Limits / Continuity

Evaluate a limit:

1) graph (LH "y" = RH "y" or DNE)
2) algebraically (plug in x)
   if denom = 0, try:
   - Factoring
   - Synthetic division
   - Rationalize (if $\sqrt{\ }$)
3) if denom zero can't be cancelled
   → infinite limit case: $+\infty$ or $-\infty$

$$\lim_{x \to 5^-} \frac{x+2}{x-5} = \frac{(4.9)+2}{(4.9)-5} \quad \frac{(+)}{(-)} = \boxed{-\infty}$$

(try 4.9)

4) limit at infinity:

$$\lim_{x\to\infty} \frac{3x^2+x-1}{2x^2-x} = \lim_{x\to\infty} \frac{3+\cancel{\frac{1}{x}}^{\,0}-\cancel{\frac{1}{x^2}}^{\,0}}{2-\cancel{\frac{1}{x}}^{\,0}} = \boxed{\frac{3}{2}}$$

Continuity

a) graph 
must be no "break"

b) algebraically (3 things must be true)

1) $f(c)$ must exist "there must be a point at this x"
2) $\lim_{x\to c} f(x)$ must exist

$$\lim_{x\to c^-} f(x) = \lim_{x \to c^+} f(x)$$
(LH) (RH)

"left side y approaching this y must equal right side y"

3) $f(c) = \lim_{x\to c} f(x)$ "point must fit into the hole" (y values match)

L'Hôpital's Rule

if $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{0}{0}, \frac{\infty}{\infty}$, $\lim_{x\to c}\frac{f(x)}{g(x)} = \lim_{x\to c}\frac{f'(x)}{g'(x)}$

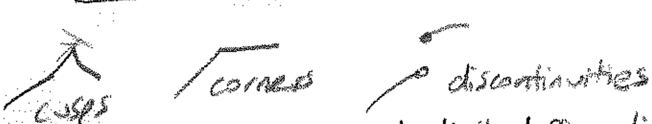
Finding Derivatives

limit definition:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

1st derivative $= f'(x) = y' = \frac{dy}{dx}$ is instantaneous rate of change "slope"

derivatives do not exist at:



derivative shortcuts

$\frac{d}{dx}[x^n] = nx^{n-1}$

$\frac{d}{dx}[e^x] = e^x$

$\frac{d}{dx}[\ln x] = \frac{1}{x}$

$\frac{d}{dx}[\text{const}] = 0$

$\frac{d}{dx}[e^{5x}] = e^{5x}(5)$

not emphasized:

$\frac{d}{dx}[a^x] = a^x \ln a$

$\frac{d}{dx}[\log_a x] = \frac{1}{x\ln a}$

$\frac{d}{dx}[\sin x] = \cos x$

$\frac{d}{dx}[\cos x] = -\sin x$

$\frac{d}{dx}[\tan x] = \sec^2 x$

derivative properties

$(cf)' = cf'$

$(f \pm g)' = f' \pm g'$

$(fg)' = fg' + gf'$

$\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$

chain rule (multiply by deriv of inside)

ex
$y = (3x^2+x)^5$

$y' = 5(3x^2+x)^4 \frac{d}{dx}[3x^2+x]$

$y' = 5(3x^2+x)^4(6x+1)$

$y = \ln(3x^2+x)$

$y' = \frac{1}{3x^2+x}\frac{d}{dx}[3x^2+x]$

$y' = \frac{1}{3x^2+x}(6x+1)$

implicit differentiation

(term by term, multiply by $\frac{dy}{dx}$ if letter mismatch)

ex $x^2 + 4y^2 + xy = 3e^x$

$\frac{d}{dx}[x^2] + \frac{d}{dx}[4y^2] + (x)\frac{d}{dx}[y] + (y)\frac{d}{dx}[x] = \frac{d}{dx}[3e^x]$
(prod. rule)

$2x + 8y\frac{dy}{dx} + x(1\frac{dy}{dx}) + y(1) = 3e^x$

(solve for $\frac{dy}{dx}$)

