Honors Finite Mathematics – Lesson Notes: Unit 5 (multiple chapters) Combinatorics

7.1 – Sets

Set = a collection of distinct objects we wish to treat as a whole.

Set Notation - There are 2 ways to denote a set:

Roster method - List all elements in the set:

$$D = \{3,5,7,9,11,13,15\}$$

Set-builder notation - Provides a rule for building the set:

$$D = \{x \mid x \text{ is an odd integer between } 3 \text{ and } 15 \text{ inclusive}\}$$

This is read: "*D is the set of all x, such that x is an odd integer between 3 and 15 inclusive.*"

Set terms, symbols, and properties:

Empty or Null Set: A set that contains no elements.
Denoted as { } or $\varnothing$.

Universal Set: The set containing all elements under consideration.
Denoted as U.

Equality of Sets: Two sets are equal if they have the same elements (order of listing does not matter). $\{1,3,5,7\} = \{7,1,5,3\}$

Set terms, symbols, and properties:

Subset: A is a subset of B if every element in A is contained in B (the A=B case is allowed). Denoted $A \subseteq B$

$$\{dog, cat, bird\} \subseteq \{dog, cat, bird, elephant\}$$
$$\{dog, cat, bird\} \subseteq \{dog, cat, bird\}$$

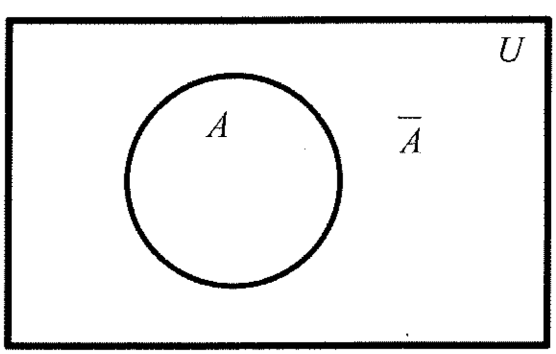
Proper Subset: A is a subset of B if every element in A is contained in B but there is at least one element in B not contained in A (the A=B case is not allowed). Denoted $A \subset B$

$$\{cat, bird\} \subset \{dog, cat, bird\}$$
$$\{dog, cat, bird\} \not\subset \{dog, cat, bird\}$$

Negation: A line through the symbol designates 'not true'.

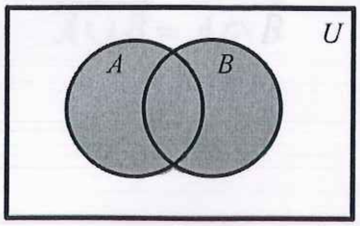
Venn Diagrams:

A convenient way to visually represent sets and their interactions.



Operations on Sets:

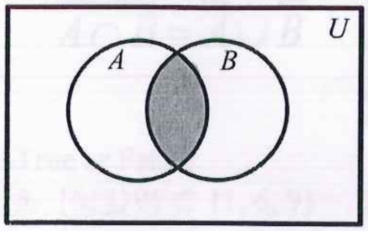
Union: $A \cup B$



- $A \cup B = \{x \mid x \text{ is in } A \text{ or } x \text{ is in } B\}$
- A union B
- Those elements in A or B
- Those elements in A and B combined

$\cup$ = 'or'

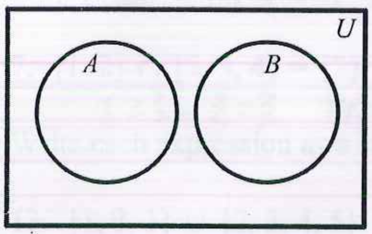
Intersection: $A \cap B$



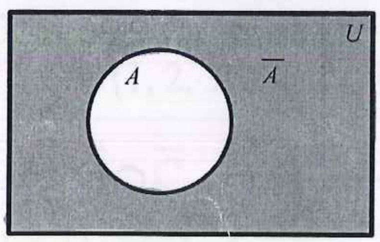
- $A \cap B = \{x \mid x \text{ is in } A \text{ and } x \text{ is in } B\}$
- A intersect B
- Those elements in A and B
- Those elements in common to both A and B

$\cap$ = 'and'

Disjoint Sets: Sets having no elements in common. $A \cap B = \varnothing$



Complement: The set consisting of elements in the universal set not in A.



$\overline{A} = \{x \mid x \text{ is not in } A\}$

Denoted $\overline{A}$ or A'

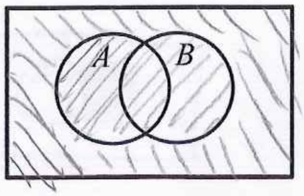
True for all sets:

$\varnothing \subseteq \text{every set}$	The null set is a subset of every set.
$A \subseteq A$	Every set is a subset of itself.
$A \subseteq U$	Every set is a subset of the universal set.
$\overline{A} \cup A = U$	The union of a set and its complement is the universal set.
$\overline{A} \cap A = \varnothing$	The intersection of a set and its complement is the null set.
$\overline{\overline{A}} = A$	A complement of a complement is the original set.

