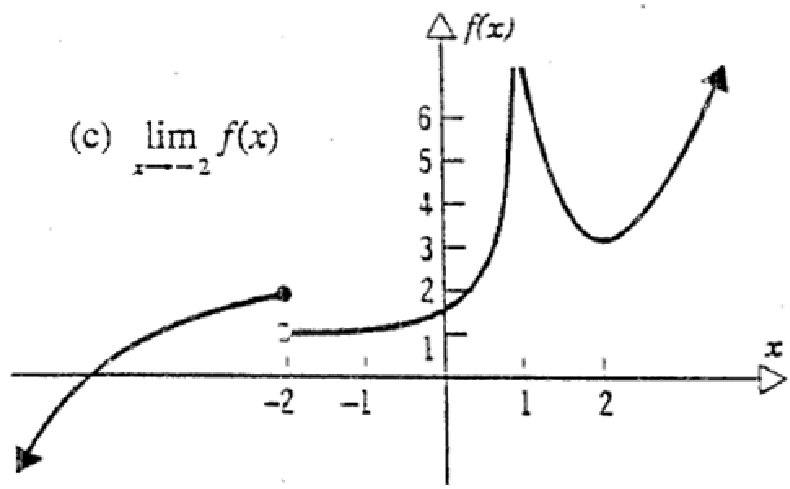
$\lim_{x \to -1^+} f(x)$ $\lim_{x \to 1^+} f(x)$ $\lim_{x \to 2^+} f(x)$

$\lim_{x \to -1} f(x)$ $\lim_{x \to 1} f(x)$ $\lim_{x \to 2} f(x)$

$f(-1)$ $f(1)$ $f(2)$

37.

(a) $\lim_{x \to -2^+} f(x)$ (b) $\lim_{x \to -2^-} f(x)$ (c) $\lim_{x \to -2} f(x)$



(d) $\lim_{x \to 1^+} f(x)$ (e) $\lim_{x \to 1^-} f(x)$ (f) $\lim_{x \to 1} f(x)$

(g) $\lim_{x \to 2^+} f(x)$ (h) $\lim_{x \to 2^-} f(x)$ (i) $\lim_{x \to 2} f(x)$

3.2 – Algebraic Techniques for Finding Limits

Groups: Find the indicated limit.

#1) $\lim_{x \to 1} 4$ #2) $\lim_{x \to -2} (3x+2)$ #3) $\lim_{x \to 1} \sqrt{3x^2+1}$ #4) $\lim_{x \to -2} \frac{x+2}{3x-5}$

These illustrate general properties of limits….

$$\lim_{x\to c} b = b$$

$$\lim_{x\to c} x = c$$

$$\lim_{x\to c}\left[f(x) \pm g(x)\right] = \lim_{x\to c} f(x) \pm \lim_{x\to c} g(x)$$

$$\lim_{x\to c}\left[f(x) \cdot g(x)\right] = \lim_{x\to c} f(x) \cdot \lim_{x\to c} g(x)$$

$$\lim_{x\to c}\left[f(x)\right]^n = \left[\lim_{x\to c} f(x)\right]^n$$

$$\lim_{x\to c} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to c} f(x)}$$

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$$

$$\left(\textit{if } \lim_{x\to c} g(x) \neq 0 \right)$$

#5) $\lim_{x \to -1} \frac{x^2-1}{x+1}$

This limit <u>does</u> exist. The function can have a limit at an x-value even if the function is undefined at this value.

Try graphing this function with your calculator. Is the graph shape surprising? Does this suggest a way to handle cases like this?

If $f(x) = 2x^2 + x$

a) Find:

$$\lim_{x \to 4} \frac{f(x) - f(1)}{x-4}$$

b) Find:

$$\lim_{x \to 4} \frac{f(x) - f(4)}{x-4}$$

If $f(x) = 3 - 4x$

find $\lim_{x \to 2} \frac{f(x) - f(2)}{x-2}$

#6) $\lim_{x \to 4} \frac{2x^2 - 32}{x^3 - 4x^2}$

#7) $\lim_{x \to 3}\left(\frac{x^3 - 27}{x - 3}\right)$

#8) $\lim_{x \to 4} \frac{\sqrt{x} - 2}{x - 4}$

#9) $\lim_{x \to 3} \left[\frac{3}{x-3} - \frac{x}{x-3}\right]$

#10) Find $\lim_{x \to 2} f(x)$ and $f(2)$,

when $f(x) = \begin{cases} 4x^3 + x & \text{if } x \neq 2 \\ 8 & \text{if } x = 2 \end{cases}$

#11) Find $\lim_{x \to 1} f(x)$ and $f(1)$,

when $f(x) = \begin{cases} \frac{4x^3 + x - 5}{x - 1} & \text{if } x \neq 1 \\ 8 & \text{if } x = 1 \end{cases}$

