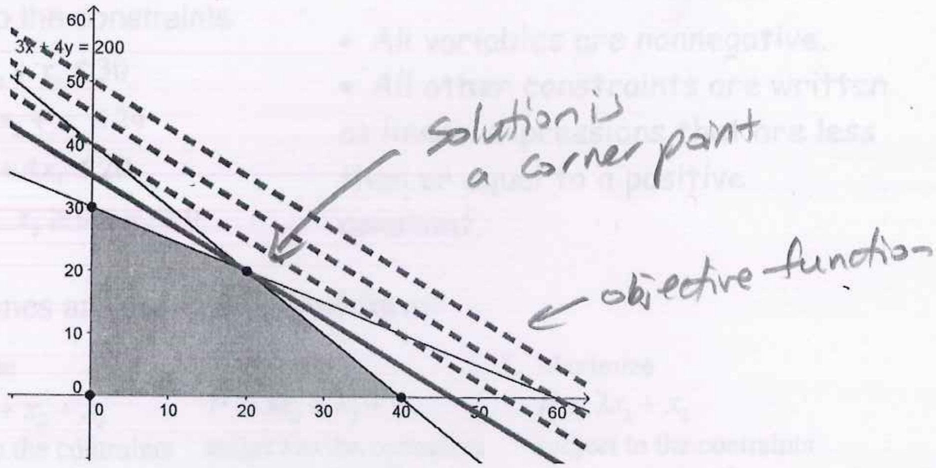
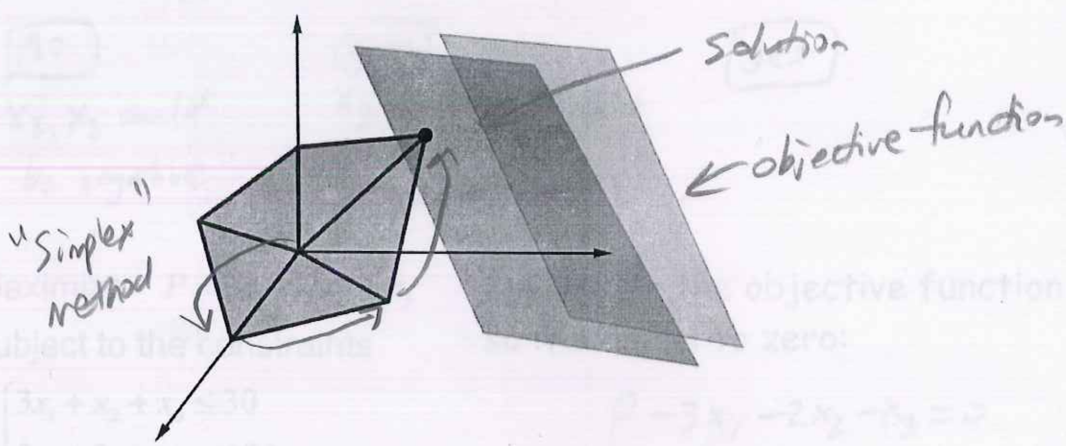
Honors Finite Mathematics – Lesson Notes: Unit 4 (Ch5) The Simplex Method

5.1 – Simplex Tableau, Pivoting

For 2D systems of inequalities, it is possible to graph the system, and the solution (if it exists) can be found manually by evaluating the objective function at corner points:



For higher dimensions with linear inequalities, the equations represent the equivalent of planes, and the feasible region is a multi-dimensions volume, but the solutions will still occur at corner points:



But it is difficult or impossible to graph systems of inequalities in higher dimensions. We need a different way to solve.

One thing we can do is solve systems of <u>equations</u> in higher dimensions. So we convert systems of inequalities into systems of equations in a specific way. The result is something called a **Simplex Tableau**.

5.1 - Converting systems of inequalities to Simplex Tableau, Pivot operation.
5.2 - Using the procedure to solve some maximizing problems.
5.4 - Extending the procedure to solve all maximizing or minimizing problems.

The idea: if all variables are positive, and all inequalities are <, then you can convert to equations by introducing a 'slack variable' to take up the slack:

Maximize $P = 3x_1 + 2x_2 + x_3$

subject to the constraints

$$\begin{cases} 3x_1 + x_2 + x_3 \le 30 \\ 5x_1 + 2x_2 + x_3 \le 24 \\ x_1 + x_2 + 4x_3 \le 20 \\ x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 \ge 0 \end{cases}$$

This problem is in **standard form:**

- All variables are nonnegative.
- All other constraints are written as linear expressions that are less than or equal to a positive constant.

