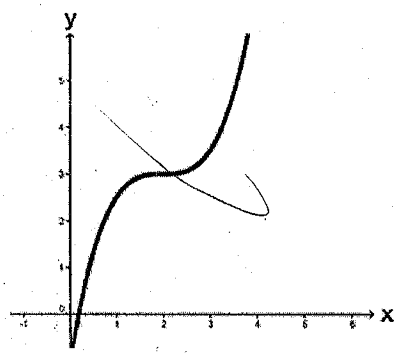
Honors Brief Calculus – Lesson Notes: Unit 17 (Ch8) – Multivariate Calculus

8.1/8.2 – Functions of Two or More Variables

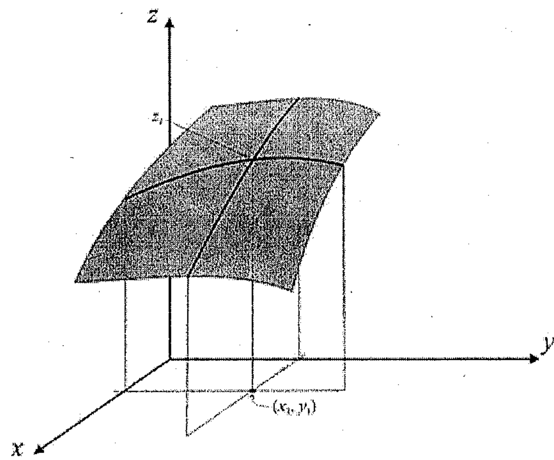
The functions we've studied so far have one output (dependent) variable which varies as a function of one other variable (input or independent variable). But we could define a relationship between 3 variables: 1 dependent variable (z) which depends upon 2 independent variables (x and y):

$$y = f(x) \qquad\qquad z = f(x, y)$$

$$y = \frac{1}{2}(x-2)^3 + 3 \qquad\qquad z = 5 - 2x^2 - y^2 + xy$$



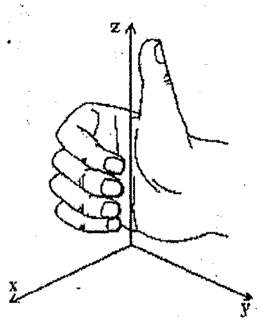
The graph is a 2-dimensional curve of each y associated with each x value.



The graph is a 3-dimensional surface of each z associated with each (x,y) point.

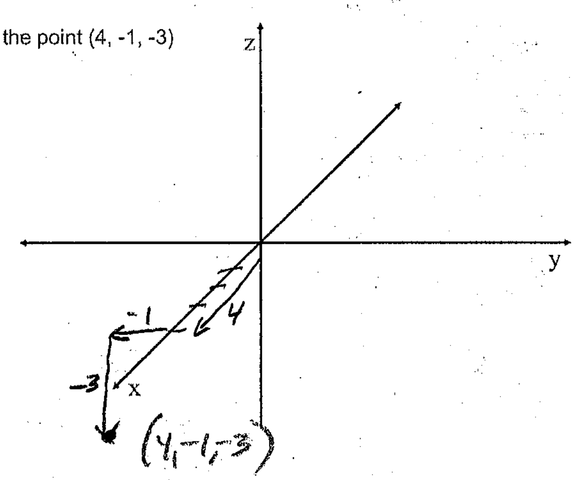
For 3-D graphing, convention is to use the 'right hand rule'

Fingers in direction of +x axis, curl around to +y axis, then thumb points in direction of +z axis:



(we also usually position z to be 'up' on the paper)

Plot the point (4, -1, -3)



#10 Opposite vertices of a rectangular box whose edges are parallel to the coordinate axes are given. List the coordinates of the other six vertices of the box.

(5, 6, 1), and (3, 8, 2)

(5, 6, 2)

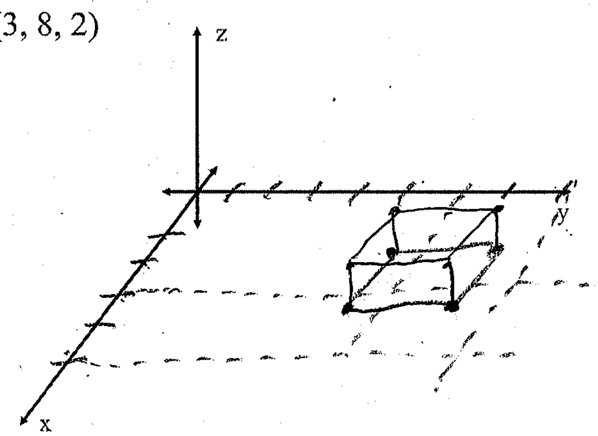
(5, 8, 1)

(5, 8, 2)

(3, 6, 1)

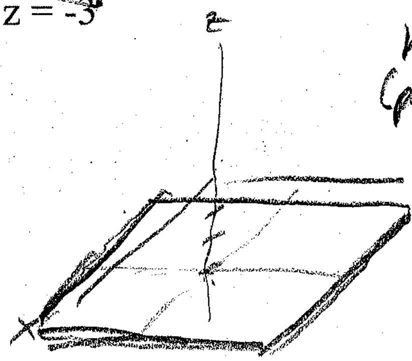
(3, 6, 2)

(3, 8, 1)



#14 Describe in words the set of all points (x,y,z) that satisfy the given conditions.

