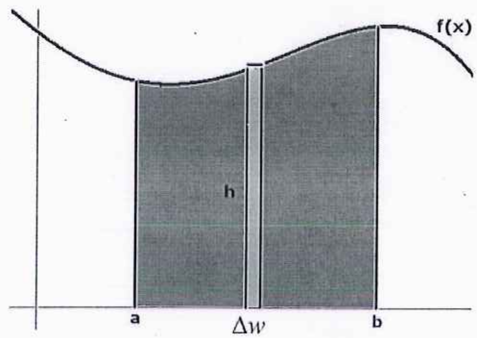
Volumes of Solids of Revolution – Intro; Disc Method (and Washer Method)

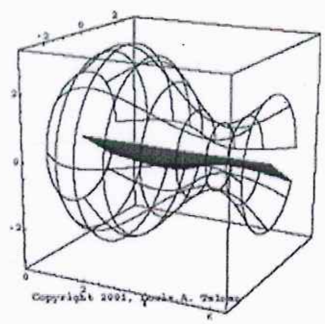
We can use an integral to find the area under a function curve (between the curve and the x-axis):



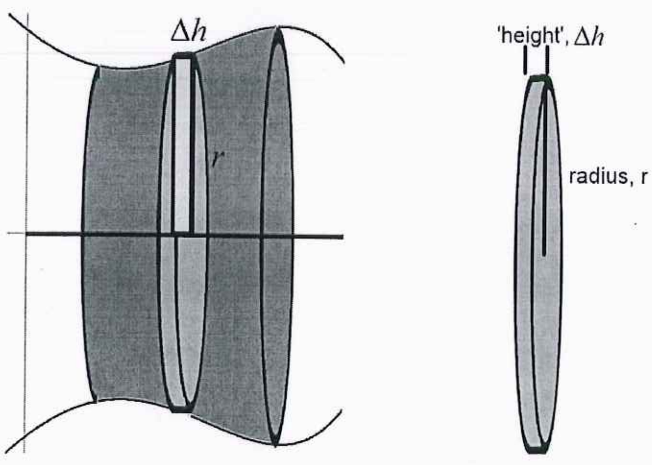
The area is a summation of an infinite number of small rectangles:

$$A = \sum (\text{area of rectangle})$$
$$A = \sum \text{height} \bullet \text{width}$$
$$A = \int_a^b h \bullet \Delta w$$
$$A = \int_a^b f(x)\,dx$$

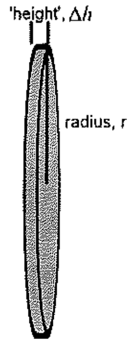
If we rotate this area around the x-axis, we form a 3 dimensional volume called a 'solid of revolution':



We can use an integral to find the 3-D volume of a solid of revolution, by computing the summation of an infinite number of small shapes, but instead of the shapes being 2-D rectangles, the 2-D rectangles would also revolve around the axis and form 3-D cylinders called 'discs':

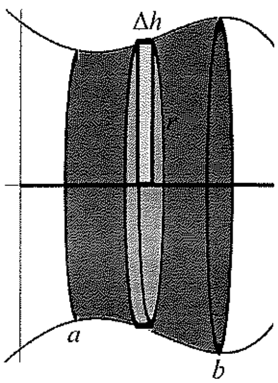


From geometry, the volume of a right circular cylinder is $V_{cylinder} = \pi r^2 h$ so the volume in our small (infinitely thin) cylinder is:



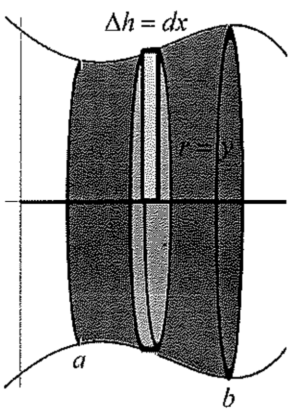
$$V_{thin\ cylinder} = \pi r^2 \Delta h$$

We can therefore use an integral to find the summation of a series of these cylinder volumes to find the volume of the solid of revolution:



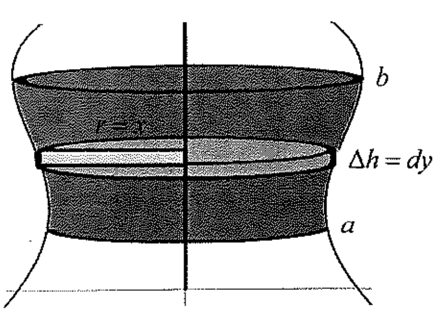
$$V = \sum (volume\ of\ cylinder)$$
$$V = \sum \pi r^2 \bullet height$$
$$V = \int_a^b \pi r^2 \Delta h$$

For this solid of revolution, the radius, r, is also 'y' which is f(x), and the small height, Δh, is a small change in 'x', which we would write as 'dx':



$$V = \sum (volume\ of\ cylinder)$$
$$V = \sum \pi r^2 \bullet height$$
$$V = \int_a^b \pi r^2 \Delta h$$
$$V = \int_a^b \pi y^2 dx$$
$$\boxed{V = \int_a^b \pi \left[f(x) \right]^2 dx}$$