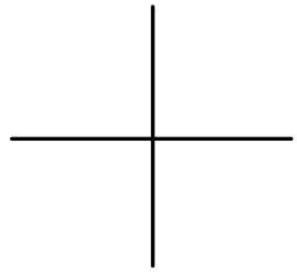
#12) Assume that $\lim_{x \to c} f(x) = 5$ and $\lim_{x \to c} g(x) = 2$ to find each limit

$$\lim_{x \to c} \frac{f(x)}{g(x) - f(x)}$$

3.4 – Limits at Infinity, Infinite Limits

Limits at Infinity

Graph: $f(x) = \dfrac{1}{x}$



a. $\lim\limits_{x\to\infty} \dfrac{1}{x} =$ b. $\lim\limits_{x\to-\infty} \dfrac{1}{x} =$

c. $\lim\limits_{x\to\infty} \dfrac{1}{x^n} =$ d. $\lim\limits_{x\to-\infty} \dfrac{1}{x^n} =$

What about this one? (Try graphing)

$$\lim_{x\to\infty}\frac{3x-2}{4x-1}$$

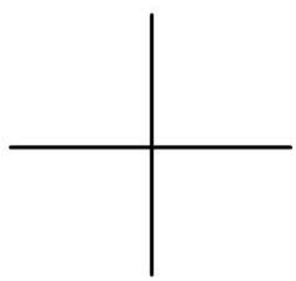
Finding LIMITS at INFINITY of a RATIONAL FUNCTION

1. DIVIDE each term of BOTH the numerator and the denominator by the highest power of x that appears in the denominator.
2. Take the limit of the new numerator and denominator.

$\lim\limits_{x\to\infty} \dfrac{2x^2-5x+2}{5x^2+7x-1}$ $\lim\limits_{x\to\infty} \dfrac{x^2-2x+1}{x^3+5x+4}$ $\lim\limits_{x\to\infty} \dfrac{5x^3-1}{x^2+1}$

Infinite Limits

Graph: $f(x) = \dfrac{1}{x}$



$$\lim_{x\to0^-} \frac{1}{x} =$$

$$\lim_{x\to0^+} \frac{1}{x} =$$

Where do infinite limits occur?

$$\lim_{x\to 0^+}\frac{1}{x} \qquad \lim_{x\to 1^-}\frac{x-1}{x^2-3x+2} \qquad \lim_{x\to 1^-}\frac{1}{x}$$

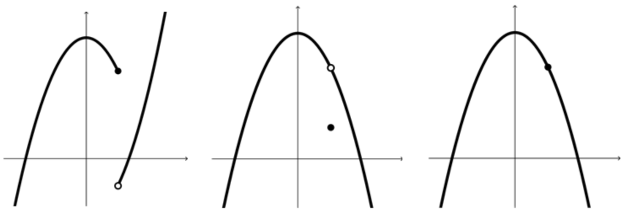
$$\lim_{x\to 1^-}\frac{x+1}{x^2-3x+2} \qquad \lim_{x\to 5^+}\frac{x+1}{5-x} \qquad \lim_{x\to 0^+}\frac{x\left(x^2-1\right)}{x^2}$$

Finding VERTICAL and HORIZONTAL ASYMPTOTES

1. Graphically
2. Using LIMITS --- this will be discussed in a later section.

3.3 – Continuous Functions

One of these things is not like the other...



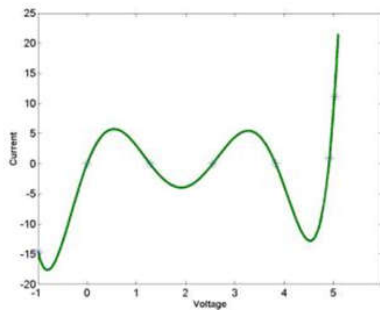
Conditions for a Function to Be Continuous at c
To summarize, a function f is continuous at c provided that three conditions are met:

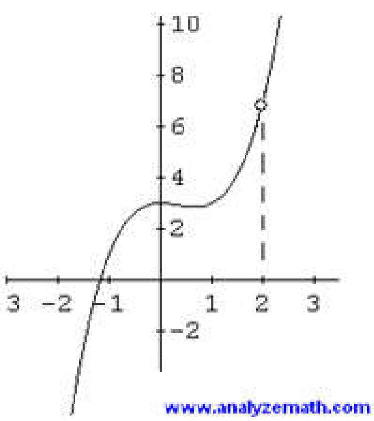
Condition 1 $f(c)$ is defined;
that is, c is in the domain of the function

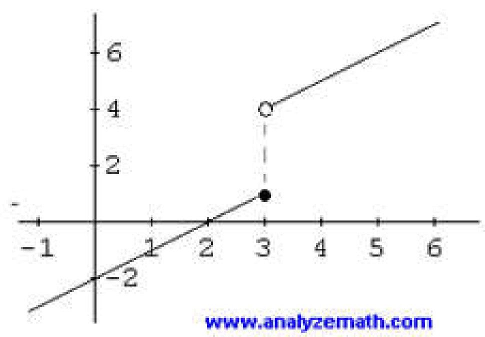
Condition 2 $\lim_{x \to c} f(x)$ exists