z = -3



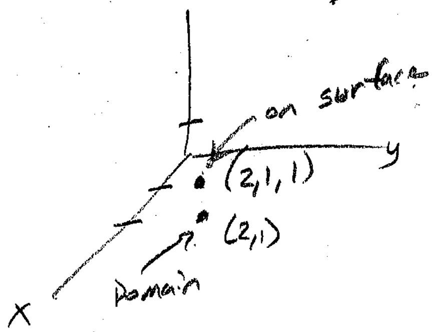
horizontal plane (parallel to x and y axes) of all points with z = -3

#20 Evaluate

$f(2,1)$ if $f(x, y) = x - y^2$

$f(2,1) = (2) - (1)^2$

$= \boxed{1}$



#31 Let $f(x,y) = \sqrt{xy} + x$

Find:

(a) $f(0,0)$

$= \sqrt{(0)(0)} + 0$

$= \boxed{0}$

(b) $f(0, 1)$

$= \sqrt{(0)(1)} + (0)$

$= \boxed{0}$

(c) $f(a^2, t^2)$

$(a \geq 0, t \geq 0)$

$= \sqrt{(a^2)(t^2)} + (a^2)$

$= at + a^2$

$= \boxed{a(t+a)}$

(d) $f(x, y + \Delta y)$

$= \sqrt{(x)(y+\Delta y)} + (x)$

$= \boxed{\sqrt{xy + x\Delta y} + x}$

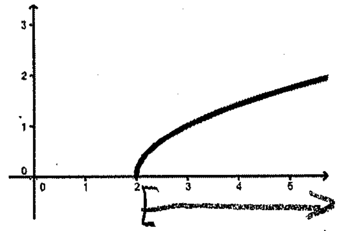
(e) $f(x + \Delta x, y)$

$= \sqrt{(x+\Delta x)y} + (x + \Delta x)$

$= \boxed{\sqrt{xy + y\Delta x} + x + \Delta x}$

What is the domain of this function?

$$y = f(x)$$
$$y = \sqrt{x-2}$$



Set of x values on number line (x-axis)

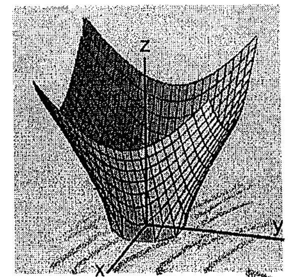
$x - 2 \geq 0$

$x \geq 2$

$$\boxed{D = \{x \mid x \geq 2\}}$$

What is the domain of this function?

$$z = f(x,y)$$
$$z = \sqrt{x^2+y^2-16}$$

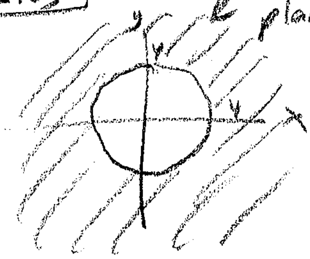


$x^2+y^2-16 \geq 0$

$x^2+y^2 \geq 16$

$$\boxed{D = \{(x,y) \mid x^2+y^2 \geq 16\}}$$

domain is a region of x-y plane



Find the domain of the function:

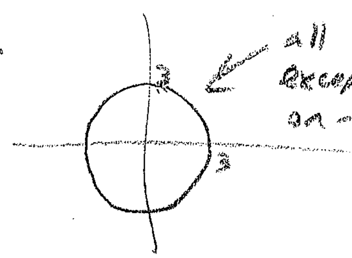
$$z = f(x,y) = \frac{4}{9-x^2-y^2}$$

$9 - x^2 - y^2 \neq 0$

$-x^2 - y^2 \neq -9$

$x^2 + y^2 \neq 9$

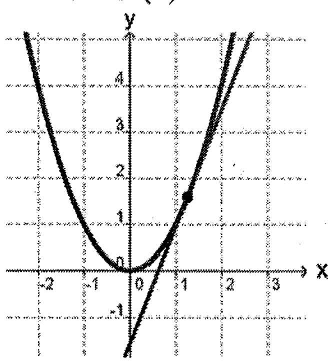
$$\boxed{D = \{(x,y) \mid x^2+y^2 \neq 9\}}$$



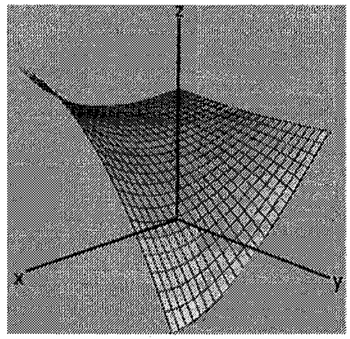
all (x,y) except points on this circle

8.3 – Partial Derivatives

$y = f(x)$



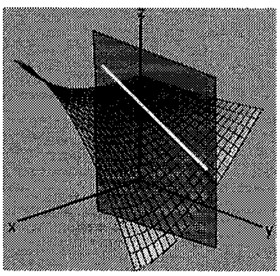
$z = f(x,y)$



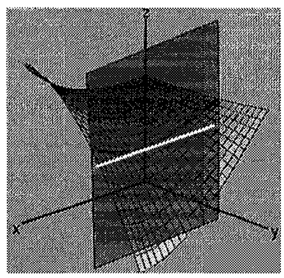
Derivative = slope of line tangent to curve

What would derivative mean for a surface?

