Triangle Trigonometry……

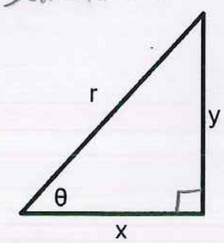
Right Triangles

SOH CAH TOA

$$\cos\theta = \frac{adj}{hyp} = \frac{x}{r}$$

$$\sin\theta = \frac{opp}{hyp} = \frac{y}{r}$$

$$\tan\theta = \frac{opp}{adj} = \frac{y}{x}$$



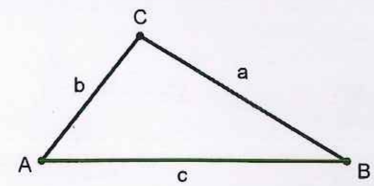
All triangles (including non-right triangles)

Law of Sines:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

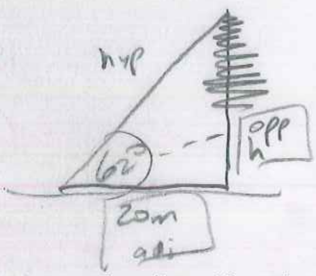
Law of Cosines:

$$a^2 = b^2 + c^2 - 2bc\cos A$$



'<u>angle of elevation</u>' = angle <u>above</u> a horizontal line 'angle of depression = angle <u>below</u> a horizontal line

#1) You are standing 20 m from the base of a large tree. The angle of elevation from your feet to the top of the tree is 62°. How tall is the tree?



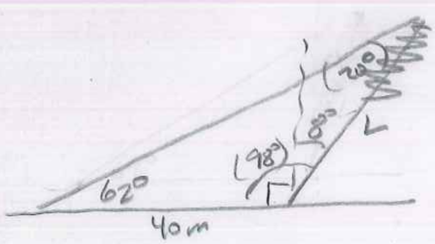
$\tan\theta = \frac{opp}{adj}$

$\tan 62° = \frac{h}{20}$

$h = 20\tan 62°$

$\boxed{h = 37.6m}$

#2) You are standing 40 m from the base of a tree that is leaning 8° from vertical away from you. The angle of elevation from your feet to the top of the tree is 62°. How tall (slant height) is the tree?

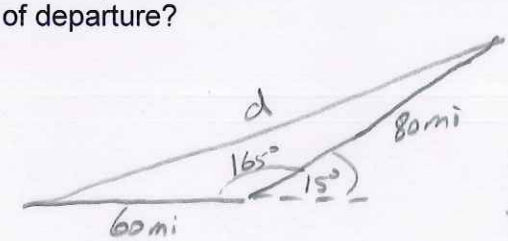


$180° - 62° - 98° = 20°$

Law of sines:

$\frac{L}{\sin 62°} = \frac{40}{\sin 20°}$ $L = \frac{40\sin 62°}{\sin 20°} = \boxed{103.3\ m}$

#3) A ship travels 60 miles, then adjusts its course 15°. After traveling 80 miles in the new direction, how far is the ship from its point of departure?



Law of cosines:

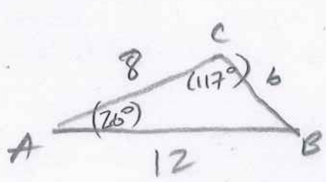
$d^2 = 60^2 + 80^2 - 2(60)(80)\cos 165°$

$d^2 = 19277.88793$

$\boxed{d = 138.8\ miles}$

#4) The lengths of the sides of a triangle are 6, 8, and 12. Find all the interior angles of this triangle.