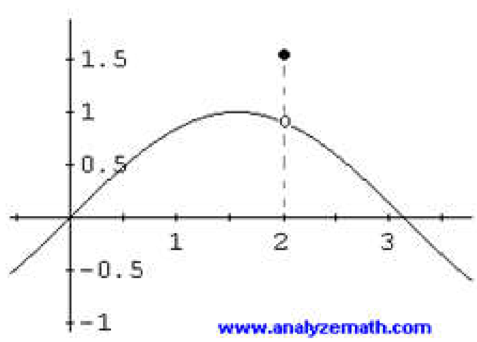
Condition 3 $\lim_{x \to c} f(x) = f(c)$

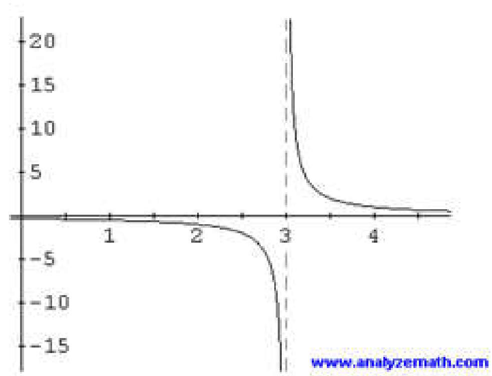
A **polynomial** function f is **continuous** at every real #.











Determine whether the function ***f*** is continuous at ***c***.

#2 $f(x)=\begin{cases}1-3x^2 & \text{if } x\neq 0\\ 1 & \text{if } x=0\end{cases}$ at $c = 0$

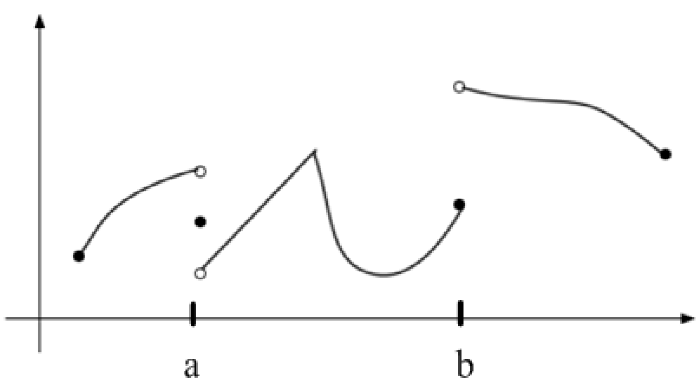
#6. $f(x)=\begin{cases}2x+1 & \text{if } x\leq 0\\ 2x & \text{if } x>0\end{cases}$ at $c = 0$

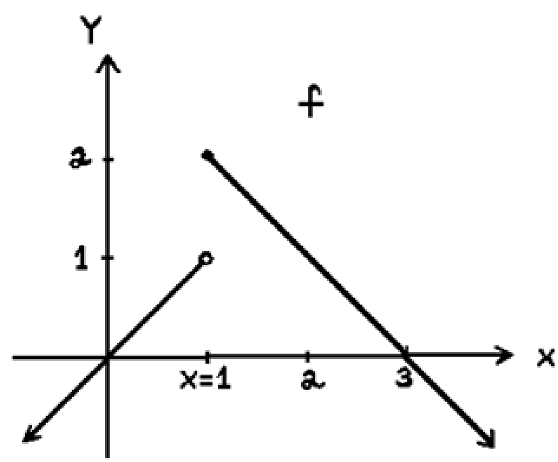
Determine the value of the constant ***k*** that will make the function ***f*** continuous for all ***x***.

$$f(x)=\begin{cases}1-4x & \text{if } x<2\\ kx^2-3x+2 & \text{if } 2\leq x\end{cases}$$

Is the function *f* defined by $f(x)=\dfrac{x^2+x-12}{x-3}$ continuous at 3?

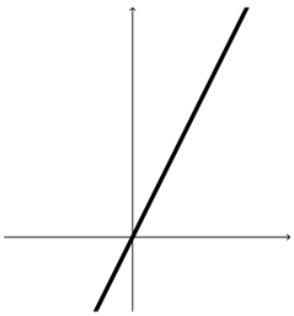
If not, can *f* be redefined at 3 to make it continuous at 3?