$z = f(x,y)$



$z = f(x,y)$



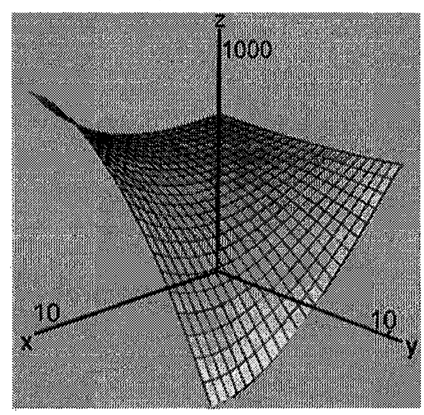
$f_y(x,y) = \frac{\partial z}{\partial y}$ means:

"Differentiate f **with respect to y** while treating **x as a constant.**"

$f_x(x,y) = \frac{\partial z}{\partial x}$ means:

"Differentiate f **with respect to x** while treating **y as a constant.**"

$z = f(x,y) = x^3 + 2y^2 - 3xy^2$



$f_x(x,y) =$ $3x^2 - 3y^2$

$f_x(5,5) =$ $3(5)^2 - 3(5)^2 = 0$

$f_y(x,y) =$ $4y - 6xy$

$f_y(5,5) =$ $4(5) - 6(5)(5) = 20 - 150 = -130$

Try it: Find f_x, f_y, $f_x(2,-1)$, $f_y(-2,3)$

If $f(x, y) = 2x^3 - 3y + x^2$

$f_x = 6x^2 + 2x$ $\quad f_x(2,-1) = 6(2)^2 + 2(2) = 24 + 4 = \boxed{28}$

$f_y = -3$ $\quad f_y(-2,3) = \boxed{-3}$

Try it: Find f_x, f_y, $f_x(2,-1)$, $f_y(-2,3)$

If $f(x,y) = 2x^3 - 3y + x^2$

$\boxed{f_x = 6x^2 + 2x}$ $\quad f_x(2,-1) = 6(2)^2 + 2(2) = 24 + 4 = \boxed{28}$

$\boxed{f_y = -3}$ $\quad \boxed{f_y(-2,3) = -3}$

Higher-order partial derivatives

$f_x(x,y)$ *and* $f_y(x,y)$ are known as **'first-order'** partial derivatives.
But these partial derivatives are also function of x and y, so they form their own 'surfaces'. We could take partial derivatives of these, which are called **'second-order'** partial derivatives:

Starting with $f_x(x,y)$	Starting with $f_y(x,y)$
$f_{xx}(x,y) = \frac{\partial}{\partial x}\frac{\partial z}{\partial x} = \frac{\partial^2 z}{\partial x^2}$	$f_{yy}(x,y) = \frac{\partial}{\partial y}\frac{\partial z}{\partial y} = \frac{\partial^2 z}{\partial y^2}$
$f_{xy}(x,y) = \frac{\partial}{\partial y}\frac{\partial z}{\partial x} = \frac{\partial^2 z}{\partial y \partial x}$	$f_{yx}(x,y) = \frac{\partial}{\partial x}\frac{\partial z}{\partial y} = \frac{\partial^2 z}{\partial x \partial y}$

$f_{yx}(x,y)$ differentiate with respect to y first

$\frac{\partial^2 z}{\partial x \partial y}$ differentiate with respect to y first

Find f_x, f_y, f_{xx}, f_{yy}, f_{yx}, and f_{xy}.

#8 $f(x,y) = x^3 - xy + 10y^2 x$

$\boxed{f_x = 3x^2 - y + 10y^2}$ $\quad \boxed{f_y = -x + 20yx}$

$\boxed{f_{xx} = 6x}$ $\quad \boxed{f_{yy} = 20x}$

$\boxed{f_{xy} = -1 + 20y}$ $\quad \boxed{f_{yx} = -1 + 20y}$

#14 $f(x,y) = \ln(x^2 - y^2)$

$f_x = \frac{1}{x^2-y^2}(2x)$ $\quad f_y = \frac{1}{x^2-y^2}(-2y)$

$\boxed{f_x = \frac{2x}{x^2-y^2}}$ $\quad \boxed{f_y = \frac{-2y}{x^2-y^2}}$

$f_{xx} = \frac{(x^2-y^2)(2)-(2x)(2x)}{(x^2-y^2)^2}$ $\quad f_{yy} = \frac{(x^2-y^2)(-2)-(-2y)(-2y)}{(x^2-y^2)^2}$

$\boxed{f_{xx} = \frac{-2x^2-2y^2}{(x^2-y^2)^2}}$ $\quad \boxed{f_{yy} = \frac{-2x^2-2y^2}{(x^2-y^2)^2}}$

$f_{xy} = 2x\left(-(x^2-y^2)^{-2}(-2y)\right)$ $\quad f_{yx} = (-2y)\left(-(x^2-y^2)^{-2}(2x)\right)$

$\boxed{f_{xy} = \frac{4xy}{(x^2-y^2)^2}}$ $\quad \boxed{f_{yx} = \frac{4xy}{(x^2-y^2)^2}}$

The idea of partial differentiation may be extended to a function of three or more variables:

Find f_x, f_y, and f_z.

#24. $f(x,y,z) = 3xy + 4yz + 8z^2$

$\boxed{f_x = 3y}$ $\quad \boxed{f_y = 3x + 4z}$ $\quad \boxed{f_z = 4y + 16z}$

Find the slope of the tangent line to the curve of intersection of the surface $z = f(x, y)$ with the given **plane** at the indicated **point**.