Which ones are not in standard form?

3. Maximize
$P = 3x_1 + x_2 + x_3$
subject to the contraints
$x_1 + x_2 + x_3 \le 6$
$2x_1 + 3x_2 + 4x_3 \le 10$
$x_1 \ge 0$

no

x_2, x_3 could be negative

5. Maximize
$P = 3x_1 + x_2 + x_3$
subject to the contraints
$x_1 + x_2 + x_3 \le 8$
$2x_1 + x_2 + 4x_3 \ge 6$
$x_1 \ge 0 \quad x_2 \ge 0$

no

x_3 could be negative and 2^{nd} constraint $\ge$

7. Maximize
$P = 2x_1 + x_2$
subject to the contraints
$x_1 + x_2 \ge -6$
$2x_1 + x_2 \le 4$
$x_1 \ge 0 \quad x_2 \ge 0$

yes

Maximize $P = 3x_1 + 2x_2 + x_3$

subject to the constraints

$$\begin{cases} 3x_1 + x_2 + x_3 \le 30 \\ 5x_1 + 2x_2 + x_3 \le 24 \\ x_1 + x_2 + 4x_3 \le 20 \\ x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 \ge 0 \end{cases}$$

1) Rewrite the objective function so it is equal to zero:

$P - 3x_1 - 2x_2 - x_3 = 0$

2) Add slack variables for each constraint and convert to equations:

$$\begin{aligned} 3x_1 + x_2 + x_3 + s_1 &= 30 \\ 5x_1 + 2x_2 + x_3 + s_2 &= 24 \\ x_1 + x_2 + 4x_3 + s_3 &= 20 \\ P - 3x_1 - 2x_2 - x_3 &= 0 \end{aligned}$$

3) Write in Simplex Tableau format:

$$\begin{array}{ccccccc|c} P & x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & RHS \end{array}$$

$$\left[\begin{array}{ccccccc|c} 0 & 3 & 1 & 1 & 1 & 0 & 0 & 30 \\ 0 & 5 & 2 & 1 & 0 & 1 & 0 & 24 \\ 0 & 1 & 1 & 4 & 0 & 0 & 1 & 20 \\ \hline 1 & -3 & -2 & -1 & 0 & 0 & 0 & 0 \end{array}\right]$$

In the next section, we will learn the procedure for how to use the Simplex Method to find the maximum value of P at the corner points of a tableau. This method uses a procedure called **pivoting**.

Steps for pivoting:

1) In the pivot row, divide each entry by the pivot element (for now, the pivot element will be given to us).

2) Obtain zeros elsewhere in the pivot column by performing row operations using the pivot row.

Perform a pivot operation for the indicated pivot element

$$\begin{array}{c c} & \begin{array}{ccccccc|c} P & x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & RHS \end{array} \\ \begin{array}{c} \\ \\ \frac{1}{3}r_3 \\ \\ \end{array} & \left[\begin{array}{ccccccc|c} 0 & 1 & 2 & 1 & 1 & 0 & 0 & 6 \\ 0 & 2 & 3 & 1 & 0 & 1 & 0 & 12 \\ 0 & 1/3 & -2/3 & 3 & 0 & 0 & 1/3 & 0 \\ \hline 1 & -1 & -2 & -3 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

Current Values?

$s_1=6$, $s_2=12$, $s_3=0$, $P=0$

$x_1=0$, $x_2=0$, $x_3=0$

$$\begin{array}{c c} & \begin{array}{ccccccc|c} P & x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & RHS \end{array} \\ \begin{array}{c} -r_3+r_1 \\ -r_3+r_2 \\ \\ \\ \end{array} & \left[\begin{array}{ccccccc|c} 0 & 2/3 & 8/3 & 0 & 1 & 0 & -1/3 & 6 \\ 0 & 5/3 & 11/3 & 0 & 0 & 1 & -1/3 & 12 \\ 0 & 1/3 & -2/3 & 1 & 0 & 0 & 1/3 & 0 \\ \hline 1 & 0 & -4 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

Current Values after each pivot??