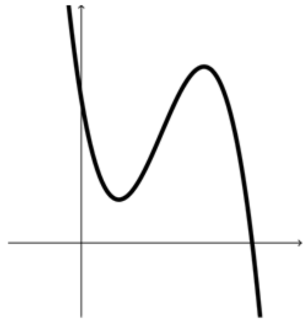


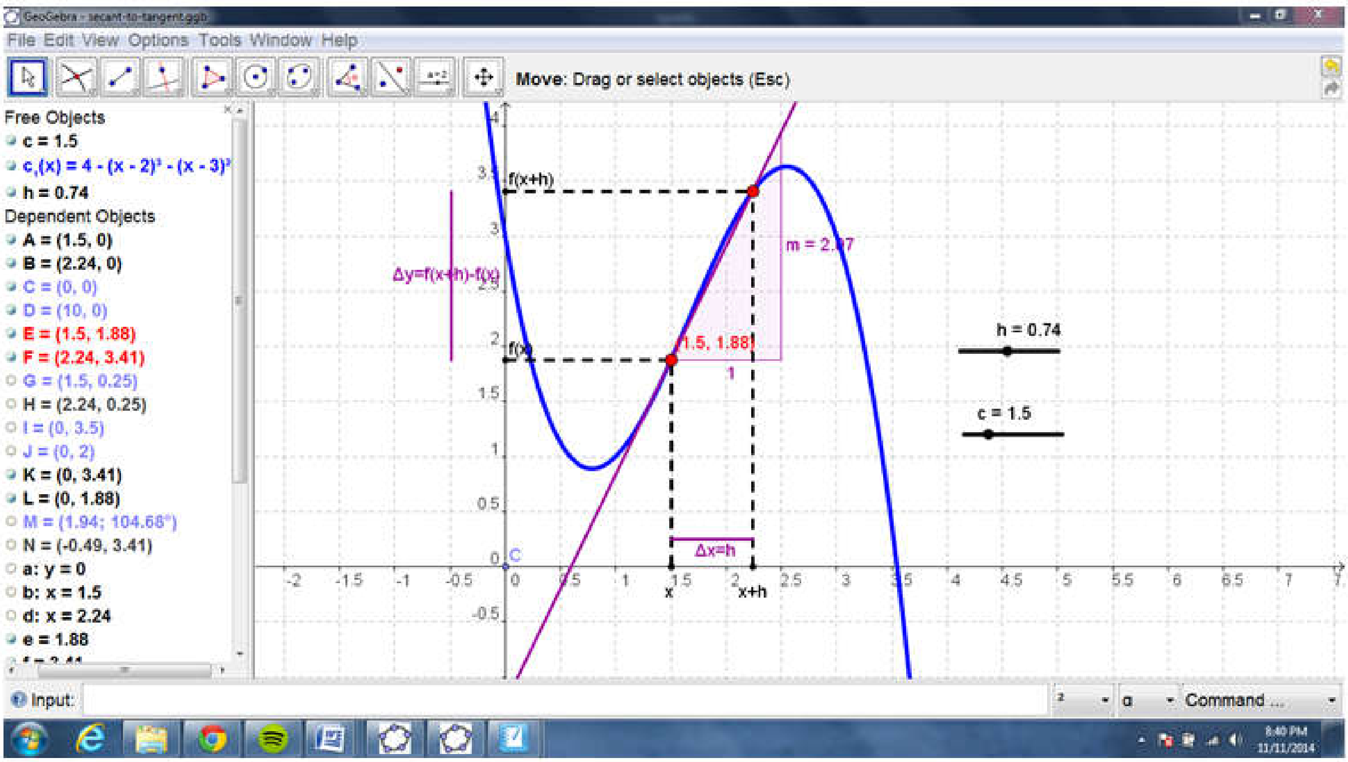


4.1 day 1 – Tangent to a curve, The Derivative

What is the slope (rate of change) of these functions?







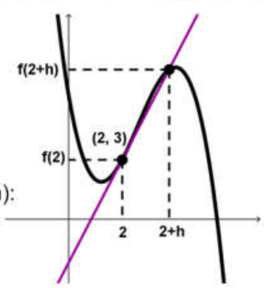
- Slope (rate of change) of a curve at a point is the slope of the line tangent to the curve at that point.
- Compute the slope: limit of difference quotient as h approaches 0.

$$f'(2) = \lim_{h\to 0} \frac{f(2+h) - f(2)}{h}$$

$f'(2)$ *is slope of* f *at* $x = 2$

For any value c (x-value in the domain):

$$f'(c) = \lim_{h\to 0} \frac{f(c+h) - f(c)}{h}$$



$$f'(c) = \lim_{h\to 0} \frac{f(c+h) - f(c)}{h}$$

"f prime of x at c" means all of the following:

- The 'derivative' of the function f(x) at x=c.
- The rate of change of f as x changes, at x=c.
- The slope of the line tangent to f(x) at the point (c, f(c)).

Find the slope of the tangent line to the graph of *f* at the given point. Then find the equation of this tangent line. Graph *f* and the tangent line.

$f(x) = x^2 + 4$ at $(1, 5)$

Procedure:
- Graph the curve, locate the point, roughly sketch the tangent line.
- Compute $f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$ with c equal to the point's x coordinate.
- The result is the slope of the tangent line, $m_{\tan}$
- Find the equation of the tangent line using point-slope form, the tangent slope and point coordinates.