$z = f(x, y) = e^x \ln y$; plane: $x = 0$; point: $(0, 1, 0)$
(Constant)
so $f_y(x,y)$

$f_y(x,y) = e^x\left(\frac{1}{y}\right)$

$f_y(0,1) = e^0\left(\frac{1}{1}\right)$

$= 1(1)$

$= \boxed{1}$

$z = f(x, y) = 5x^2 + 3y^2$; plane: $y = 3$; point: $(2, 3, 47)$
(Constant)
So $f_x(x,y)$

$f_x(x,y) = 10x$

$f_x(2,3) = 10(2)$

$= \boxed{20}$

Finding Marginal **Cost** and Marginal **Productivity**

46

Marginal Productivity The production function of a certain commodity is given by $P = 8I - I^2 + 3Ik + 50k - k^2$

where I and k are the labor and capital inputs, respectively. Find the marginal **productivities** of I and k at $I = 2$ and $k = 5$.

$\frac{\partial P}{\partial I} = 8 - 2I + 3K$

$\left.\frac{\partial P}{\partial I}\right|_{\substack{I=2 \\ K=5}} = 8 - 2(2) + 3(5)$

$= 8 - 4 + 15$

$= 19$

means production will increase 19 units for each 1 unit added labor.

$\frac{\partial P}{\partial k} = 3I + 50 - 2K$

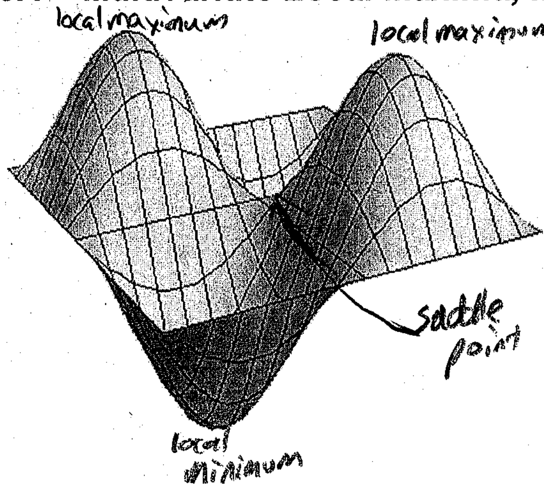
$\left.\frac{\partial P}{\partial K}\right|_{\substack{I=2 \\ K=5}} = 3(2) + 50 - 2(5)$

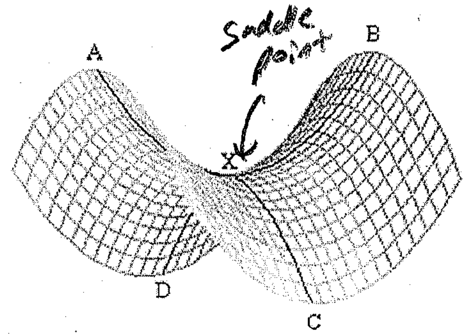
$= 6 + 50 - 10$

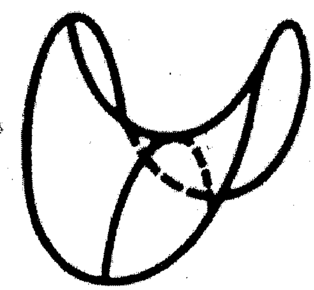
$= 46$

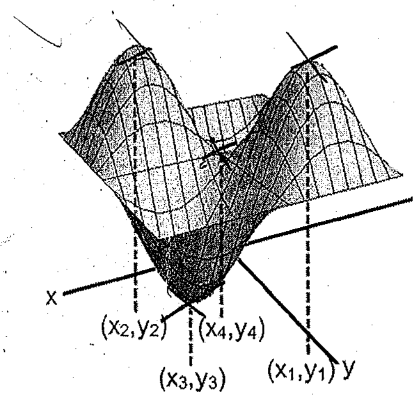
means production will increase 46 units for each 1 unit added capital.

8.4 – Multivariate Local Maxima, Minima, and Saddle Points









Critical points

Critical points occur when

$f_x(x,y)=0$ *and* $f_y(x,y)=0$

Determining whether a critical point is a minimum, maximum, or saddle point:

$\boxed{\text{Find } f_{xx}(x,y),\ f_{yy}(x,y),\ \textit{and } f_{xy}(x,y)}$

For each critical point...

Compute the Hessian Determinant:

$$D = f_{xx}(x_0,y_0)\cdot f_{yy}(x_0,y_0) - \left[f_{xy}(x_0,y_0)\right]^2$$

$D<0$: (x_0,y_0) *is a saddle point*

f_{xx}, f_{yy} one − and one +

$D>0$:

- $f_{xx}(x_0,y_0)<0$: (x_0,y_0) *is a local maximum*

  f_{xx} & f_{yy} both −

- $f_{xx}(x_0,y_0)>0$: (x_0,y_0) *is a local minimum*

  f_{xx} & f_{yy} both +

$D=0$: *inconclusive*

anything else

p. 904

#4 Find all the critical points of the function.