$s_1=6$, $s_2=12$, $x_3=0$, $P=0$

$s_2=0$, $x_1=0$, $x_2=0$

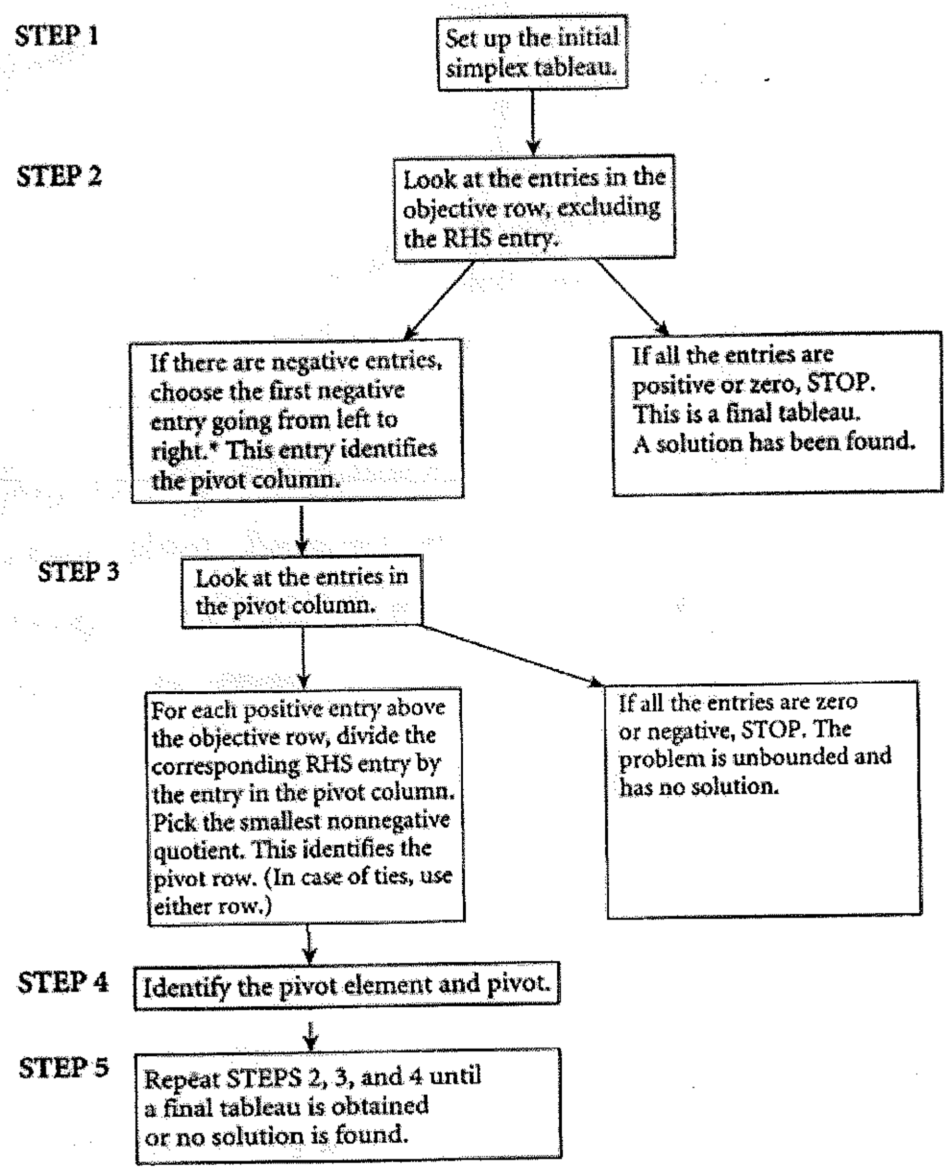
Load the pivot program for your calculator...

.2 day 1 – *Simplex Method for solving maximization problems in standard form*

The Simplex Method is an iterative procedure that selects specific pivot elements and performs pivot operations on a simplex tableau.

It works for maximization problems in standard form, and provides the maximum value for P (the objective function) along with the values of the nonbasic variables that produce that maximum value.