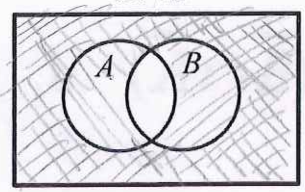
DeMorgan's Properties:

$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$

$\overline{A \cup B}$

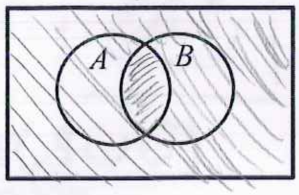


$\overline{A} \cap \overline{B}$

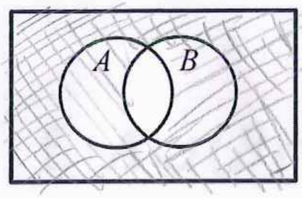


$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$

$\overline{A \cap B}$



$\overline{A} \cup \overline{B}$



True or False?

5. $\{6, 7, 9\} \subset \{1, 6, 9\}$ FALSE

6. $\{4, 8, 1\} \subset \{2, 4, 6, 8\}$ FALSE

7. $\{1, 2\} \cap \{2, 3, 4\} = \{2\}$

$\{2\} = \{2\}$ TRUE

Write each expression as a single set.

13. $\{1, 2, 3\} \cup \{2, 3, 4, 5\} = \{1,2,3,4,5\}$

14. $\{0, 1, 2\} \cap \{2, 3, 4\} = \{2\}$

20. If U = Universal Set = $\{1, 2, 3, 4, 5\}$ and if $A = \{3, 5\}$, $B = \{1, 2, 3\}$, and $C = \{2, 3, 4\}$, find:

a. $\overline{A} \cap \overline{C}$

$\{1,2,4\} \cap \{1,5\} = \boxed{\{1\}}$

b. $(A \cup B) \cap C$

$\{1,2,3,5\} \cap \{2,3,4\} = \boxed{\{2,3\}}$

c. $A \cup (B \cap C)$

$\{3,5\} \cup \{2,3\} = \boxed{\{2,3,5\}}$

d. $(A \cup B) \cap (A \cup C)$

$\{1,2,3,5\} \cap \{2,3,4,5\} = \boxed{\{2,3,5\}}$

e. $\overline{A \cap C}$

$\overline{\{3\}} = \boxed{\{1,2,4,5\}}$

f. $\overline{A \cup B}$

$\overline{\{1,2,3,5\}} = \boxed{\{4\}}$

g. $\overline{A} \cap \overline{B}$

$\{1,2,4\} \cap \{4,5\}$

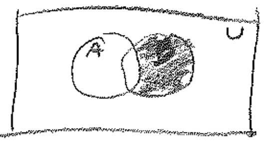
$= \boxed{\{4\}}$

h. $(A \cap B) \cup C$

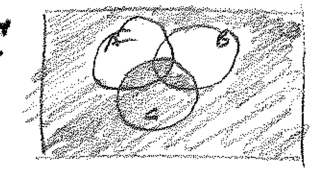
$\{3\} \cup \{2,3,4\} = \boxed{\{2,3,4\}}$

23. Use Venn diagrams to illustrate the following sets:

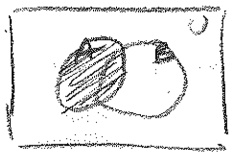
a. $\overline{A} \cap B$



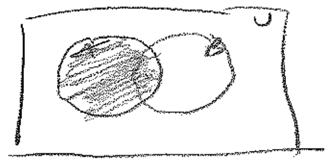
b. $(\overline{A} \cap \overline{B}) \cup C$



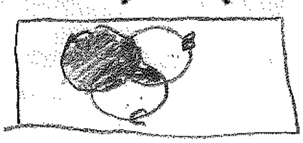
c. $A \cap (A \cup B)$



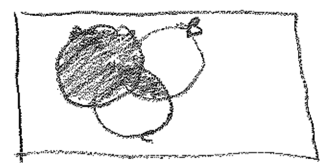
d. $A \cup (A \cap B)$



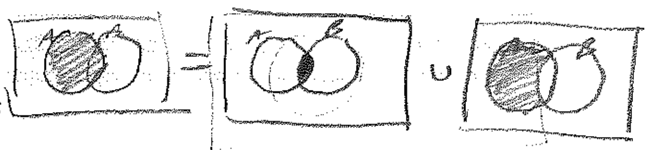
e. $(A \cup B) \cap (A \cup C)$



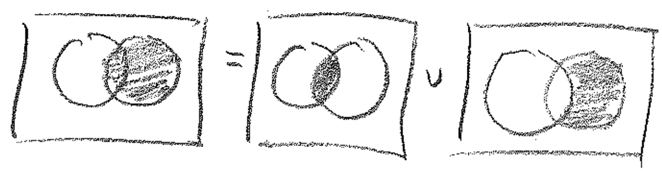
f. $A \cup (B \cap C)$



g. $A = (A \cap B) \cup (A \cap \overline{B})$



h. $B = (A \cap B) \cup (\overline{A} \cap B)$



If

$A =$ {x | x is a customer of IBM}
$B =$ {x | x is a secretary employed by IBM}
$C =$ {x | x is a computer operator at IBM}
$D =$ {x | x is a stockholder of IBM}
$E =$ {x | x is a member of the board of directors of IBM}