(find largest angle first)



law of cosines

$12^2 = 6^2 + 8^2 - 2(6)(8)\cos C$

$144 = 100 - 96\cos C$

$-100 \quad -100$

$44 = -96\cos C$

$\cos C = -0.4583..$

$C = \cos^{-1}(-0.4583) = \boxed{117°}$

law of sines

$\frac{12}{\sin 117°} = \frac{6}{\sin A}$

$12\sin A = 6\sin 117°$

$\sin A = \frac{6\sin 117°}{12}$

$\sin A = 0.4455..$

$A = \sin^{-1}(0.4455) \approx \boxed{26°}$

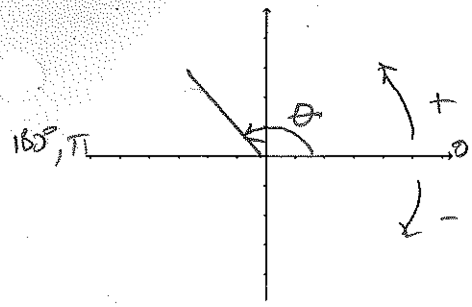
angles sum to 180°

$B = 180 - 117 - 26$

$B = \boxed{37°}$

General angles, Arc Length, Unit Circle definitions of Sine and Cosine....

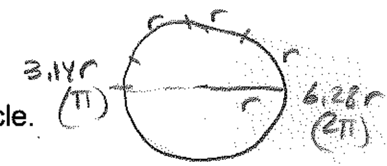
Angles in standard position:



0 reference to right (x-axis), positive CCW, negative CW.

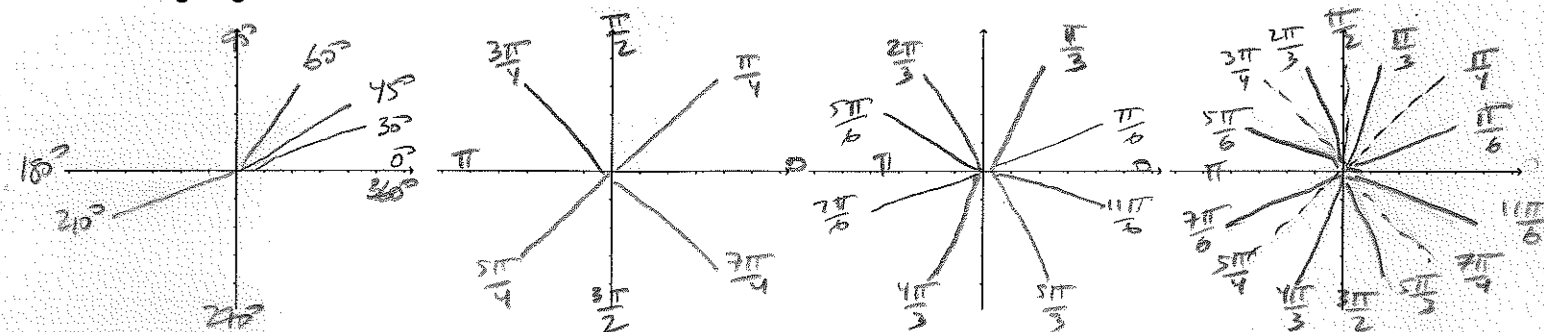
degrees (360 in a circle)

radians (2π in a circle)
1 radian = distance of one radius around the circle.



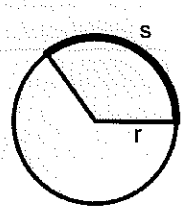
Converting: $180° = \pi$

Locating angles: Use $180°$ or π as a reference:



Coterminal angles: add or subtract multiples of 2π *or* $360°$. $60°, 420°$ coterminal $2\pi, 4\pi, -2\pi$ coterminal

Arc length problems: $s = r\theta, \quad \theta \text{ in radians}$



1 radian = distance of one radius around the circle.

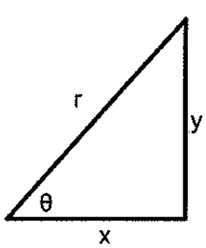
What is distance between 2 points on a circle of radius 5m, if angle between points is 45 degrees?