$8y\frac{dy}{dx} + \frac{dy}{dx} = 3e^x - 2x - y$

$\frac{dy}{dx}(8y+1) = 3e^x - 2x - y$

$\frac{dy}{dx} = \frac{3e^x - 2x - y}{8y+1}$

logarithmic differentiation (not emphasized) logarithmic differentiation (not emphasized)

(add logs in order to break apart complex structures)

$$y=\frac{(x^2+1)^3(x-1)^2}{(x^4+x)^5}$$

$$\ln(y)=\ln\left(\frac{(x^2+1)^3(x-1)^2}{(x^4+x)^5}\right)$$

$$\ln(y)=\ln(x^2+1)^3+\ln(x-1)^2-\ln(x^4+x)^5$$

$$\ln(y)=3\ln(x^2+1)+2\ln(x-1)-5\ln(x^4+x)$$

$$\frac{d}{dx}[\ln(y)]=3\frac{d}{dx}[\ln(x^2+1)]+2\frac{d}{dx}[\ln(x-1)]-5\frac{d}{dx}[\ln(x^4+x)]$$

$$\frac{1}{y}y'=3\frac{1}{x^2+1}(2x)+2\frac{1}{x-1}(1)-5\frac{1}{x^4+x}(4x^3+1)$$

$$\frac{1}{y}y'=\frac{6x}{x^2+1}+\frac{2}{x-1}-\frac{5(4x^3+1)}{x^4+x} \quad (\text{multiply by } y)$$

$$\boxed{y'=\left[\frac{6x}{x^2+1}+\frac{2}{x-1}-\frac{5(4x^3+1)}{x^4+x}\right]\left(\frac{(x^2+1)^3(x-1)^2}{(x^4+x)^5}\right)}$$

partial derivatives

ex $f=3xe^y\ln z+x^4\ln y\, z^5$

$\frac{\partial}{\partial x}$ $3\boxed{x}e^y\ln z+\boxed{x^4}\ln y\, z^5$

$$\frac{\partial f}{\partial x}=f_x=3(1)e^y\ln z+4x^3\ln y\, z^5$$

$3e^y\ln z+4\boxed{x^3}\ln y\, z^5$

$$\frac{\partial^2 f}{\partial x^2}=f_{xx}=12x^2\ln y\, z^5$$

$3\boxed{e^y}\ln z+4x^3\boxed{\ln y}\, z^5$

$$\frac{\partial^2 f}{\partial y\partial x}=f_{xy}=3e^y\ln z+4x^3\frac{1}{y}z^5$$

$3x\boxed{e^y}\ln z+x^4\boxed{\ln y}\, z^5$

$$\frac{\partial f}{\partial y}=f_y=3xe^y\ln z+x^4\frac{1}{y}z^5$$

$3x\boxed{e^y}\ln z+x^4\boxed{y^{-1}}z^5$

$$\frac{\partial^2 f}{\partial y^2}=f_{yy}=3xe^y\ln z+x^4(-y^{-2})z^5$$

$3xe^y\boxed{\ln z}+x^4y^{-1}\boxed{z^5}$

$$\frac{\partial^2 f}{\partial z\partial y}=f_{yz}=3xe^y\frac{1}{z}+x^4y^{-1}(5z^4)$$

Applications of derivatives

Instantaneous rate of change $= f'(x)$ (use calculus)

Average rate of change (use algebra)

avg rate of change $= \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$

$f(x)$ = "y value"

$f'(x)$ = "instantaneous rate of change" "slope"

$f''(x)$ = "concavity" "curvature"

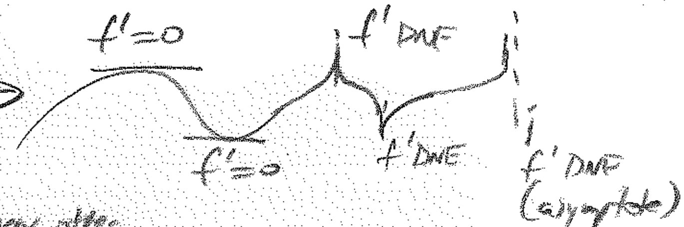
distance: $s(t)$

velocity: $v(t) = s'(t)$

acceleration: $a(t) = v'(t) = s''(t)$

Curve sketching

- interesting things happen when $f'(x) = 0$ or DNE →



- use $f''(x)$ value at that x to find concavity to determine min, max, other

(when $f''(x) = 0$ called an "inflection point" (curvature is changing))
$\cap\cup$ or $\cup\cap$