Describe in words:

25. $A \cap E$

set of people who are customers and also member of board

26. $B \cap D$

Secretaries of IBM who also hold stock

27. $A \cup D$

people either customers or stockholders.

28. $C \cap E$

computer operators who are also members of board.

29. $\overline{A} \cap D$

stockholders who are not customers

30. $A \cup \overline{D}$

customers who are not stockholders

7.2 – Number of Elements in a Set

Notation: $c(A)$

$A = \{1, 23, 43, 82, 123, 144, 156\}$ $\qquad c(A) = 7$

$c(\varnothing) = 0$

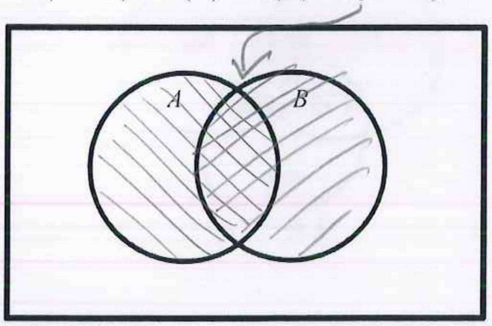
Finite set: The number of elements is zero or a positive integer.

Infinite set: Not a finite number of elements.

Counting Formula

Let A and B be two finite sets, then

$$c(A \cup B) = c(A) + c(B) - c(A \cap B)$$



Two methods of solving counting problems

1) Use the Counting Formula
Write out the formula, substitute what you are given and solve for what is missing.

2) Use a Venn Diagram
Draw a generic Venn diagram with circles for each category. Fill in counts beginning with most overlapped region, working outward.

8. Find $c(A \cup B)$, given that
$c(A) = 14$, $c(B) = 11$, and $c(A \cap B) = 6$.

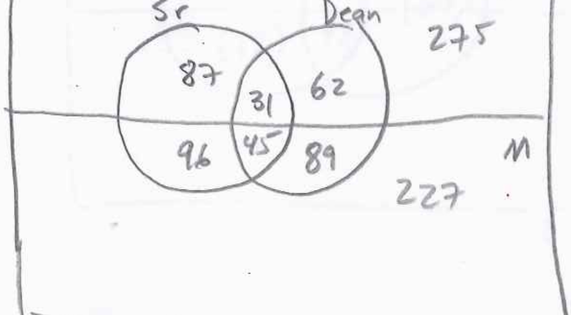
$c(A \cup B) = c(A) + c(B) - c(A \cap B)$

$c(A \cup B) = 14 + 11 - 6$

$c(A \cup B) = \boxed{19}$

27. At a small Midwestern college:
31 female seniors were on the dean's list
62 women were on the dean's list who were not seniors
45 male seniors were on the dean's list
87 female seniors were not on the dean's list
96 male seniors were not on the dean's list
275 women were not seniors and were not on the dean's list
89 men were on the dean's list who were not seniors
227 men were not seniors and were not on the dean's list

a) How many were seniors?
b) How many were women?
c) How many were on the dean's list?
d) How many were seniors on the dean's list?
e) How many were female seniors?
f) How many were women on the dean's list?
g) How many were students at the college?



a) $(87+31)+(96+45) = \boxed{259}$

b) $87+31+62+275 = \boxed{455}$

c) $(31+62)+(45+89) = \boxed{227}$

d) $31+45 = \boxed{76}$

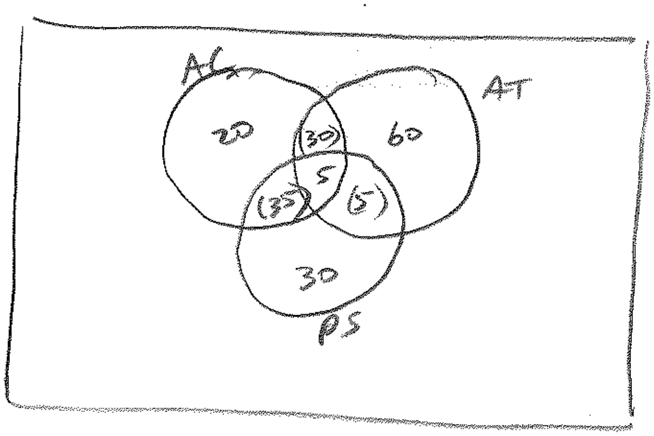
e) $87+31 = \boxed{118}$

f) $31+62 = \boxed{93}$

g) add all $= \boxed{912}$

29. **Car Sales** Of the cars sold during the month of July, 90 had air conditioning, 100 had automatic transmissions, and 75 had power steering. Five cars had all three of these extras. Twenty cars had none of these extras. Twenty cars had only air conditioning; 60 cars had only automatic transmissions; and 30 cars had only power steering. Ten cars had both automatic transmission and power steering.