$\theta = 45°\left(\frac{\pi}{180°}\right) = \frac{\pi}{4}$ radians (0.785 radians)

$s = r\theta = (5m)\left(\frac{\pi}{4}\right) = \boxed{3.927m}$

Two ways to view sine, cosine:

<u>Triangle / ratio view</u>

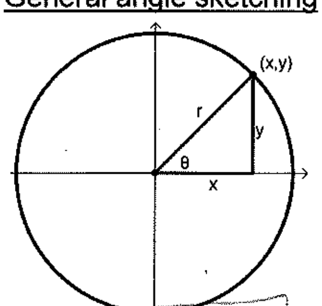


$$\cos\theta = \frac{adj}{hyp} = \frac{x}{r}$$

$$\sin\theta = \frac{opp}{hyp} = \frac{y}{r}$$

$$\tan\theta = \frac{opp}{adj} = \frac{y}{x}$$

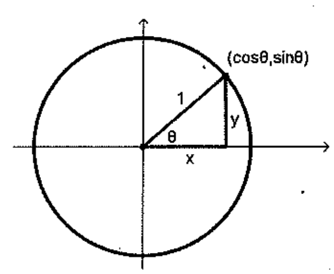
<u>General angle sketching</u>



$$\boxed{\cos\theta = \frac{x}{r} \quad \sin\theta = \frac{y}{r} \quad \tan\theta = \frac{y}{x}}$$

sketching rules

<u>Unit circle / function view</u>



$\cos\theta = x\ coordinate$

$\sin\theta = y\ coordinate$

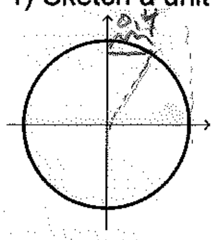
of a point on unit circle

The other 4 trig functions are defined from sine and cosine:

$$\tan(\theta)=\frac{\sin(\theta)}{\cos(\theta)} \qquad \csc(\theta)=\frac{1}{\sin(\theta)} \qquad \sec(\theta)=\frac{1}{\cos(\theta)} \qquad \cot(\theta)=\frac{1}{\tan(\theta)}$$

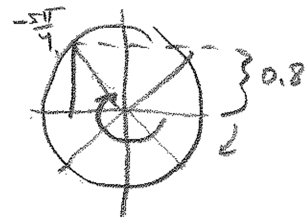
Computing output of a sine or cosine function. Interpret the input as an angle in standard position on a unit circle. Output is x or y value of point on the circle at that angle. 3 ways:

1) Sketch a unit circle and measure the x or y distance:



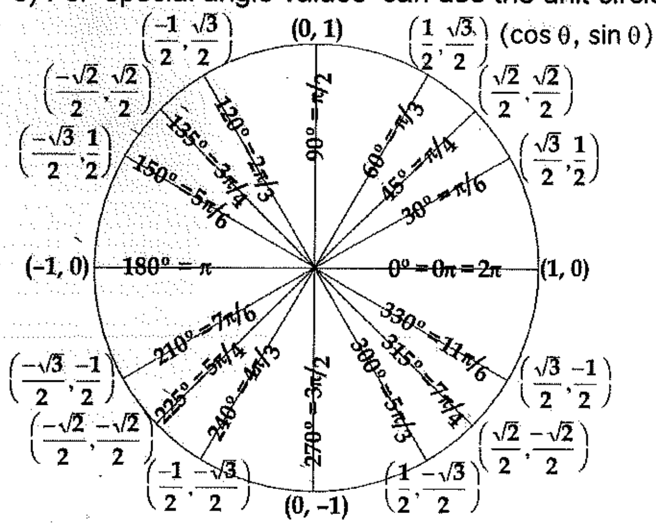
Find $\cos(70°) \simeq 0.4$ $\qquad \sin\left(-\frac{5\pi}{4}\right)$

$\simeq 0.8$



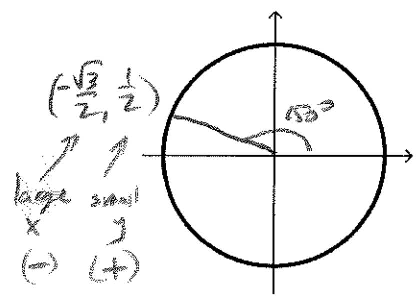
2) Use a calculator (make sure mode is set correctly). $\cos(70°) \approx 0.342$ $\quad \sin\left(-\frac{5\pi}{4}\right) \approx 0.7071 \left(\frac{\sqrt{2}}{2}\right)$