---

optimizing what are we optimizing? (objective function)

2^{nd} function = tool kit, substitute to have obj. function w/ 1 variable

$f' = 0$ to find max, min

Related rates - need an equation relating quantities in problem (usually geometry)

- take $\frac{d}{dt}$ (implicit)
- Solve by plugging in snapshot values

Multi-variate

optimization w/o constraint

obj. func: $f(x,y)$

Critical points $\in \begin{cases} f_x = 0 \\ f_y = 0 \end{cases}$

$D = (f_{xx})(f_{yy}) - (f_{xy})^2$

if $D < 0$ C.P. is saddle pt

$D > 0$ C.P. is max/min

use f_{xx} (concavity)

$f_{xx} +$ $\cup$ (min)

$f_{xx} -$ $\cap$ (max)

$D = 0$ inconclusive

Optimization w/ constraint (Method of Lagrange Multipliers)

obj. func: $f(x,y)$ constraint: $g(x,y) = 0$

$F = f(x,y) + \lambda g(x,y)$

Critical points $\in \begin{cases} F_x = 0 \\ F_y = 0 \\ F_\lambda = 0 \end{cases}$

C.P.	obj. function
(x,y)	20 ← max
(x,y)	10
(x,y)	-5 ← min

Evaluating integrals

Shortcuts:

$\int 5\,dx = 5x + K$

$\int x^2 dx = \frac{x^3}{3} + K$

$\int \frac{1}{x} dx = \ln|x| + K$

$\int e^x dx = e^x + K$

$\int e^{5x} dx = \frac{e^{5x}}{5} + K$

More complicated?

1) algebraic simplification

$$\int \left(\frac{\sqrt{x} + 3x}{\sqrt{x}}\right) dx = \int \left(\frac{\sqrt{x}}{\sqrt{x}} + \frac{3x}{\sqrt{x}}\right) dx = \int (1 + 3x^{1/2}) dx$$

2) int. by substitution

$$\int x^2 e^{(x^3)} dx$$

$u = x^3$ (inside function)

$\frac{du}{dx} = 3x^2$ substitute into orig. integral

$du = \boxed{3x^2 dx}$

$\frac{1}{3} du = x^2 dx$

$\int e^u \boxed{x^2 dx}$

$\int e^u \frac{1}{3} du$

$\frac{1}{3} \int e^u du$

$\frac{1}{3} e^u + K$

$\boxed{\frac{1}{3} e^{x^3} + K}$

3) int by parts

$$\int \underbrace{x}_{u}\, \underbrace{e^{-3x} dx}_{dv}$$

$u = x \quad dv = e^{-3x} dx$

$\frac{du}{dx} = 1 \quad \int dv = \int e^{-3x} dx$

$du = dx \quad v = \frac{e^{-3x}}{-3}$

substitute into $uv - \int v\,du$

$(x)\left(\frac{e^{-3x}}{-3}\right) - \int \left(\frac{e^{-3x}}{-3}\right) dx$

$-\frac{1}{3} x e^{-3x} + \frac{1}{3} \int e^{-3x} dx$

$-\frac{1}{3} x e^{-3x} + \frac{1}{3} \frac{e^{-3x}}{-3} + K$

$\boxed{-\frac{1}{3} x e^{-3x} + \frac{1}{9} e^{-3x} + K}$

Properties of integrals

$\int_a^b f(x)dx = -\int_b^a f(x)dx$

$\int_a^a f(x)dx = 0$

$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

(c is between a & b)