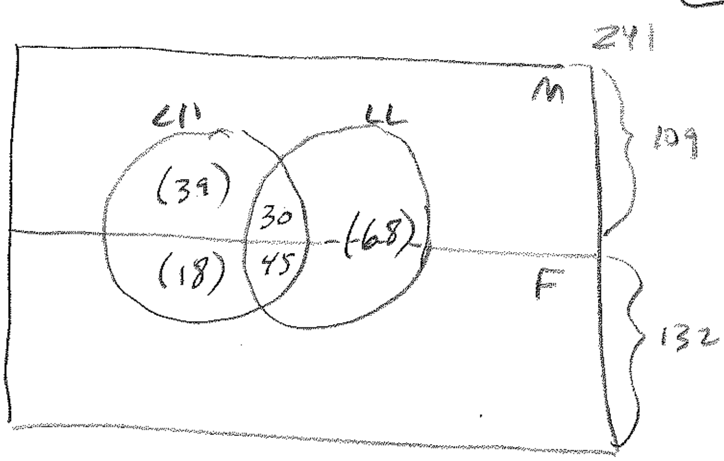
a) How many cars had both power steering and air conditioning? $\boxed{40}$
b) How many had both automatic transmission and air conditioning? $\boxed{35}$
c) How many had neither power steering ~~nor au~~tomatic transmission? $\boxed{40}$
d) How many cars were sold in July? $\boxed{205}$
e) How many had automatic transmission or air conditioning or both? $\boxed{155}$



32. **Survey Analysis** A survey of 52 families from a suburb of Chicago indicated that there was a total of 241 children below the age of 18. Of these, 109 were male; 132 were below the age of 11; 143 played Little League; 69 males were below the age of 11; 45 females under 11 had played Little League; and 30 males under 11 had played Little League.

(a) How many children over 11 and under 18 had played Little League. $\boxed{68}$

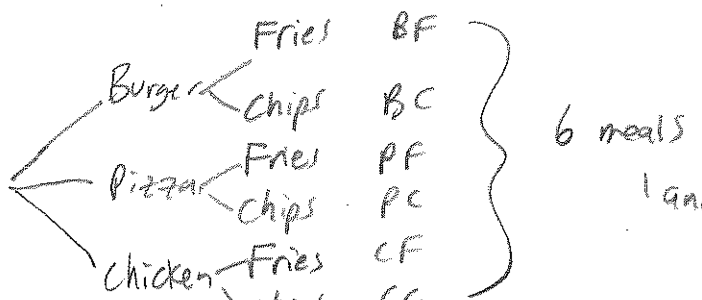
(b) How many females under 11 did not play Little League? $\boxed{18}$



7.3 – The Multiplication Principle

Draw a tree diagram representing this situation, and solve:

A snack bar offers a choice of a burger, pizza, or chicken as a main dish, with a side choice of fries or chips. How many different meals can be ordered?



'and' = multiply

number of ways to choose: $\underset{\text{main dish}}{3} \cdot \underset{\text{side}}{2} = \boxed{6 \text{ meals}}$

The multiplication principle:

If we can perform a first task in p different ways, a second task in q different ways, a third task in r different ways,..., then the total act of performing the first, then second, then third task can be done in p*q*r...different ways.

2. A woman has 4 blouses and 5 skirts. How many different outfits can she wear?

$4 \cdot 5 = \boxed{20}$

8. A woman has 4 pairs of gloves. In how many ways can she select a right-hand glove and a left-hand glove that do not match?

$\underset{\text{left}}{4} \cdot \underset{\text{right}}{3} = 12$

16. Find the number of 7-digit telephone numbers:
    a. With no repeated digits (lead 0 is allowed)
    b. With no repeated digits (lead 0 not allowed)
    c. With repeated digits allowed including a lead 0

a) $10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 = \boxed{604{,}800}$

b) $9 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 = \boxed{544{,}320}$

c) $10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = \boxed{10{,}000{,}000}$

32. How many different 3-letter code words are possible using the first 10 letters of the alphabet if
    a. No letter can be repeated?
    b. Letters can be repeated?
    c. Adjacent letters cannot be the same?

a) $10 \cdot 9 \cdot 8 = \boxed{720}$

b) $10 \cdot 10 \cdot 10 = \boxed{1000}$

c) $10 \cdot 9 \cdot 9 = \boxed{810}$

22. A system has 7 switches each of which may be opened or closed. The state of the system described by indicating for each switch whether it is open or closed. How many different states of the system are there?