3) For 'special angle values' can use the unit circle to lookup sin or cos:



Find $\cos(\cancel{210°})$ $\boxed{-\frac{\sqrt{3}}{2}}$ $\qquad \sin\left(-\frac{7\pi}{4}\right)$ $\boxed{\frac{\sqrt{2}}{2}}$ $\qquad \tan\left(\frac{2\pi}{3}\right)$

$$\frac{\sin\left(\frac{2\pi}{3}\right)}{\cos\left(\frac{2\pi}{3}\right)} = \frac{\sqrt{3}/2}{-1/2} = \boxed{-\sqrt{3}}$$



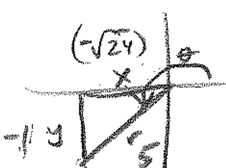
all values are $0, 1, \frac{1}{2}, \frac{\sqrt{2}}{2}$, or $\frac{\sqrt{3}}{2}$

You can also find $\sin(\theta)$ and $\cos(\theta)$ given a point (even if it isn't on the unit circle) by sketching:

Example: If $\csc\theta=-5$ *and* $\pi<\theta<\frac{3\pi}{2}$, find $\tan\theta$

$\frac{1}{\sin\theta} = -5$ $\qquad$ quadrant III

$\sin\theta = -\frac{1}{5} = \frac{y}{r}$



$x^2+y^2=r^2$

$x^2+1^2=5^2$

$x^2+1=25$

$x^2=24$

$x=\sqrt{24}$

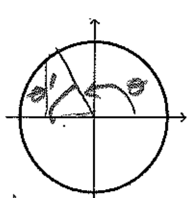
$\tan\theta = \frac{y}{x} = \frac{-1}{-\sqrt{24}}$

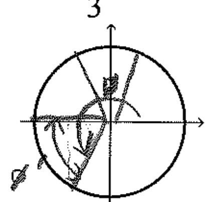
$= \frac{\sqrt{24}}{-24} = \frac{-\sqrt{4}\sqrt{6}}{-24}$

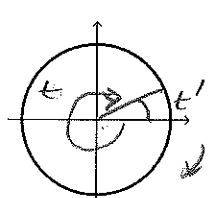
$= \frac{-2\sqrt{6}}{-24} = \boxed{\frac{\sqrt{6}}{12}}$

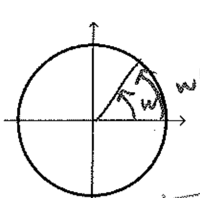
<u>Reference angle</u>: The acute angle from the terminal side to the nearest x-axis. Find reference angles:

$\theta=120°$ $\qquad \phi=\frac{4\pi}{3}$ $\qquad t=-330°$ $\qquad w=50°$









$\theta' = 180° - 120°$ $= \boxed{60°}$

$\frac{4\pi}{3}$ $\quad \phi' = \frac{4\pi}{3} - \pi = \boxed{\frac{\pi}{3}}$

$t' = \boxed{30°}$

$w' = w = \boxed{50°}$

Trigonometric Identities....

Equation: $\sin(u)=\frac{1}{2}$ Identity: $\tan u=\frac{\sin u}{\cos u}$

One side of an identity can be substituted for the other, so trigonometric identities can be used to simplify expressions.

If you know that $\sin(2t+3v)=14$ and you want to simplify, is it valid to do this: $\begin{array}{l}\sin(2t+3v)=14\\ \sin(2t)+\sin(3v)=14\end{array}$?

No...

$$\sin(u+v)\neq\sin(u)+\sin(v)$$

$$\textit{Instead}:\ \sin(u+v)=\sin(u)\cos(v)+\cos(u)\sin(v)$$

This is a trigonometric identity because it is true for all input values.