$f(x,y) = x^3 + 6xy + 3y^2 + 8$

$f_x(x,y) = 3x^2 + 6y$ $\qquad f_y(x,y) = 6x + 6y$

$\begin{cases} 3x^2 + 6y = 0 \\ 6x + 6y = 0 \end{cases}$ $\qquad 6x + 6y = 0$

$6y = -6x$

$y = -x$

$3x^2 + (-6x) = 0$

$3x(x-2) = 0$

$x = 0 \qquad x = 2$

$y = 0 \qquad y = -2$

$\boxed{(0,0) \quad (2,-2) \text{ critical points}}$

#6 Find all the critical points.

$f(x,y) = xy + \frac{2}{x} + \frac{4}{y} = xy + 2x^{-1} + 4y^{-1}$

$f_x(x,y) = y - 2x^{-2}$ $\qquad f_y(x,y) = x - 4y^{-2}$

$y - \frac{2}{x^2} = 0$ $\qquad x - \frac{4}{y^2} = 0$

$x = \frac{4}{y^2}$

$y - \frac{2}{\left(\frac{4}{y^2}\right)^2} = 0$

$y - \frac{2y^4}{16} = 0$

$y - \frac{1}{8}y^4 = 0$

$y\left(1 - \frac{1}{8}y^3\right) = 0$

$y = 0 \qquad 1 - \frac{1}{8}y^3 = 0$

$x = \frac{4}{y^2} \qquad \frac{1}{8}y^3 = 1$

$x = \frac{4}{0^2} \qquad y^3 = 8$

undef. $\qquad y = 2 \Rightarrow x = \frac{4}{y^2}$

so (no point) $\qquad x = \frac{4}{2^2} = 1$

$\boxed{(1,2) \text{ is critical point}}$

Find all the critical points and determine whether they are a local maximum, a local minimum, or a saddle point.

#8. $f(x,y) = x^2 - 2xy + 3y^2$

critical points

$f_x(x,y) = 2x - 2y$ $\qquad f_y(x,y) = -2x + 6y$

$2x - 2y = 0$ $\qquad -2x + 6y = 0$

$2x = 2y \rightarrow -(2y) + 6y = 0$

$4y = 0$

$y = 0,\ x = 0$ $\boxed{(0,0) \text{ critical}}$

D:

$f_{xx}(x,y) = 2$ $\qquad f_{yy}(x,y) = 6$ $\qquad f_{xy}(x,y) = -2$

$f_{xx}(0,0) = 2$ $\qquad f_{yy}(0,0) = 6$ $\qquad f_{xy}(0,0) = -2$

$D = f_{xx}(0,0) \cdot f_{yy}(0,0) - [f_{xy}(0,0)]^2$

$= 2 \cdot 6 - (-2)^2$

$= 12 - 4$

$D = 8 > 0$ so max or min

$f_{xx}(0,0) = 2 > 0$ (+ concavity $\cup$)

so $\boxed{(0,0) \text{ is a local minimum}}$

Student

#10. $f(x,y) = x^2 + y^2 - 6y + 10$

critical points

$f_x(x,y) = 2x$ $\qquad f_y(x,y) = 2y - 6$

$2x = 0$ $\qquad 2y - 6 = 0$ $\qquad \boxed{(0,3) \text{ critical}}$

$x = 0$ $\qquad 2y = 6$

$y = 3$

D:

$f_{xx}(x,y) = 2$ $\qquad f_{yy}(x,y) = 2$ $\qquad f_{xy}(x,y) = 0$

$f_{xx}(0,3) = 2$ $\qquad f_{yy}(0,3) = 2$ $\qquad f_{xy}(0,3) = 0$

$D = 2 \cdot 2 - 0^2 = 4 > 0$ so max or min

$f_{xx}(0,3) = 2$ + $\cup$)

$\boxed{\text{so } (0,3) \text{ is a local minimum}}$

3

#12. $f(x,y)=x^2-y^2-2x+4y$

critical points

$f_x(x,y)=2x-2$ $\quad$ $f_y(x,y)=-2y+4$

$2x-2=0$ $\quad$ $-2y+4=0$

$2x=2$ $\quad$ $-2y=-4$

$x=1$ $\quad$ $y=2$

$\boxed{(1,2)\text{ critical}}$

D:

$f_{xx}(x,y)=2$ $\quad$ $f_{yy}(x,y)=-2$ $\quad$ $f_{xy}(x,y)=0$

$f_{xx}(1,2)=2$ $\quad$ $f_{yy}(1,2)=-2$ $\quad$ $f_{xy}(1,2)=0$

$D=2\cdot(-2)-0^2=-4<0$

so $\boxed{(1,2)\text{ is a saddle point}}$

#20. $f(x,y)=x^3-3xy-y^3$

critical points

$f_x(x,y)=3x^2-3y$ $\quad$ $f_y(x,y)=-3x-3y^2$

$3x^2-3y=0$ $\quad$ $-3x-3y^2=0$

$3(-y^2)^2-3y=0$ $\quad$ $3x=-3y^2$

$3y^4-3y=0$ $\quad$ $x=-y^2$

$3y(y^3-1)=0$ $\quad$ $y=0\Rightarrow x=0$

$y=0 \quad y^3=1$ $\quad$ $y=1\Rightarrow x=-1$

$\boxed{(0,0)\ (-1,1)}$ critical

~~$(-1,-1)$~~

D:

$f_{xx}(x,y)=6x$ $\quad$ $f_{yy}(x,y)=-6y$ $\quad$ $f_{xy}(x,y)=-3$

$f_{xx}(0,0)=0$ $\quad$ $f_{yy}(0,0)=0$ $\quad$ $f_{xy}(0,0)=-3$

$f_{xx}(-1,1)=-6$ $\quad$ $f_{yy}(-1,1)=-6$ $\quad$ $f_{xy}(-1,1)=-3$

$D(0,0)=0\cdot0-(-3)^2=-9<0$ so $\boxed{(0,0)\text{ is a saddle point}}$

$D(-1,1)=-6\cdot-6-(-3)^2=36-9=27>0$

$f_{xx}(-1,1)=-6<0$

so $\boxed{(-1,1)\text{ is a local maximum}}$

skip

#22. $f(x,y)=3y^2-x^2y+x$

critical pts

$f_x(x,y)=-2xy+1$ $\quad$ $f_y(x,y)=6y-x^2$

$-2xy+1=0$ $\quad$ $6y-x^2=0$

$-2x(\frac{1}{6}x^2)+1=0$ $\quad$ $6y=x^2$

$-\frac{1}{3}x^3+1=0$ $\quad$ $y=\frac{1}{6}x^2$

$\frac{1}{3}x^3=1$ $\quad$ $y=\frac{1}{6}(\sqrt[3]{9})$

$x^3=3$

$x=\sqrt[3]{3}$

$\boxed{(\sqrt[3]{3},\frac{1}{6}\sqrt[3]{9})\text{ is critical}}$

$(1.4422, .3467)$

D:

$f_{xx}(x,y)=-2y$ $\quad$ $f_{yy}(x,y)=6$ $\quad$ $f_{xy}(x,y)=-2x$

$f_{xx}(1.4422,.3467)=-.6934$ $\quad$ $f_{yy}(1.4422,.3467)=6$

$f_{xy}(1.4422,.3467)=-2.8844$

$D=(-.6934)(6)-(-2.8844)^2=-12.48<0$

$\boxed{(1.4422,.3467)\text{ is a saddle point}}$

5

#24. $f(x,y)=\dfrac{x}{x+y}=x(x+y)^{-1}$

critical pts

$f_x(x,y)=\dfrac{(x+y)(1)-(x)(1)}{(x+y)^2}$ $\quad$ $f_y(x,y)=\dfrac{(x+y)(0)-(x)(1)}{(x+y)^2}$

$=\dfrac{y}{(x+y)^2}=0$ $\quad$ $=\dfrac{-x}{(x+y)^2}=0$

$y=0$ $\quad$ $x=0$

but if $x=0,\ y=0$ $\quad f(x,y)=\frac{0}{0}$ undef

so $\boxed{\text{no critical points}}$

8.5 – *Optimization using Lagrange Multipliers*

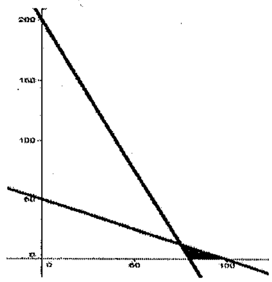
We've learn various ways to 'optimize' things in this course.
First, we did linear programming (and Simplex method for more than 2 variables):

Example: A diet must be constructed which provides at least 40g of carbohydrates, and no more than 200g of fat. The diet can be made up of two foods (A and B). Each unit of food A provides 5g carbs and 2g fat. Each unit of food B provides 4g carbs and 2g fat. Food A costs \$0.20 per unit and Food B costs \$0.15 per unit. How many units of each food should be used if we want to minimize cost?

Solution:

$$5x+2y\geq 400$$
$$2x+4y\leq 200$$

$$C=0.20x+0.15y$$

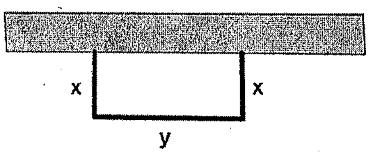


Corners	$C=0.20x+0.15y$
$\boxed{(80,0)}$	$\boxed{\$16}$
$(100,0)$	$\$20$
$(75,12.5)$	$\$16.875$

But this method requires that all the functions involved must be linear.

Then we learned that you can find the optimum value of any function using derivatives:

Example: A farmer wants to enclose a rectangular plot that borders on a straight river with a fence, not fencing in the side along the river. If the farmer has 4000 m of fencing, what is the largest area that can be enclosed?



$A=xy$ (equation to be optimized)

$2x+y=4000$ (constraint)

solution:

$$A=x(4000-2x)$$
$$A=4000x-2x^2$$
$$A'=4000-4x=0$$
$$x=1000\ m$$

But this was solvable because:
1) It was easy to solve for y in terms of x.
2) After substituting, it was easy to find the derivative.

For cases with non-linear equations, where:

- It is *not* easy to solve the constraint equation for a variable,

- or -

- After substituting, the equation to be optimized is hard to differentiate,

...we need the **Method of Lagrange Multipliers:**

For function $z=f(x,y)$ *subject to constraint* $g(x,y)=0$

1) Form the function: $F(x,y,\lambda)=f(x,y)+\lambda g(x,y)$

2) Take partial derivatives of this function with respect to each variable, including λ and set each to zero, forming a set of equations: $\dfrac{\partial F}{\partial x}=0,\ \dfrac{\partial F}{\partial y}=0,\ \dfrac{\partial F}{\partial z}=0,\ \dfrac{\partial F}{\partial \lambda}=0$

3) Find all points which are solutions of this system, and evaluate the objective function at each point to find extrema.