Find the derivative of f at the given number.

$f(x) = 3x^2$ *at* 2

Procedure:
- Compute $f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$ with c equal to the given number.
- The result is the derivative of f at the value.

Find the derivative of f at the given number.

Procedure:

- Compute $f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$ with c equal to the given number.
- The result is the derivative of f at the value.

$f(x) = -x^2 + 2x - 1$ at - 1

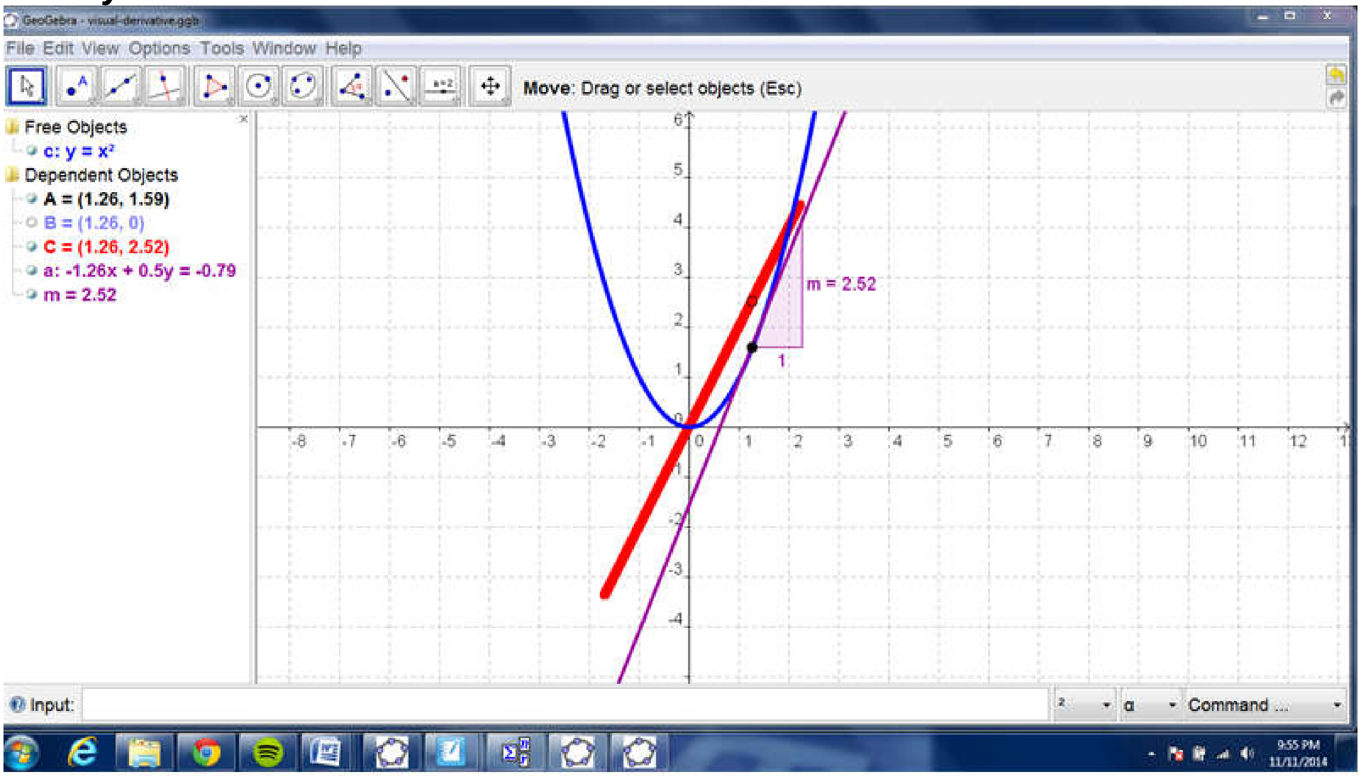
Find the derivative of f at the given number.

Procedure:

- Compute $f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$ with c equal to the given number.
- The result is the derivative of f at the value.

$f(x) = x^5 - 3x + 1$ at 0

4.1 day 2 – The Derivative as a Function



Geogebra - Visual Derivative

Since we can use f(x) to compute the value of the derivative, f '(c) at any point x=c, that means the derivative is also a function of x.

Derivative function: $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

- The function $f'(x)$ is 'the derivative of f at x'.
- 'Differentiate f' means to 'find the derivative of f'.
- For a given x, the derivative function output is the slope of f(x) at the given x.