$2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$

$2^7 = \boxed{128}$

8.4 – Permutations

Factorial:

$$n! = n(n-1)(n-2)....(3)(2)(1)$$

$$0! = 1$$

example: $5! = 5 \bullet 4 \bullet 3 \bullet 2 \bullet 1$

$5! = 120$

on calculator: 5, MATH/PRB, !

why does $0! = 1$?

$$n! = n \cdot (n-1)! \quad for\ n = 1:$$

$$1! = 1 \cdot (1-1)!$$

$$1! = 1 \cdot (0)!$$

$$1 = 0!$$

How many different ways can you select and line up 3 books from a pile of 9 books?

$$\underline{9} \cdot \underline{8} \cdot \underline{7} = \frac{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = \frac{9!}{6!} = \frac{9!}{(9-3)!}$$

Permutations

Permutation = An ordered arrangement of r objects chosen from n objects in which:

- The n objects are all different.
- No object is repeated in an arrangement.
- Order is important.

$${}_nP_r = P(n,r) = \frac{n!}{(n-r)!}$$

1. Find the number of ways of choosing five people from a group of 10 and arranging them in a line.

$$P(10,5) = {}_{10}P_5 = \frac{10!}{(10-5)!} = \frac{10!}{5!} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot \cancel{5!}}{\cancel{5!}}$$

$$= 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6$$

$$= \boxed{30{,}240}$$

2. Find the number of six-letter "words" that can be formed with no letter repeated.

$$P(26,6) = {}_{26}P_6 = \frac{26!}{(26-6)!} = \frac{26!}{20!}$$

$$= \frac{26 \cdot 25 \cdot 24 \cdot 23 \cdot 22 \cdot 21 \cdot \cancel{20!}}{\cancel{20!}}$$

$$= \boxed{165{,}765{,}600}$$

3. Find the number of seven-digit telephone numbers, with no repeated digit (allow 0 for a first digit).

$$P(10,7) = {}_{10}P_7 = \frac{10!}{(10-7)!} = 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4$$

$$= \boxed{604{,}800}$$

4. Find the number of ways of arranging eight people in a line.

$$P(8,8) = \frac{8!}{(8-8)!} = \frac{8!}{0!} = \frac{8!}{1} = 8!$$

$$= \boxed{40{,}320}$$

The number of permutations (arrangements) of n different objects using all n of them $P(n, n) = n!$

26. There are 5 different French books and 5 different Spanish books. How many ways are there to arrange them on a shelf if
    a. Books of the same language must be grouped together, French on the left, Spanish on the right?
    b. French and Spanish books must alternate in the grouping, beginning with a French book?

FFFFF SSSSS

$P(5,5) \cdot P(5,5)$

$5! \cdot 5!$

$120 \cdot 120$

$\boxed{14{,}400}$

F S F S F S F S F S

$\underline{5}\ \underline{5}\ \underline{4}\ \underline{4}\ \underline{3}\ \underline{3}\ \underline{2}\ \underline{2}\ \underline{1}\ \underline{1}$

$\boxed{14{,}400}$

10. $\dfrac{9!}{3!6!}$

$\dfrac{9 \cdot 8 \cdot 7 \cdot \cancel{6!}}{3 \cdot 2 \cdot \cancel{6!}}$

$\dfrac{3 \cdot \cancel{3} \cdot \cancel{2} \cdot 4 \cdot 7}{\cancel{3} \cdot \cancel{2}}$

$\boxed{84}$

11. $P(7, 2)$

$\dfrac{7!}{(7-2)!}$

$\dfrac{7!}{5!}$

$7 \cdot 6$

$\boxed{42}$

16. $P(6, 4)$

$\dfrac{6!}{(6-4)!}$

$\dfrac{6!}{2!}$

$6 \cdot 5 \cdot 4 \cdot 3$

$\boxed{360}$

34. How many ways are there to seat 4 people in a 6-passenger automobile?

of 6 seats, choose the 4 to be occupied

$\underset{\text{person 1}}{\underline{6}} \cdot \underset{2}{\underline{5}} \cdot \underset{3}{\underline{4}} \cdot \underset{4}{\underline{3}} = P(6,4)$

$\boxed{360}$

8.5 – Combinations

Combinations vs. Permutations

On a sports team with 10 players, you need to choose a team captain, a co-captain and an equipment manager. Each person has a different job.

On a sports team with 10 players, you need to choose 3 players to make a 'leadership team' that work together to share all jobs to lead the team.