The Simplex Method for Solving a Maximum Problem in Standard Form



Maximize

$$P = 4x_1 + 2x_2 + 5x_3$$

$$x_1 + 3x_2 + 2x_3 \le 30$$

$$2x_1 + x_2 + 3x_3 \le 12$$

$$x_1 \ge 0 \quad x_2 \ge 0 \quad x_3 \ge 0$$

$$P - 4x_1 - 2x_2 - 5x_3 = 0$$

$$\begin{array}{c} \begin{array}{ccccccc} P & x_1 & x_2 & x_3 & s_1 & s_2 & RHS \end{array} \\ \left[\begin{array}{cccccc|c} 0 & 1 & 3 & 2 & 1 & 0 & 30 \\ 0 & 2 & 1 & 3 & 0 & 1 & 12 \\ \hline 1 & -4 & -2 & -5 & 0 & 0 & 0 \end{array}\right] \end{array} \quad \begin{array}{l} 30/1 = 30 \\ 12/2 = 6 \end{array}$$

$$\begin{array}{c} \begin{array}{ccccccc} P & x_1 & x_2 & x_3 & s_1 & s_2 & RHS \end{array} \\ \left[\begin{array}{cccccc|c} 0 & 0 & 5/2 & 1/2 & 1 & -1/2 & 24 \\ 0 & 1 & 1/2 & 3/2 & 0 & 1/2 & 6 \\ \hline 1 & 0 & 0 & 1 & 0 & 2 & 24 \end{array}\right] \end{array}$$

no negatives – done

equations:

$P + x_3 + 2s_2 = 24$

so $P = -x_3 - 2s_2 + 24$

but we want maximum P, so we want x_3 and s_2 to be zero.

this is why all columns which don't have all zeros except a single 1 are set to zero.

this column tells us $P = 24$

this column tells us $x_1 = 6$

this column tells us $s_1 = 24$

all other variables are zero

so

$$\boxed{\begin{array}{l} P_{max} = 24 \\ \text{when } x_1 = 6 \\ \quad\quad\; x_2 = 0 \\ \quad\quad\; x_3 = 0 \end{array}}$$

and $\left(\begin{array}{l} s_1 = 24 \\ s_2 = 0 \end{array} \right)$

Maximize

$$P = 2x_1 + 4x_2 + x_3$$

subject to

$$-x_1 + 2x_2 + 3x_3 \le 6$$

$$-x_1 + 4x_2 + 5x_3 \le 5$$

$$-x_1 + 5x_2 + 7x_3 \le 7$$

$$x_1 \ge 0 \quad x_2 \ge 0 \quad x_3 \ge 0$$

$$P - 2x_1 - 4x_2 - x_3 = 0$$

$$\begin{array}{cccccccc} P & x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & RHS \end{array}$$

$$\left[\begin{array}{ccccccc|c} 0 & -1 & 2 & 3 & 1 & 0 & 0 & 6 \\ 0 & -1 & 4 & 5 & 0 & 1 & 0 & 5 \\ 0 & -1 & 5 & 7 & 0 & 0 & 1 & 7 \\ \hline 1 & -2 & -4 & -3 & 0 & 0 & 0 & 0 \end{array}\right]$$

$\uparrow$

no positive values in column, so $\boxed{\text{no solution}}$

(unbounded region)

5.2 day 2 – Simplex Method max, std form word problems

Suppose that a large hospital classifies its surgical operations into three categories according to their length and charges a fee of \$600, \$900, and \$1200, respectively, for each of the categories. The average time of the operations in the three categories is 30 minutes, 1 hour, and 2 hours, respectively; and the hospital has four operating rooms, each of which can be used for 10 hours per day. If the total number of operations cannot exceed 60, how many of each type should the hospital schedule to maximize its revenues?

Objective:
$P = 600x_1 + 900x_2 + 1200x_3$

type1 (x_1) type2 (x_2) type3 (x_3)

time: $\frac{1}{2}x_1 + x_2 + 2x_3 \le 40$

number: $x_1 + x_2 + x_3 \le 60$

$P - 600x_1 - 900x_2 - 1200x_3 = 0$

$\rightarrow$

$\frac{1}{2}x_1 + x_2 + 2x_3 + s_1 = 40$

$x_1 + x_2 + x_3 + s_2 = 60$

$P - 600x_1 - 900x_2 - 1200x_3 = 0$

$$\begin{array}{c}
\begin{array}{cccccc|c} P & x_1 & x_2 & x_3 & s_1 & s_2 & RHS \end{array} \\
\left[\begin{array}{cccccc|c}
0 & 1/2 & 1 & 2 & 1 & 0 & 40 \\
0 & 1 & 1 & 1 & 0 & 1 & 60 \\
\hline
1 & -600 & -900 & -1200 & 0 & 0 & 0
\end{array}\right]
\end{array}
\begin{array}{l} 40/\tfrac{1}{2} = 80 \\ 60/1 = 60 \end{array}$$