$f(x) = x^2 + 4$

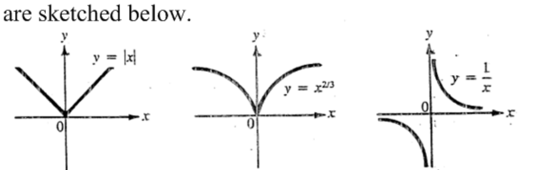
Find the slope at x=1: Find $f'(x)$, the derivative function:

Find $f'(x)$, the derivative function:

#26. $f(x) = 2x^2 + x + 1$

Differentiability and Continuity

Not all functions have a derivative for every value of x. Three functions that do not have derivatives when $x = 0$ are sketched below.



(a) "Corner" no tangent at $x = 0$

(b) "Cusp" vertical tangent at $x = 0$

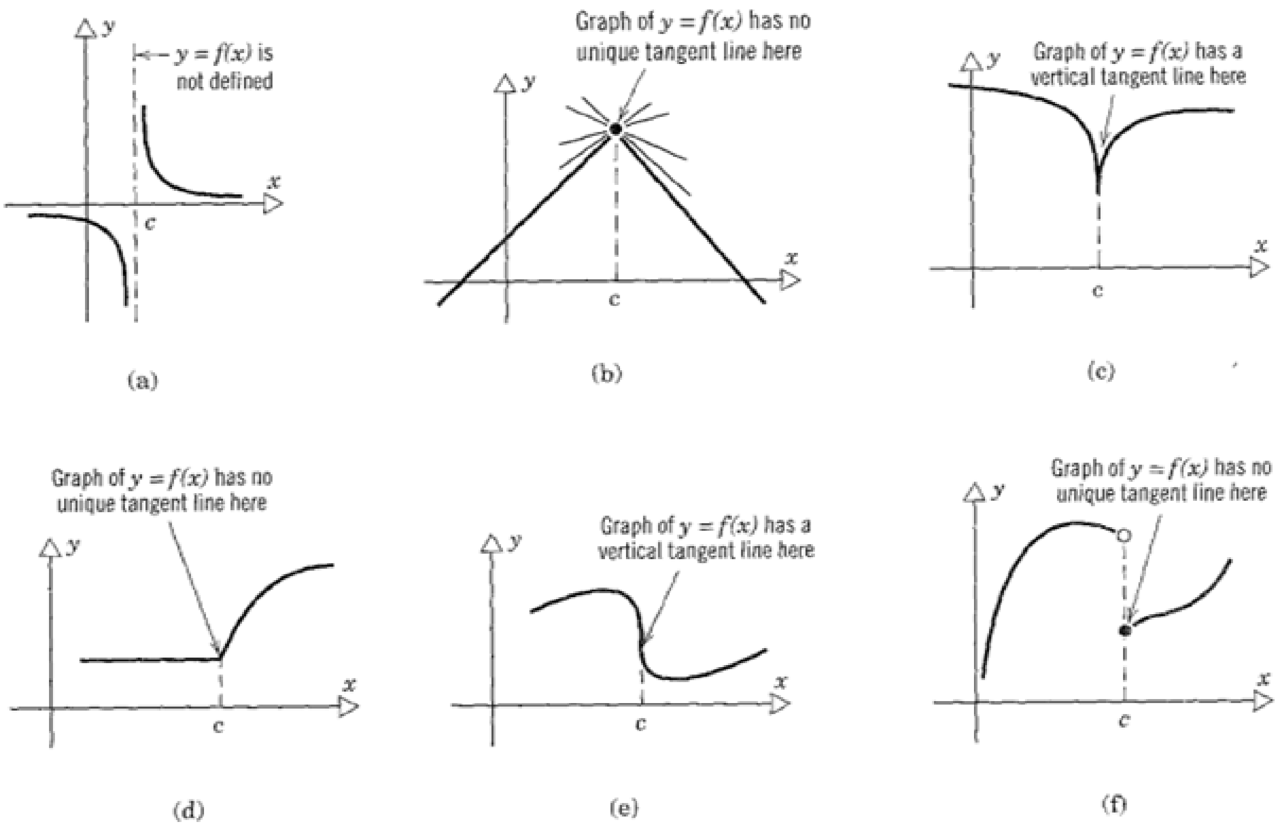
(c) Function undefined at $x = 0$

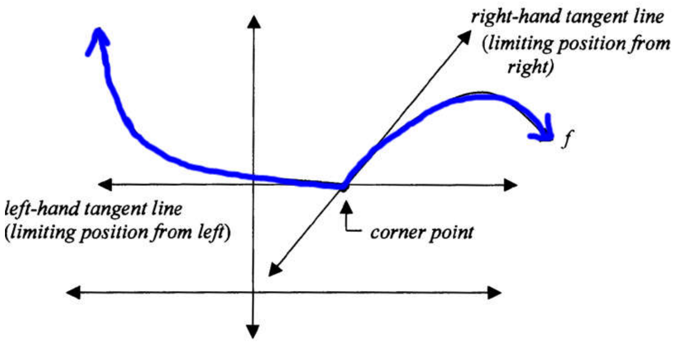
Continuity does not imply differentiability

If the derivative of a function exists at $x = c$, then $f(x)$ is continuous at $x = c$.