Order matters - 'Permutation'

$\underset{C}{\underline{10}} \cdot \underset{CC}{\underline{9}} \cdot \underset{EM}{\underline{8}} = \boxed{720}$

$P(10,3) = {}_{10}P_3 = \frac{10!}{(10-3)!} = \frac{10!}{7!}$

$= \frac{10 \cdot 9 \cdot 8 \cdot \cancel{7!}}{\cancel{7!}}$

$= 10 \cdot 9 \cdot 8 = 720$

$P(n,r) = {}_nP_r = \frac{n!}{(n-r)!}$

Order does not matter - 'Combination'

$\frac{\text{ways to pick players}}{\text{ways those players can be arranged}} \quad \frac{10 \cdot 9 \cdot 8}{3 \cdot 2 \cdot 1} = \frac{720}{6}$

$= \boxed{120}$

Jill	Bob	Jane
Jill	Jane	Bob
Bob	Jill	Jane
Bob	Jane	Jill
Jane	Bob	Jill
Jane	Jill	Bob

$C(10,3) = {}_{10}C_3 = \frac{10!}{(10-3)!\,3!} = \frac{10!}{7!\,3!} = \frac{10 \cdot 9 \cdot 8 \cdot 7!}{3 \cdot 2 \cdot 1\; 7!} = 120$

$C(n,r) = {}_nC_r = \frac{n!}{(n-r)!\,r!}$

You could use 'boxes' to solve, but for Combinations (order doesn't matter) you would need to divide by boxes for how many ways those you choose can be rearranged.

Combinations

Combination = An unordered selection of r objects chosen from n objects in which:

- The n objects are all different.
- No object is repeated in an arrangement.
- **Order is not important.**

$${}_nC_r = C(n,r) = \binom{n}{r} = \frac{n!}{(n-r)!\,r!}$$

Examples of Combinations

1. Find the number of ways of selecting four people from a group of six.
2. Find the number of committees of six that can be formed from the U.S. Senate (100 members).
3. Find the number of ways of selecting five courses from a catalog containing 200 courses.

32. **Investment Selection** An investor is going to invest $21,000 in 3 stocks from a list of 12 prepared by hisbroker. How many different investments are possible if:

a. $7000 is to be invested in each stock?
b. $10,000 is to be invested in one stock, $6000 in another, and $5000 in the third?
c. $8000 is to be invested in each of 2 stocks and $5000 in a third stock?

(a) one choice

${}_{12}C_3 = \frac{12!}{(12-3)!\,3!}$

$= \frac{12!}{9!\,3!}$

$= \boxed{220}$

(b) three choice, 1 each

${}_{12}C_1 \cdot {}_{11}C_1 \cdot {}_{10}C_1$

$\frac{12!}{(12-1)!\,1!} \cdot \frac{11!}{(11-1)!\,1!} \cdot \frac{10!}{(10-1)!\,1!}$

$\frac{12!}{11!} \cdot \frac{11!}{10!} \cdot \frac{10!}{9!}$

$\frac{12!}{9!}$

$\boxed{1320}$

(c) 2 choices

${}_{12}C_2 \cdot {}_{10}C_1$

$\frac{12!}{(12-2)!\,2!} \cdot \frac{10!}{(10-1)!\,1!}$

$\frac{12!}{10!\,2!} \cdot \frac{10!}{9!}$

$\frac{12!}{2!\,9!}$

$\boxed{660}$

30. **Congressional Committees** In the U.S. Congress a conference committee is to be composed of 5 senators and 4 representatives. In how many ways can this be done? (There are 435 representatives and 100 senators).

$\binom{435}{4} \cdot \binom{100}{5}$

$\frac{435!}{(435-4)!4!} \cdot \frac{100!}{(100-5)!5!}$

$\frac{435!}{431!4!} \cdot \frac{100!}{95!5!}$

$\frac{435 \cdot 434 \cdot 433 \cdot 432 \cdot \cancel{431!}}{4! \cdot \cancel{431!}} \cdot \frac{100 \cdot 99 \cdot 98 \cdot 97 \cdot 96 \cdot \cancel{95!}}{5! \cdot \cancel{95!}} \approx \boxed{1.1 \cdot 10^{17}}$

24. How many 5-card poker hands contain all spades? (A deck of 52 cards contains 13 spades). $C(13,5) = \boxed{1287}$

18. **Stock Trading** Of 1520 stocks traded in 1 day on the New York Stock Exchange, 841 advanced, 434 declined, and the remainder were unchanged. In how many ways can this happen? $1520 - 841 = 679$

$C(1520, 841) \cdot C(679, 434)$

$\frac{1520!}{\cancel{679!}\,841!} \cdot \frac{\cancel{679!}}{245!\,434!} = \boxed{\frac{1520!}{841!\,245!\,434!}}$ too large to evaluate in a calculator

(14 significant figure, floating point, base 10)

16. **Basketball Teams** On a basketball team of 12 players, 2 play only center, 3 play only guard, and the rest play forward (7) (5 players on a team: 2 forwards, 2 guards, and 1 center). How many different starting lineups are possible, assuming it is not possible to distinguish left and right guards and left and right forwards?

forward $\binom{7}{2}$ guard $\binom{3}{2}$ center $\binom{2}{1}$

$21 \cdot 3 \cdot 2 = \boxed{126}$

Permutation or Combination?