$\int c f(x)dx = c \int f(x) dx$

$\int [f(x) \pm g(x)] dx = \int f(x) dx \pm \int g(x) dx$

Definite Integrals (Fundamental Theorem of Calculus)

$$\int_a^b f(x)dx = F(b) - F(a)$$

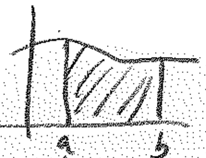
(antiderivatives)

ex $\int_1^3 \frac{1}{x} dx = [\ln x]_1^3 = \boxed{\ln(3) - \ln(1)}$

or use calculator (MATH→9)

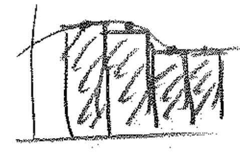
Application of Integrals

area under a curve:



$$A = \int_a^b f(x)\,dx$$

is the smoothed version of a Riemann Sum:

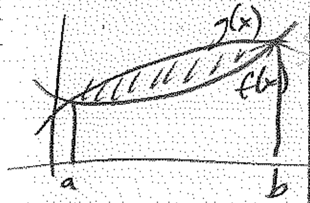


$$A = \sum_{n \text{ strips}} f(x_i)\cdot \Delta x_i$$

area between curves:

$$A = \int_a^b [(g(x) - f(x)]\,dx$$

top − bottom



Volumes of solids of revolution:

disc	shell
no hole: $V = \int \pi r^2 \Delta h$	no hole: $V = \int 2\pi r h \Delta r$
w/ hole: $V = \int \pi r_1^2 \Delta h - \int \pi r_2^2 \Delta h$	w/ hole: $V = \int 2\pi r_1 h \Delta r - \int 2\pi r_2 h \Delta r$

rect ⊥ axis of rotation

rect ∥ axis of rotation

area under curve = accumulated quantity of antiderivative

ex if curve is velocity:

then $\int_0^1 v(t)\,dt = 50$ miles the accumulated distance

Average value:

$$AV = \frac{\int_a^b f(x)\,dx}{b-a}$$

f(x) is the value (a–b entire interval)

Expected value:

$$EV = \int_a^b x f(x)\,dx$$

x is the value
f(x) is PDF
(a–b is entire interval)

solving separable differential equations:

$$\frac{dy}{dx} = x^2 + 3x^4$$

Boundary condition: $y = 6$ when $x = 0$

separate variables:

$$dy = (x^2 + 3x^4)\,dx$$

$$\int dy = \int (x^2 + 3x^4)\,dx$$

$$y = \frac{x^3}{3} + \frac{3x^5}{5} + K$$

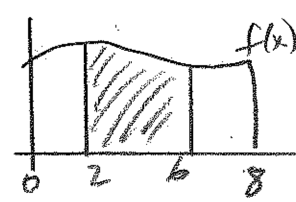
(general solution)

$$(6) = \frac{(0)^3}{3} + 3\frac{(0)^5}{5} + K$$

$$6 = K$$

So $y = \frac{x^3}{3} + 3\frac{x^5}{5} + 6$ (particular solution)

Continuous Probability



if f(x) is a Probability Density Function (PDF) for $0 \le x \le 8$

then $\int_0^8 f(x)\,dx = 1$ (100%)

So $P(2 < x < 6) = \int_2^6 f(x)\,dx$

Discrete Probability

X	0 1 2 3 4 5 6 . . . . .
P	poissoncdf(λ,5)

if no n → Poisson

poissonpdf(λ,5)

$P(x=5) = \text{poissonpdf}(\lambda,5)$
$P(x\le 5) = \text{poissoncdf}(\lambda,5)$
$P(x>5) = 1 - \text{poissoncdf}(\lambda,5)$

no fixed n? → Poisson
fixed n? → Binomial

X	0 1 2 3 4 5 6 . . . . 10 ← fixed n trials
	binomcdf(10,p,5) binompdf(10,p,5)

$P(x=5) = \text{binompdf}(10,p,5)$
$P(x\le 5) = \text{binomcdf}(10,p,5)$
$P(x>5) = 1 - \text{binomcdf}(10,p,5)$