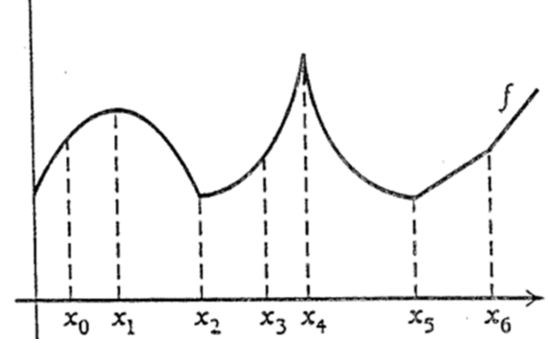
A function is said to be **differentiable at** $x = c$, if it has a derivative when $x = c$.

These functions do not have tangent lines at $x = c$, and, hence, the derivative does not exist at $x = c$.

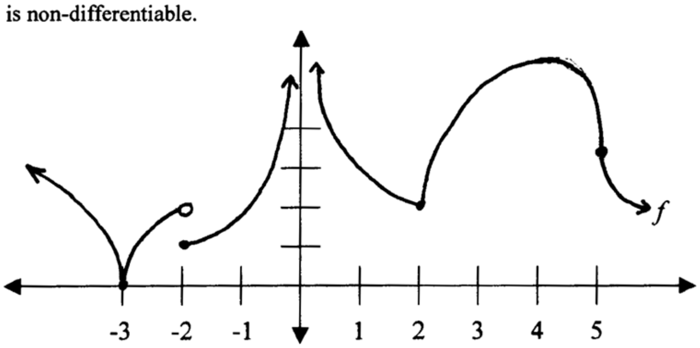


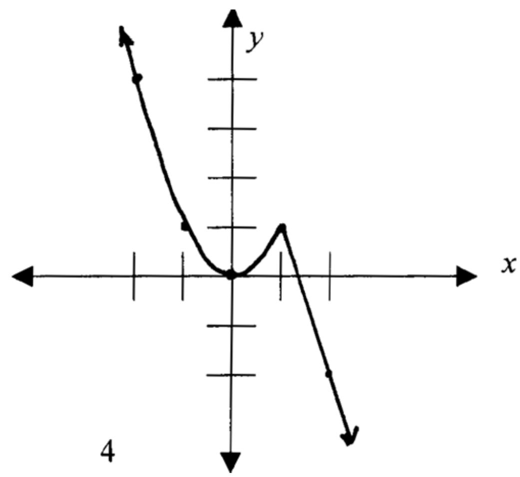


List the points at which the following function is not differentiable.

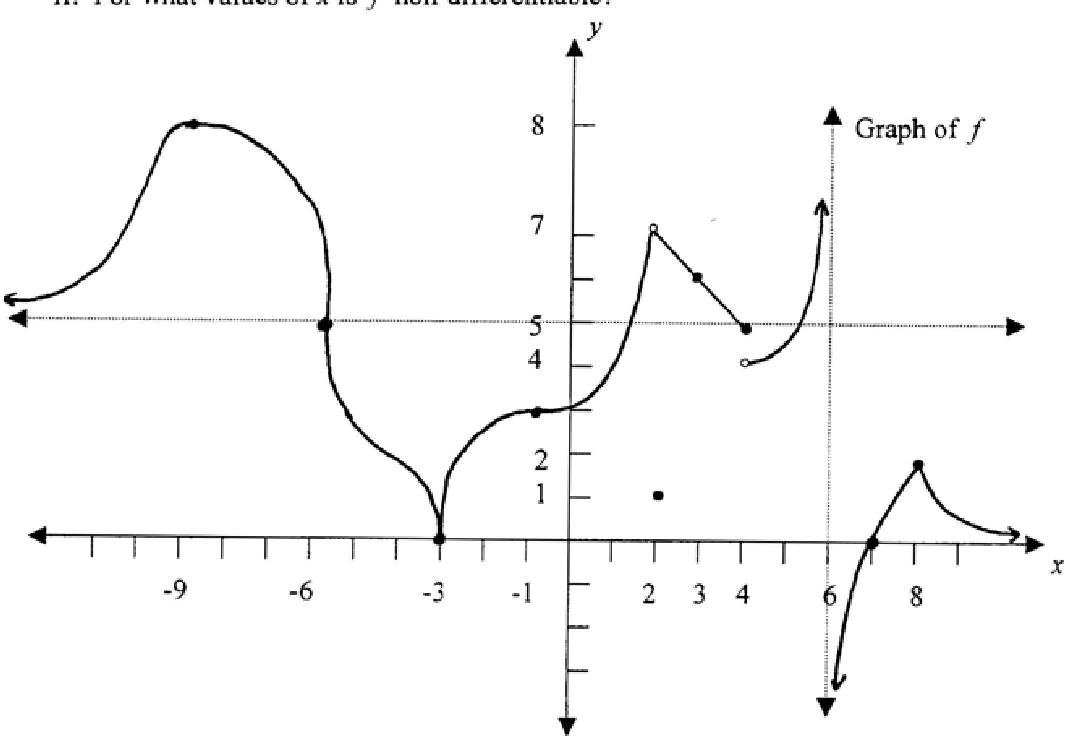


Example 1: Given below is the graph of $y = f(x)$. Determine the points where f is non-differentiable.