1. Find the number of five-card unordered poker hands. C
2. Find the number of six-letter "words" that can be formed with no letter repeated. P
3. Find the number of committees of six that can be formed from the U.S. Senate (100 members). C
4. Find the number of ways of selecting four people from a group of six. C
5. Find the number of seven-digit telephone numbers, with no repeated digit (allow 0 for a first digit). P
6. Find the number of ways of choosing five people from a group of 10 and arranging them in a line. P
7. Find the number of ways of arranging eight people in a line. P
8. Find the number of ways of selecting five courses from a catalog containing 200 courses. C

8.5 day 2 – Counting problems useful for probability calculations

Toss a coin six times.

a. How many different outcomes are possible?

$\underline{2} \cdot \underline{2} \cdot \underline{2} \cdot \underline{2} \cdot \underline{2} \cdot \underline{2} = 2^6 = 64$

b. How many different outcomes have exactly 3 heads?

HHHTTT
HTTHHT
TTHTHH
⋮

we can choose which spots the heads will occupy

${}_6C_3 = \boxed{20}$

c. How many different outcomes have 4 heads or 5 heads?

4 heads 'or' 5 heads

${}_6C_4 + {}_6C_5$

$15 + 6 = \boxed{21}$

d. How many different outcomes have at least 2 heads? 2 ways to calculate.

2 heads OR 3 heads OR 4 heads OR 5 heads or 6 heads

${}_6C_2 + {}_6C_3 + {}_6C_4 + {}_6C_5 + {}_6C_6$

$15 + 20 + 15 + 6 + 1$

57

opposite (complement) of at least 2 heads is 0 heads or 1 head

${}_6C_0 + {}_6C_1$

$1 + 6$

7

So ways for at least 2 heads
= (total outcomes) − (# ways for 0 or 1 head)
= 64 − 7
= 57

$c(A) = c(U) - c(\bar{A})$

2 pants, 3 shirts
how many outfits?

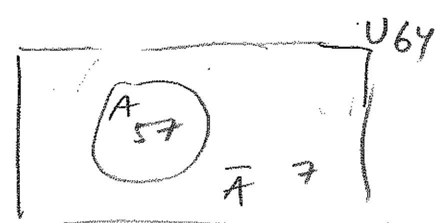
$\underset{\text{pants}}{\underline{2}}$ 'and' $\underset{\text{shirt}}{\underline{3}}$

both occur, so each first choice occurs with second

how many ways to roll
4 heads or 5 heads?

$\underset{\text{4 heads}}{\underline{15}}$ 'or' $\underset{\text{5 heads}}{\underline{6}}$

each 4 head case doesn't occur with a 5 head case.



$\boxed{\begin{array}{l}\text{'and' = multiply} \\ \text{'or' = add}\end{array}}$

Problems calculating 'at least' an amount are often most easily solved by finding the count of the 'opposite' (complement) and subtracting from the total number of outcomes.

Distinguishable permutations:

How many 'words' can be formed using the letters in MAMMAL?

__ __ __ __ __ __

- Choose positions for the Ms $C(6,3)$
- Choose positions for the As $C(3,2)$ (3 spots left)
- Choose positions for the L $C(1,1)$ (1 spot left)

$C(6,3)\cdot C(3,2)\cdot C(1,1)$

$\frac{6!}{3!3!}\cdot\frac{\cancel{3!}}{\cancel{1!}2!}\cdot\frac{\cancel{1!}}{0!1!}$ ← always 0!

$\frac{6!}{3!2!1!}$ — (in general)

$$\#\,\textit{distinguishable permutations} = \frac{n!}{n_1!n_2!n_3!\ldots}$$

4. Urn contains 15 red balls and 10 white balls. Five balls are selected. In how many ways can the 5 balls be drawn from the total of 25 balls

   a. If all the balls are red?

   5 red 'and' 0 white
   $C(15,5)\cdot C(10,0)$
   $3003\cdot 1 = \boxed{3003}$

   b. If 3 balls are red and 2 are white?

   3 red 'and' 2 white
   $C(15,3)\cdot C(10,2)$
   $455\cdot 45 = \boxed{20475}$

   c. If at least 4 are red balls?

   (4 red 'and' 1 white) 'or' (5 red 'and' 0 white)
   $C(15,4)\cdot C(10,1) + C(15,5)\cdot C(10,0)$
   $1365\cdot 10 + 3003\cdot 1$
   $\boxed{16653}$

6. How many different ways can 3 red, 4 yellow, and 5 blue bulbs be arranged in a string of Christmas tree lights with 12 sockets?