2. Find max $z = f(x,y) = 3xy$

subject to the constraint $g(x,y) = x^2 + y^2 - 4 = 0$

$F = 3xy + \lambda(x^2+y^2-4)$

$\frac{\partial F}{\partial x} = 3y + 2x\lambda = 0 \qquad \frac{\partial F}{\partial y} = 3x + 2y\lambda = 0 \qquad \frac{\partial F}{\partial \lambda} = x^2+y^2-4=0$

$2x\lambda = -3y \qquad 2y\lambda = -3x \qquad x^2+y^2=4$

$\lambda = \frac{-3y}{2x} \qquad \lambda = \frac{-3x}{2y} \qquad x^2+x^2=4$

$\frac{-3y}{2x} = \frac{-3x}{2y} \qquad 2x^2=4$

$-6y^2=-6x^2 \qquad x^2=2$

$x^2=y^2 \qquad x=\pm\sqrt{2}$

$(\pm\sqrt{2})^2=y^2$

$y^2=2$

$y=\pm\sqrt{2}$

(x,y)	z
$(-\sqrt{2},-\sqrt{2})$	$3(-\sqrt{2})(-\sqrt{2})=6$
$(-\sqrt{2},\sqrt{2})$	$3(-\sqrt{2})(\sqrt{2})=-6$
$(\sqrt{2},-\sqrt{2})$	$3(\sqrt{2})(-\sqrt{2})=-6$
$(\sqrt{2},\sqrt{2})$	$3(\sqrt{2})(\sqrt{2})=6$

$\boxed{\text{max of } 6 \text{ at } (-\sqrt{2},-\sqrt{2}) \text{ and } (\sqrt{2},\sqrt{2})}$

6. Find max $z = f(x,y) = xy$

subject to the constraint $g(x,y) = x + y - 8 = 0$

$F = xy + \lambda(x+y-8)$

$\frac{\partial F}{\partial x} = y+\lambda=0 \qquad \frac{\partial F}{\partial y} = x+\lambda=0 \qquad \frac{\partial F}{\partial \lambda} = x+y-8=0$

$\lambda=-y \qquad \lambda=-x \qquad x+x-8=0$

$-y=-x \qquad 2x=8$

$x=y \qquad x=4$

$y=4$

(x,y)	z
$(4,4)$	$(4)(4)=16$

$\boxed{\text{Max of } 16 \text{ at } (4,4)}$

4. Find min $z = f(x,y) = 3x + 4y$

subject to the constraint $g(x,y) = x^2 + y^2 - 9 = 0$

$F = 3x+4y+\lambda(x^2+y^2-9)$

$\frac{\partial F}{\partial x} = 3+2\lambda x=0 \qquad \frac{\partial F}{\partial y} = 4+2\lambda y=0 \qquad \frac{\partial F}{\partial \lambda} = x^2+y^2-9=0$

$2\lambda x=-3 \qquad 2\lambda y=-4 \qquad x^2+y^2=9$

$\lambda=\frac{-3}{2x} \qquad \lambda=\frac{-4}{2y} \qquad x^2+\left(\frac{4}{3}x\right)^2=9$

$\frac{-3}{2x}=\frac{-4}{2y} \qquad x^2+\frac{16}{9}x^2=9$

$-6y=-8x \qquad \frac{25}{9}x^2=9$

$y=\frac{4}{3}x \qquad \sqrt{25x^2}=\sqrt{81}$

$5x=\pm 9 \qquad x=\pm\frac{9}{5}$

$y=\pm\frac{4}{3}\cdot\frac{9}{5}=\pm\frac{12}{5}$

(x,y)	z
$\left(\frac{9}{5},\frac{12}{5}\right)$	$\frac{27}{5}+\frac{48}{5}=\frac{75}{5}=15$
$\left(-\frac{9}{5},-\frac{12}{5}\right)$	$-\frac{27}{5}-\frac{48}{5}=-15$

$\boxed{\text{min} = -15 \text{ @ } \left(-\frac{9}{5},-\frac{12}{5}\right)}$

8. Find min $z = f(x,y) = x^2 + y^2$

subject to the constraint $g(x,y) = 2x + 3y - 4 = 0$

$F = x^2+y^2+\lambda(2x+3y-4)$

$\frac{\partial F}{\partial x} = 2x+2\lambda=0 \qquad \frac{\partial F}{\partial y} = 2y+3\lambda=0 \qquad \frac{\partial F}{\partial \lambda} = 2x+3y-4=0$

$\lambda=-x \qquad \lambda=\frac{-2y}{3} \qquad 2\left(\frac{2y}{3}\right)+3y=4$

$-x=\frac{-2y}{3}, \quad x=\frac{2y}{3} \qquad \frac{4y}{3}+3y=4$

$4y+9y=12$

$13y=12$

$y=\frac{12}{13}$

$x=\frac{2}{3}\cdot\frac{12}{13}=\frac{8}{13}$

(x,y)	z
$\left(\frac{8}{13},\frac{12}{13}\right)$	$\left(\frac{8}{13}\right)^2+\left(\frac{12}{13}\right)^2 = \frac{64}{169}+\frac{144}{169} = \frac{208}{169}$

$\boxed{\text{min of } \frac{208}{169} \text{ at } \left(\frac{8}{13},\frac{12}{13}\right)}$