Some Trigonometric Identities:

$$\sin\theta=\frac{1}{\csc\theta}\quad \cos\theta=\frac{1}{\sec\theta}\quad \tan\theta=\frac{1}{\cot\theta}\qquad \tan\theta=\frac{\sin\theta}{\cos\theta}\qquad \sin^2\theta+\cos^2\theta=1$$

$$\csc\theta=\frac{1}{\sin\theta}\quad \sec\theta=\frac{1}{\cos\theta}\quad \cot\theta=\frac{1}{\tan\theta}\qquad \cot\theta=\frac{\cos\theta}{\sin\theta}\qquad 1+\tan^2\theta=\sec^2\theta$$

$$1+\cot^2\theta=\csc^2\theta$$

$$\sin(u+v)=\sin u\cos v+\cos u\sin v$$

$$\sin(u-v)=\sin u\cos v-\cos u\sin v$$

$$\sin 2u=2\sin u\cos u$$

$$\sin\frac{u}{2}=\pm\sqrt{\frac{1-\cos u}{2}}=\pm\sqrt{\frac{1}{2}(1-\cos u)}$$

$$\cos\frac{u}{2}=\pm\sqrt{\frac{1+\cos u}{2}}=\pm\sqrt{\frac{1}{2}(1+\cos u)}$$

$$\cos(u+v)=\cos u\cos v-\sin u\sin v$$

$$\cos(u-v)=\cos u\cos v+\sin u\sin v$$

$$\cos 2u=\cos^2 u-\sin^2 u$$

$$\cos 2u=2\cos^2 u-1$$

$$\cos 2u=1-2\sin^2 u$$

$$\tan\frac{u}{2}=\frac{1-\cos u}{\sin u}=\frac{\sin u}{1+\cos u}$$

$$\tan(u+v)=\frac{\tan u+\tan v}{1-\tan u\tan v}$$

$$\tan(u-v)=\frac{\tan u-\tan v}{1+\tan u\tan v}$$

$$\tan 2u=\frac{2\tan u}{1-\tan^2 u}$$

$$\sin^2 u=\frac{1-\cos 2u}{2}$$

$$\cos^2 u=\frac{1+\cos 2u}{2}$$

$$\tan^2 u=\frac{1-\cos 2u}{1+\cos 2u}$$

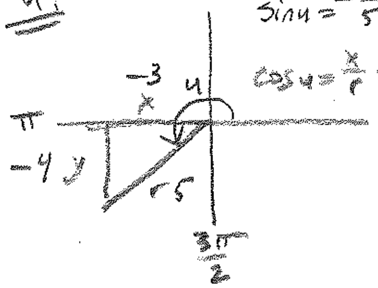
If $\sin u = -\frac{4}{5}$ where $\pi < u < \frac{3\pi}{2}$ and $\sin v = \frac{8}{17}$ where $0 < v < \frac{\pi}{2}$

evaluate $\cos(u - v)$ (answer in exact form).

u:

$\sin u = \frac{-4}{5} \frac{(y)}{(r)}$

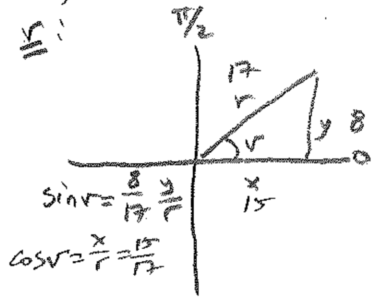
$\cos u = \frac{x}{r} = \frac{-3}{5}$



v:

$\sin v = \frac{8}{17} \frac{y}{r}$

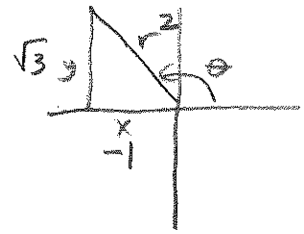
$\cos v = \frac{x}{r} = \frac{15}{17}$



$$\cos(u - v) = \cos u \cos v + \sin u \sin v$$
$$= \left(\frac{-3}{5}\right)\left(\frac{15}{17}\right) + \left(\frac{-4}{5}\right)\left(\frac{8}{17}\right)$$
$$= \frac{-45}{85} + \frac{-32}{85}$$
$$= \boxed{\frac{-77}{85}}$$