   2 methods

choosing which spots to place colors:
$C(12,3)\cdot C(9,4)\cdot C(5,5)$
$220\cdot 126\cdot 1$
$\boxed{27720}$

distinguishable permutations
RRRYYYYBBBBB
$\frac{12!}{3!4!5!} = \boxed{27720}$

18. Eight couples (husband and wife) are present at a meeting where a committee of 3 is to be chosen. How many ways can this be done so that the committee

    a. contains a couple? $\underbrace{C(8,1)}_{\text{ways to choose the couple}}\cdot\underbrace{C(14,1)}_{\text{ways to choose 3rd person}} = 8\cdot 14 = \boxed{112}$

    b. contains no couple? ways for no couple = total ways − way for containing a couple
    $= C(16,3) - 112$
    $= 560 - 112$
    $= \boxed{448}$

A.5 – Binomial Theorem, Subsets of a set

$(x+y)^0 = 1$

$(x+y)^1 = x + y$

$(x+y)^2 = x^2 + 2xy + y^2$

$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$

$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$

$(x+y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$

Find all patterns

What would the expansion be for $(x+y)^6$?

$x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6$

Pascal's triangle:

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

$$\binom{n}{r} = \frac{n!}{(n-r)!r!}$$

The Binomial Theorem:

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \ldots + \binom{n}{r}x^{n-r}y^r + \ldots + \binom{n}{n}y^n$$

where $\binom{n}{r} = \frac{n!}{(n-r)!r!}$ or a row from Pascal's triangle $= {}_nC_r$

'x' and 'y' can be more complex...

$$(2x-3)^4 = \binom{4}{0}(2x)^4(-3)^0 + \binom{4}{1}(2x)^3(-3)^1 + \binom{4}{2}(2x)^2(-3)^2 + \binom{4}{3}(2x)^1(-3)^3 + \binom{4}{4}(2x)^0(-3)^4$$

$$= (1)(16x^4)(1) + (4)(8x^3)(-3) + (6)(4x^2)(9) + (4)(2x)(-27) + (4)(1)(81)$$

$$= \boxed{16x^4 - 96x + 216x^2 - 216x + 81}$$

A. Use the binomial theorem to expand the expression $(2x - y)^4$

$\binom{4}{0}(2x)^4(-y)^0+\binom{4}{1}(2x)^3(-y)^1+\binom{4}{2}(2x)^2(-y)^2+\binom{4}{3}(2x)^1(-y)^3+\binom{4}{4}(2x)^0(-y)^4$

$(1)(16x^4)(1)+(4)(8x^3)(-y)+(6)(4x^2)(y^2)+(4)(2x)(-y^3)+(1)(1)(y^4)$

$16x^4-32x^3y+24x^2y^2-8xy^3+y^4$

B. Expand $(4x-1)^5$

$\binom{5}{0}(4x)^5(-1)^0+\binom{5}{1}(4x)^4(-1)^1+\binom{5}{2}(4x)^3(-1)^2+\binom{5}{3}(4x)^2(-1)^3+\binom{5}{4}(4x)^1(-1)^4+\binom{5}{5}(4x)^0(-1)^5$

$1(1024x^5)(1)+5(256x^4)(-1)+(10)(64x^3)(1)+(10)(16x^2(-1))+(5)(4x)(1)+(1)(1)(-1)$

$1024x^5-1280x^4+640x^3-160x^2+20x-1$

C. Find the coefficient of $x^{12}y^3$ in the expansion of $(4x-5y)^{15}$.

$\binom{15}{0}(4x)^{15}(-5y)^0+\cdots\cdots\binom{15}{3}(4x)^{12}(-5y)^3$

$(455)(16777216x^{12})(-125y^3)=\boxed{-9,542,10^{11}}x^{12}y^3$

D. Find the coefficient of $x^{10}y^8$ in the expansion of $(-3x^2+2y^4)^7$.

$\binom{7}{0}(-3x^2)^7(2y^4)^0+\binom{7}{1}(-3x^2)^6(2y^4)^1+\binom{7}{2}(-3x^2)^5(2y^4)^2$

x^{14} $\quad$ 10^{12} $\quad$ $(21)(-243x^{10})(4y^8)=\boxed{-20412}x^{10}y^8$

Given $\{A, B, C, D\}$, how many different subsets can be chosen?

∅	A, B, C, D	AB, AC, AD, BC, BD, CD	ABC, ABD, ACD, BCD	ABCD	
1	4	6	4	1	= 16
$\binom{4}{0}$	$\binom{4}{1}$	$\binom{4}{2}$	$\binom{4}{3}$	$\binom{4}{4}$	2^4

Interesting facts about subsets of sets:

For a set with n elements:

$\binom{n}{0}$ = number of subsets with 0 elements.

$\binom{n}{1}$ = number of subsets with 1 element.

2^n = total number of subsets.

$\binom{n}{2}$ = number of subsets with 2 elements.

$\binom{n}{m}$ = number of subsets with m elements.