10. Find max $w = f(x,y,z) = x + y + z$

subject to the constraint $g(x,y,z) = x^2 + y^2 + z^2 - 12 = 0$

$F = x+y+z+\lambda(x^2+y^2+z^2-12)$

$\frac{\partial F}{\partial x} = 1+2x\lambda = 0$ $\quad 2x\lambda = -1$ $\quad \lambda = \frac{-1}{2x}$

$\frac{\partial F}{\partial y} = 1+2y\lambda = 0$ $\quad \lambda = \frac{-1}{2y}$

$\frac{\partial F}{\partial z} = 1+2z\lambda = 0$ $\quad \lambda = \frac{-1}{2z}$

$\frac{\partial F}{\partial \lambda} = x^2+y^2+z^2-12 = 0$

$\frac{-1}{2x} = \frac{-1}{2y} = \frac{-1}{2z}$

$2x = 2y = 2z$

$x = y = z$ $\longrightarrow$

$x^2+x^2+x^2-12 = 0$

$3x^2 = 12$

$x^2 = 4$

$x = \pm 2$

(x,y,z)	w
$(2,2,2)$	$2+2+2=6$
$(-2,-2,-2)$	$-2+-2+-2=-6$

$\boxed{\text{max of } 6 \text{ at } (2,2,2)}$

12. Find min $w = f(x,y,z) = 4x + 4y + 2z$

subject to the constraint $g(x,y,z) = x^2 + y^2 + z^2 - 9 = 0$

$F = 4x+4y+2z+\lambda(x^2+y^2+z^2-9)$

$\frac{\partial F}{\partial x} = 4+2x\lambda = 0$ $\quad 2x\lambda = -4$ $\quad \lambda = \frac{-4}{2x}$

$\frac{\partial F}{\partial y} = 4+2y\lambda = 0$ $\quad 2y\lambda = -4$ $\quad \lambda = \frac{-4}{2y}$

$\frac{\partial F}{\partial z} = 2+2z\lambda = 0$ $\quad 2z\lambda = -2$ $\quad \lambda = \frac{-2}{2z}$

$\frac{\partial F}{\partial \lambda} = x^2+y^2+z^2-9 = 0$

$\frac{-2}{x} = \frac{-2}{y} = \frac{-1}{z}$

$z = \frac{1}{2}x = \frac{1}{2}y$

$\left.\begin{array}{l} z = \frac{1}{2}x \\ y = x \end{array}\right\} \longrightarrow$

$x^2 + (x)^2 + (\frac{1}{2}x)^2 = 9$

$x^2 + x^2 + \frac{1}{4}x^2 = 9$

$\frac{9}{4}x^2 = 9$

$x^2 = 4$

$x = \pm 2$

$x=2$	$x=-2$
$y=2$	$y=-2$
$z=\frac{1}{2}(2)=1$	$z=\frac{1}{2}(-2)=-1$
$(2,2,1)$	$(-2,-2,-1)$

(x,y,z)	w
$(2,2,1)$	$4(2)+4(2)+2(1)=18$
$(-2,-2,-1)$	$4(-2)+4(-2)+2(-1)=-18$

$\boxed{\text{min of } -18 \text{ at } (-2,-2,-1)}$

14. Find two numbers x and y so that the sum of their squares is a minimum while their sum is 100.

$x+y=100$

$f(x,y)=x^2+y^2 \qquad g(x,y)=x+y-100$

$F=x^2+y^2+\lambda(x+y-100)$

$\frac{\partial F}{\partial x}=2x+\lambda=0 \qquad \frac{\partial F}{\partial y}=2y+\lambda=0 \qquad \frac{\partial F}{\partial \lambda}=x+y-100=0$

$\lambda=-2x \qquad \lambda=-2y \qquad x+(x)-100=0$

$-2x=-2y \qquad 2x=100$

$x=y \qquad x=50$

$y=50$

$\boxed{(50,50)}$

16. Find three numbers x, y, and z so that their sum is a minimum while the sum of their squares is 25.

$f(x,y,z)=x+y+z \qquad g(x)=x^2+y^2+z^2-25$

$F=x+y+z+\lambda(x^2+y^2+z^2-25)$

$\frac{\partial F}{\partial x}=1+2x\lambda=0 \qquad \frac{\partial F}{\partial y}=1+2y\lambda=0 \qquad \frac{\partial F}{\partial z}=1+2z\lambda=0 \qquad \frac{\partial F}{\partial \lambda}=x^2+y^2+z^2-25=0$

$\lambda=-\frac{1}{2x} \qquad \lambda=-\frac{1}{2y} \qquad \lambda=-\frac{1}{2z}$

$x=y=z$

$3x^2-25=0$

$3x^2=25$

$x^2=\frac{25}{3}$

$x=\pm\sqrt{\frac{25}{3}}=\pm\frac{5}{\sqrt{3}}$

$\left(\frac{5}{\sqrt3},\frac{5}{\sqrt3},\frac{5}{\sqrt3}\right) \quad \frac{15}{\sqrt3}$

$\left(\frac{-5}{\sqrt3},\frac{-5}{\sqrt3},\frac{-5}{\sqrt3}\right) \quad \frac{-15}{\sqrt3} \leftarrow$

$\boxed{\text{minimum sum of } \frac{-15}{\sqrt3} \text{ when } x=y=z=\frac{-5}{\sqrt3}}$