If $\tan x = -\sqrt{3}$ where x in an angle in quadrant II,

$\tan\theta = -\sqrt{3}$ (change x to θ to avoid confusion)



$\tan\theta = -\sqrt{3} = -\frac{\sqrt{3}}{1} \frac{(y)}{(x)}$

$(-1)^2 + (\sqrt{3})^2 = r^2$

$1 + 3 = r^2$

$r = 2$

So $\sin\theta = \frac{y}{r} = \frac{\sqrt{3}}{2}$

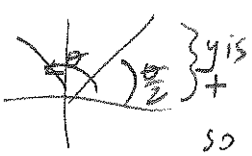
$\cos\theta = \frac{x}{r} = \frac{-1}{2}$

(a) evaluate $\cos(2x)$ (answer in exact form).

$$\cos(2\theta) = 1 - 2\sin^2\theta$$
$$= 1 - 2(\sin\theta)^2$$
$$= 1 - 2\left(\frac{\sqrt{3}}{2}\right)^2$$
$$= 1 - 2\,\frac{3}{4}$$
$$= 1 - \frac{6}{4} = \frac{4}{4} - \frac{6}{4} = \frac{-2}{4} = \boxed{\frac{-1}{2}}$$

(b) evaluate $\sin\left(\frac{1}{2}x\right)$ (answer in exact form).

$$\sin\left(\tfrac{1}{2}\theta\right) = \pm\sqrt{\frac{1 - \cos\theta}{2}}$$
$$= \pm\sqrt{\frac{1 - \left(-\frac{1}{2}\right)}{2}}$$
$$= \pm\sqrt{\frac{\left(\frac{3}{2}\right)}{2}} = \pm\sqrt{\frac{3}{4}} = \pm\frac{\sqrt{3}}{2}$$



so choose positive one $\rightarrow = \boxed{\frac{\sqrt{3}}{2}}$

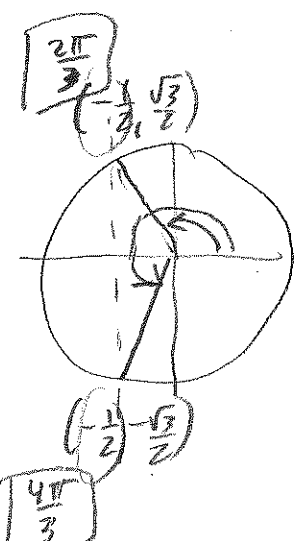
(c) evaluate $\tan^2(x)$ (answer in exact form).

$$\tan^2(\theta) = \frac{1 - \cos(2\theta)}{1 + \cos(2\theta)} = \frac{1 - \left(-\frac{1}{2}\right)}{1 + \left(-\frac{1}{2}\right)} = \frac{3/2}{1/2}$$
$$= \boxed{3}$$

Simplify and solve: $\cos x + \sin x \tan x = -2$

$$\frac{\cos x}{1} + \sin x \frac{\sin x}{\cos x} = -2$$
$$\frac{\cos^2 x}{\cos x} + \frac{\sin^2 x}{\cos x} = -2$$
$$\frac{\cos^2 x + \sin^2 x}{\cos x} = -2$$
$$\frac{1}{\cos x} = -2$$
$$\cos x = -\frac{1}{2}$$