II. For what values of x is f non-differentiable?



4.6 – The Derivative as an Instantaneous Velocity

You get into your car at noon and drive non-stop until 3:00pm. If you've driven a total of 150 miles, what was your average velocity (average speed)?

Average Velocity:

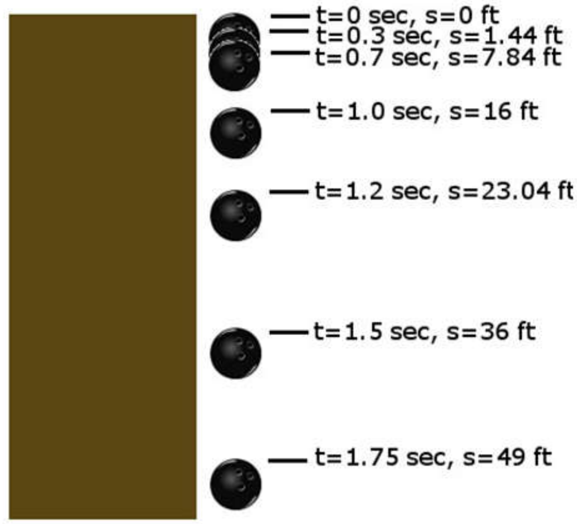
the ratio of the change in position Δs to the change in time Δt

$$v_{avg} = \frac{\Delta s}{\Delta t}$$

$$\frac{\text{change in position}}{\text{elapsed time}}$$

$$v_{avg} = \frac{s_2 - s_1}{t_2 - t_1}$$

A bowling ball is dropped from the roof of a 5 story building (wind resistance is negligible)



1) Use regression analysis to find a quadratic equation model for distance s, as a function of time t:

$$s(t) =$$

2) Estimate the velocity (speed) of the ball at t=1.0 by finding the average velocity using the following time values:

a) average velocity from t=0.7 sec to t=1.0 sec

$$v_{avg} =$$

b) average velocity from t=1.0 sec to t=1.2 sec

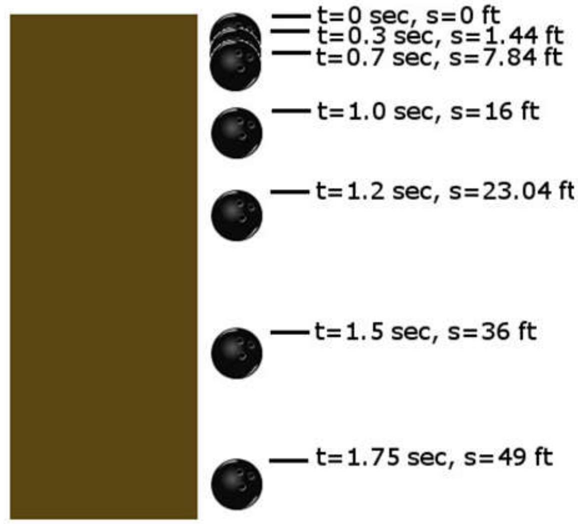
$$v_{avg} =$$

Are these good estimates of the velocity of the ball at 1.0 second? Why or why not?

How could we compute a more accurate value for the velocity at 1.0 second?

Hint: Remember, velocity is the rate of change of the distance vs. time function.

$$s = f(t) = 16t^2$$



3) Find the derivative of the distance as a function of time, and evaluate that function at t=1.0.

Average Velocity:

The ratio of the change in position Δs to the change in time Δt

$$\frac{\text{change in position}}{\text{elapsed time}}$$

$$v_{avg} = \frac{\Delta s}{\Delta t}$$

$$v_{avg} = \frac{s_2 - s_1}{t_2 - t_1}$$

Instantaneous Velocity

The rate of change of distance with time which is the slope of the distance curve at a point

If $s = f(t)$ then velocity is the derivative of the distance function:

$$f'(t) = \lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{\Delta t}$$

3. **Average Velocity** Suppose the function $s = f(t) = 16t^2$ relates the distance s (in feet) an object travels in time t (in seconds). Compute the average velocity, $\Delta s / \Delta t$, from $t = 3$ to:

a) $t = 3.5$

b) $t = 3.1$

6. **Velocity** The position s (in meters) of a particle in time t (in seconds) is given by $s = f(t) = t^2 - 4t$. Find the velocity at $t = 0$; at $t = 3$; at any time t.