$\uparrow$

$$\left[\begin{array}{cccccc|c}
0 & 0 & 1/2 & 3/2 & 1 & -1/2 & 10 \\
0 & 1 & 1 & 1 & 0 & 1 & 60 \\
\hline
1 & 0 & -300 & -600 & 0 & 600 & 36000
\end{array}\right]
\begin{array}{l} 10/\tfrac{1}{2} = 20 \\ 60/1 = 60 \end{array}$$

$\uparrow$

$$\begin{array}{c}
\begin{array}{cccccc|c} P & x_1 & x_2 & x_3 & s_1 & s_2 & RHS \end{array} \\
\left[\begin{array}{cccccc|c}
0 & 0 & 1 & 3 & 2 & -1 & 20 \\
0 & 1 & 0 & -2 & -2 & 2 & 40 \\
\hline
1 & 0 & 0 & 300 & 600 & 300 & 42000
\end{array}\right]
\end{array}$$

done (no negatives)

$P_{max} = \$42000$
when $x_1 = 40$ type1 operations
$x_2 = 20$ type2 operations
$x_3 = 0$ type3 operations

A large TV manufacturer has warehouse facilities for storing its color TVs in Chicago, New York, and Denver. Each month the city of Atlanta is shipped at most four hundred TVs. The cost of transporting each TV to Atlanta from Chicago, New York, and Denver averages \$20, \$20, and \$40, respectively, while the cost of labor required for packing averages \$6, \$8, and \$4, respectively. Suppose \$10,000 is allocated each month for transportation costs and \$3000 is allocated for labor costs. If the profit on each TV made in Chicago is \$50, in New York is \$80, and in Denver is \$40, how should monthly shipping arrangements be scheduled to maximize profit?

warehouse

Chicago NY Denver

x_1 x_2 x_3

Atlanta (sale)

$P = 50x_1 + 80x_2 + 40x_3$

total: $x_1 + x_2 + x_3 \le 400$

transport: $20x_1 + 20x_2 + 40x_3 \le 10000$

labor: $6x_1 + 8x_2 + 4x_3 \le 3000$

$P - 50x_1 - 80x_2 - 40x_3 = 0$

$$\begin{array}{cccccccc} P & x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & RHS \end{array}$$

$$\left[\begin{array}{ccccccc|c} 0 & 1 & 1 & 1 & 1 & 0 & 0 & 400 \\ 0 & 20 & 20 & 40 & 0 & 1 & 0 & 10000 \\ 0 & 6 & 8 & 4 & 0 & 0 & 1 & 3000 \\ \hline 1 & -50 & -80 & -40 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} 400 \\ 10000/20 = 500 \\ 3000/6 = 500 \\ \\ \end{array}$$

$$\left[\begin{array}{ccccccc|c} 0 & 1 & 1 & 1 & 1 & 0 & 0 & 400 \\ 0 & 0 & 0 & 20 & -20 & 1 & 0 & 2000 \\ 0 & 0 & 2 & -2 & -6 & 0 & 1 & 600 \\ \hline 1 & 0 & -30 & 10 & 50 & 0 & 0 & 20000 \end{array}\right] \begin{array}{l} 400 \\ \\ 600/2 = 300 \\ \\ \end{array}$$

$$\left[\begin{array}{ccccccc|c} 0 & 1 & 0 & 2 & 4 & 0 & -1/2 & 100 \\ 0 & 0 & 0 & 20 & -20 & 1 & 0 & 2000 \\ 0 & 0 & 1 & -1 & -3 & 0 & 1/2 & 300 \\ \hline 1 & 0 & 0 & -20 & -40 & 0 & 15 & 29000 \end{array}\right] \begin{array}{l} 50 \\ 100 \\ \\ \\ \end{array}$$

$$\left[\begin{array}{ccccccc|c} 0 & 1/2 & 0 & 1 & 2 & 0 & -1/4 & 50 \\ 0 & -10 & 0 & 0 & -60 & 1 & 5 & 1000 \\ 0 & 1/2 & 1 & 0 & -1 & 0 & 1/4 & 350 \\ \hline 1 & 10 & 0 & 0 & 0 & 0 & 10 & 30000 \end{array}\right]$$

done

$P_{max} = \$30000$

when $x_1 = 0$ TVs from Chicago

$x_2 = 350$ TVs from NY

$x_3 = 50$ TVs from Denver

5.4 – Simplex Method for minimum problems, mixed constraints

max P   min C

cost

$P = -C$

0

The procedure we've been using works for finding maximum of a problem which can be written in standard form. What if we want to find a minimum instead?