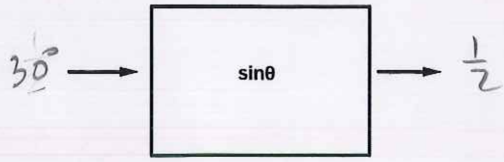
"x" is $-\frac{1}{2}$

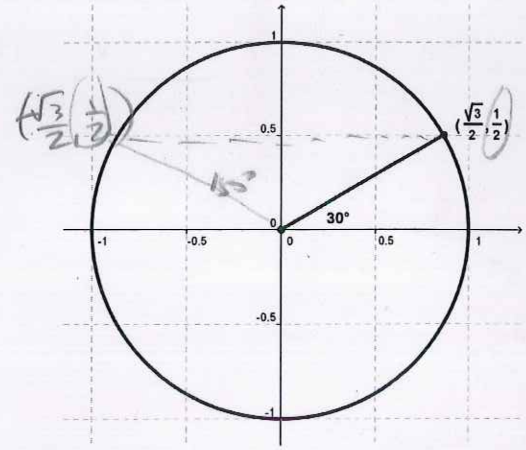


$$\boxed{\begin{aligned}(\text{the angle})\ x &= \frac{2\pi}{3}\ (+n2\pi)\\ x &= \frac{4\pi}{3}\ (+n2\pi)\end{aligned}}$$

Inverse trigonometric functions, Solving trigonometric equations....

The sine function converts an angle input to a y-value output:



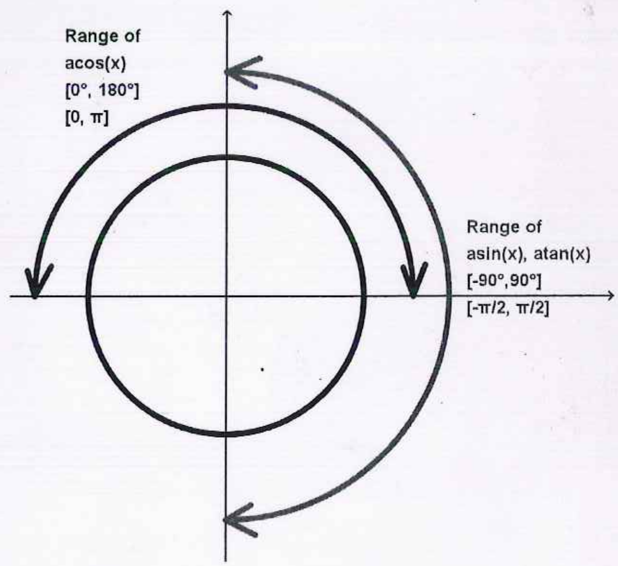


The inverse sine function converts a y-value input to an angle output:

(also 150°, but only one angle can be the output)

One complication: There are usually two angle that have a given y-value on a unit circle, but if inverse sine is a function, it can only give one angle output for a given input y-value. This is also true for inverse cosine and inverse tangent.

Range for allowable output angles for the inverse trig functions:



Solving equations containing trigonometric functions...

...usually means we are finding the angle, or angles, which make the equation true.

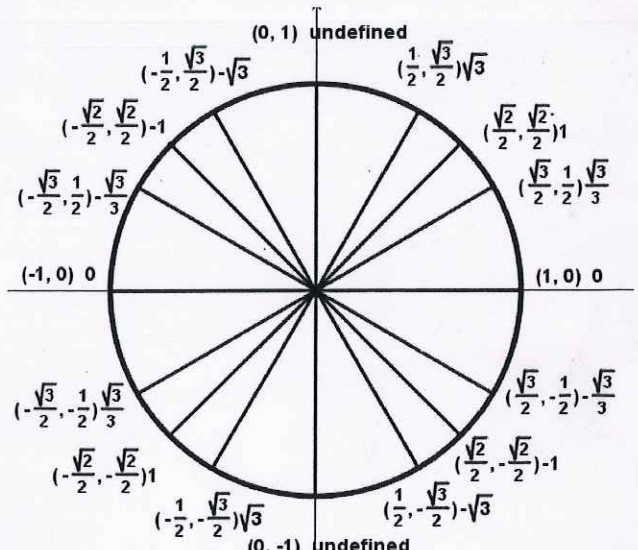
General strategy:

1) Try to isolate a single trig function on one side and a number on the other. If the equation is complicated, try to factor into multiple factors each containing a single trig functions.

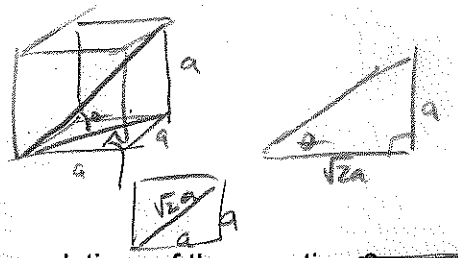
2) If the right side number is $0, \frac{1}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{2}, 1, \sqrt{3}, or\ \frac{\sqrt{3}}{3}$ look up the matching value on a unit circle chart.

If right side is anything else, use calculator inverse trig function to find angle.

3) Remember that whether you use calculator or chart, there are often two angles that have a given trig value.



#1) Determine the angle between the diagonal of a cube and the diagonal of its base.



$\tan\theta = \frac{a}{\sqrt{2}a}$

$\tan\theta = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

not a special value for tan, so use calculator

$\theta = \tan^{-1}\left(\frac{\sqrt{2}}{2}\right) = \boxed{35.3^\circ}$

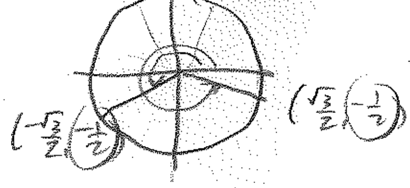
#2) Find all the solutions of the equation ~~$2\cos t + 1 = 0$~~ in the interval $[0, 2\pi)$

$2\sin t + 1 = 0$

$2\sin t + 1 = 0$

$2\sin t = -1$

$\sin t = -\frac{1}{2}$



$\boxed{t = \frac{7\pi}{6}, \frac{11\pi}{6}}$

#3) Find all the solutions of the equation $\cos^2 x - \cos x = 0$ in the interval $[0, 2\pi)$

$\cos^2 x - \cos x = 0$

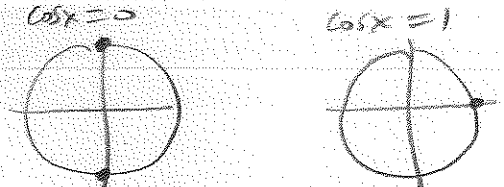
$(\cos x)(\cos x - 1) = 0$

$\cos x = 0$

$\cos x - 1 = 0$

$\cos x = 1$

$\boxed{x = 0, \frac{\pi}{2}, \frac{3\pi}{2}}$



$x = \frac{\pi}{2}, \frac{3\pi}{2}$

$x = 0$

#4) Find all the solutions of the equation $\sec^2 x - \sec x = 2$ in the interval $[0, 2\pi)$

$\sec^2 x - \sec x = 2$

$\sec^2 x - \sec x - 2 = 0$

Substitute: $u = \sec x$

$u^2 - u - 2 = 0$

$(u+1)(u-2) = 0$

$(\sec x + 1)(\sec x - 2) = 0$

$(\sec x + 1)(\sec x - 2) = 0$

$\sec x + 1 = 0$

$\sec x = -1$

$\frac{1}{\cos x} = -1$

$\cos x = -1$

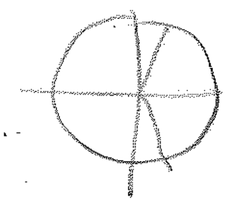
$\boxed{x = \pi}$

$\sec x - 2 = 0$

$\sec x = 2$

$\frac{1}{\cos x} = 2$

$\cos x = \frac{1}{2}$



$\boxed{x = \frac{\pi}{3}, \frac{5\pi}{3}}$

#5) Find all the solutions of the equation $\tan(4t) = 2$ in the interval $[0, 2\pi)$

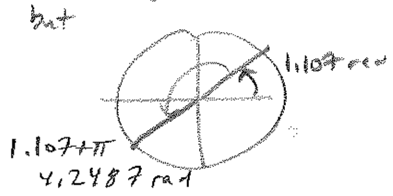
Substitute: $\theta = 4t$

$\tan\theta = 2$ ← not a special value

$\theta = \tan^{-1}(2)$

$\theta = 1.107$ radians

but



$\theta = 1.107$ rad, or 4.2487 rad

$4t = 1.107$

$4t = 4.2487$

so

$\boxed{t = 0.2768}$ $\boxed{t = 1.0622}$