A minimum problem can be changed to a maximum problem by recognizing that minimizing z is the same as maximizing -z.

Example: Minimize the cost function $C = 3x_1 - 2x_2 + x_3$
(subject to some constraints)

$P = -3x_1 + 2x_2 - x_3$
maximize P
$(C_{min} = -P_{max})$

In minimum problems we often have cases which can't be written in standard form (all nonnegative constraints $\leq$ positive numbers.) We need a more general procedure for handling mixed constraints. We start by converting all nonnegative constraint inequalities to $\leq$ even if that results in negative RHS values. Then we use an alternative method to select pivot elements.

The Simplex Method for Solving a Minimum Problem, or Problem with Mixed Constraints

STEP 1 Write each constraint, except the nonnegative constraints, as an inequality with the variables on the left side of a $\leq$ sign. (multiply by -1 if needed)

STEP 2 Introduce nonnegative slack variables on the left side of each inequality to form an equality.

STEP 3 Set up the initial simplex tableau.

Alternative Pivoting Strategy

STEP 4 Whenever negative entries occur in the right-hand column RHS of the constraint equations, the pivot element is selected as follows:

Pivot row: Identify the most negative entry in the RHS column and its corresponding basic variable (BV). (Ignore the objective row.) The basic variable BV chosen identifies the pivot row. Because the objective row is ignored, it can never be the pivot row.

Pivot column: Go from left to right along the pivot row until a negative entry is found. (Ignore the RHS.) This entry identifies the pivot column and is the pivot element. If there are no negative entries in the pivot row except the one in the RHS column, then the problem has no solution.*

STEP 5 Pivot.

1. If, in the new tableau, negative entries appear on the RHS of the constraint equations, repeat STEP 4.
2. If, in the new tableau, only nonnegative entries appear on the RHS of the constraint equations, then the tableau represents a maximum problem in standard form and the steps for a maximum problem in standard form.

Maximize $P = 5x + 4y + 2z$ $\rightarrow$ $P - 5x - 4y - 2z = 0$

subject to $x + 2y + 3z \leq 24$

$x - y + z \geq 6$ $\rightarrow$ $-x + y - z \leq -6$

$x \geq 0,\ y \geq 0,\ z \geq 0$

$$\begin{array}{ccccccc} P & x_1 & x_2 & x_3 & s_1 & s_2 & RHS \end{array}$$

$$\left[\begin{array}{cccccc|c} 0 & 1 & 2 & 3 & 1 & 0 & 24 \\ 0 & -1 & 1 & -1 & 0 & 1 & -6 \\ \hline 1 & -5 & -4 & -2 & 0 & 0 & 0 \end{array}\right] \leftarrow \text{most negative}$$

$$\left[\begin{array}{cccccc|c} 0 & 0 & 3 & 2 & 1 & 1 & 18 \\ 0 & 1 & -1 & 1 & 0 & -1 & 6 \\ \hline 1 & 0 & -9 & 3 & 0 & -5 & 30 \end{array}\right]$$

done RHS now

$\uparrow$

$$\left[\begin{array}{cccccc|c} 0 & 0 & 1 & 2/3 & 1/3 & 1/3 & 6 \\ 0 & 1 & 0 & 5/3 & 1/3 & -2/3 & 12 \\ \hline 1 & 0 & 0 & 9 & 3 & -2 & 84 \end{array}\right]$$

$\uparrow$

$$\begin{array}{ccccccc} P & x_1 & x_2 & x_3 & s_1 & s_2 & \end{array}$$

$$\left[\begin{array}{cccccc|c} 0 & 0 & 3 & 2 & 1 & 1 & 18 \\ 0 & 1 & 2 & 3 & 1 & 0 & 24 \\ \hline 1 & 0 & 6 & 13 & 5 & 0 & 120 \end{array}\right]$$

(done)

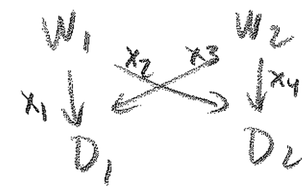
$P_{max} = 120$

when $x_1 = 24$

$x_2 = 0$

$x_3 = 0$

Sometimes, we encounter problems where some constraints are equations instead of inequalities. We can convert these into inequalities (the context of the problem determines how we do this).



A motorcycle manufacturer must fill orders from two dealers. The first dealer, D_1, has ordered 20 motorcycles, while the second dealer, D_2, has ordered 30 motorcycles. The manufacturer has the motorcycles stored in two warehouses, W_1 and W_2. There are 40 motorcycles in W_1 and 15 in W_2. The shipping costs per motorcycle are as follows: \$15 from W_1 to D_1; \$13 from W_1 to D_2; \$14 from W_2 to D_1; \$16 from W_2 to D_2. Under these conditions, find the number of motorcycles to be shipped from each warehouse to each dealer if the total shipping cost is to be held to a minimum. What is the minimum cost?

$x_1 + x_3 \geq 20$ (= 20 but more is ok)

$x_2 + x_4 \geq 30$

$x_1 + x_2 \leq 40$

$x_3 + x_4 \leq 15$

$C = 15x_1 + 13x_2 + 14x_3 + 16x_4$ ← minimize C by

so $P = -C = -15x_1 - 13x_2 - 14x_3 - 16x_4$ ← maximizing P

(and $P + 15x_1 + 13x_2 + 14x_3 + 16x_4 = 0$)

$-x_1 - x_3 \leq -20$

$-x_2 - x_4 \leq -30$

$x_1 + x_2 \leq 40$

$x_3 + x_4 \leq 15$

$P + 15x_1 + 13x_2 + 14x_3 + 16x_4 = 0$

$$\begin{array}{cccccccccc} P & x_1 & x_2 & x_3 & x_4 & S_1 & S_2 & S_3 & S_4 & RHS \end{array}$$

$$\left[\begin{array}{ccccccccc|c} 0 & -1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & -20 \\ 0 & 0 & -1 & 0 & -1 & 0 & 1 & 0 & 0 & -30 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 40 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 15 \\ \hline 1 & 15 & 13 & 14 & 16 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

← (most negative)

$$\left[\begin{array}{ccccccccc|c} 0 & -1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & -20 \\ 0 & 0 & 1 & 0 & 1 & 0 & -1 & 0 & 0 & 30 \\ 0 & 1 & 0 & 0 & -1 & 0 & 1 & 1 & 0 & 10 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 15 \\ \hline 1 & 15 & 0 & 14 & 3 & 0 & 13 & 0 & 0 & -390 \end{array}\right]$$

$$\left[\begin{array}{ccccccccc|c} 0 & 1 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 20 \\ 0 & 0 & 1 & 0 & 1 & 0 & -1 & 0 & 0 & 30 \\ 0 & 0 & 0 & -1 & -1 & 1 & 1 & 1 & 0 & -10 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 15 \\ \hline 1 & 0 & 0 & -1 & 3 & 15 & 13 & 0 & 0 & -690 \end{array}\right]$$

$$\left[\begin{array}{ccccccccc|c} 0 & 1 & 0 & 0 & -1 & 0 & 1 & 1 & 0 & 10 \\ 0 & 0 & 1 & 0 & 1 & 0 & -1 & 0 & 0 & 30 \\ 0 & 0 & 0 & 1 & 1 & -1 & -1 & -1 & 0 & 10 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 5 \\ \hline 1 & 0 & 0 & 0 & 4 & 14 & 12 & -1 & 0 & -680 \end{array}\right]$$

(10)

(5)

done RHS

$$\left[\begin{array}{ccccccccc|c} 0 & 1 & 0 & 0 & -1 & -1 & 0 & 0 & -1 & 5 \\ 0 & 0 & 1 & 0 & 1 & 0 & -1 & 0 & 0 & 30 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 15 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 5 \\ \hline 1 & 0 & 0 & 0 & 4 & 15 & 13 & 0 & 1 & -675 \end{array}\right]$$

done

$P_{max} = -675$

so $C_{min} = \$675$

when $x_1 = 5$ cycles $W_1 \to D_1$

$x_2 = 30$ cycles $W_1 \to D_2$

$x_3 = 15$ cycles $W_2 \to D_1$

$x_4 = 0$ cycles $W_2 \to D_2$