Instead of the function y=f(x), we could have a function x=f(y) and the solid could be revolving around the y-axis. In that case, the radius would be an 'x' value, and the Δh would be a change in y, dy:



$$V = \sum (volume\ of\ cylinder)$$
$$V = \sum \pi r^2 \bullet height$$
$$V = \int_a^b \pi r^2 \Delta h$$
$$V = \int_a^b \pi x^2 dy$$
$$\boxed{V = \int_a^b \pi \left[f(y) \right]^2 dy}$$

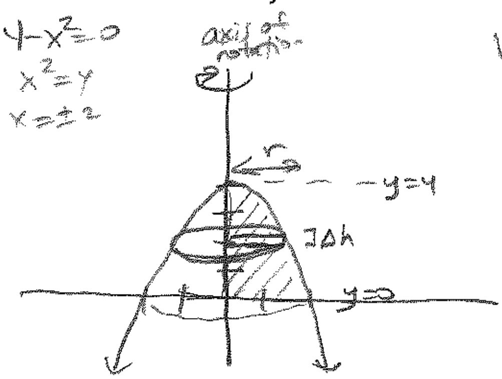
This suggests a procedure we could use to find the volume of a solid of revolution:

1) Draw a sketch (and show 3-D rotation into a solid).
2) Select method ('disc')
3) Draw the rectangle (disc: rectangle is $\perp$ to axis of rotation)
4) Rotate the rectangle to make the cylinder shape and label r, Δh
5) Make an integral using the volume equation:
   disc method: $V = \int_a^b \pi r^2 \Delta h$
6) Use geometry to change r, Δh to x's and y's.
7) The dx or dy sets the integration limits.
8) Substitute to get everything in terms of the integration variable.
9) Use a calculator to evaluate the integral to find the volume.

Find these volumes using the Disc Method:

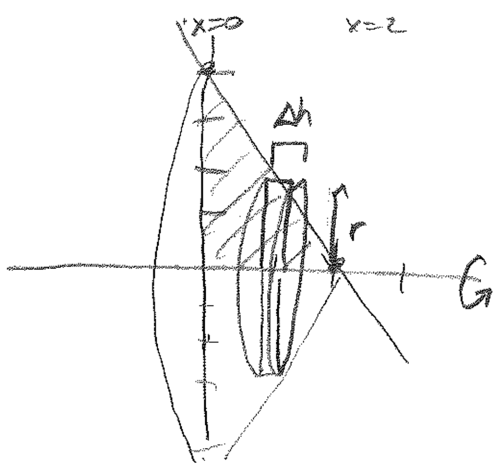
Example:
In the first quadrant and bounded by $y = 4 - x^2$, $y = 0$, $x = 0$
revolved about the y-axis.

$4 - x^2 = 0$
$x^2 = y$
$x = \pm 2$



$$V = \int \pi r^2 \Delta h$$
$$= \int \pi x^2 dy$$ integration variable is 'y' so integration limits are y values
$$= \int_0^4 \pi x^2 dy$$ and everything must use only y
$$= \int_0^4 \pi (4-y) dy$$ $y = 4 - x^2$ so $x^2 = 4 - y$
$$= \boxed{25.1327}$$
(by calculator)

You try this one:
Bounded by $2x + y = 4$, $y = 0$, $x = 0$
revolved about the x-axis.

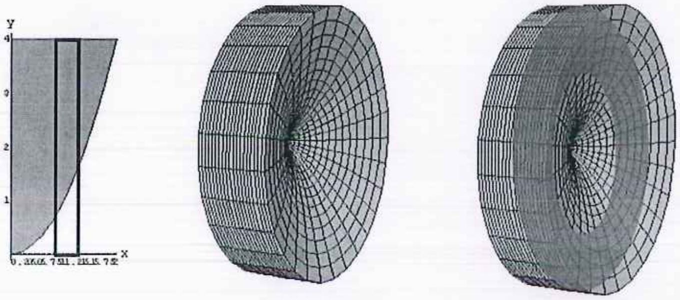


$$V = \int \pi r^2 \Delta h$$
$$= \int \pi y^2 dx$$
$$= \int_0^2 \pi (4-2x)^2 dx$$
$$= \boxed{33.5103}$$
(by calculator)

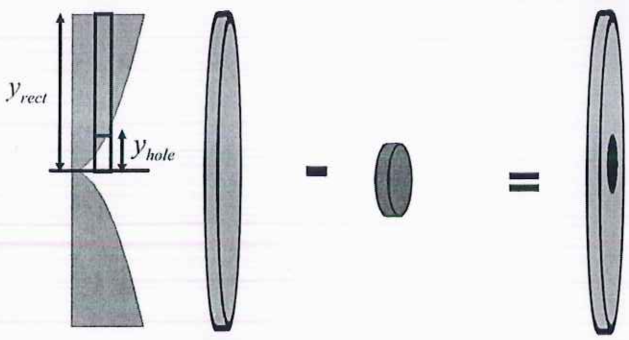
$2x + y = 4$
$y = 4 - 2x$

Find this volume using the Disc Method: $\longrightarrow$ "Washer" Method

3. Bounded by $y = x^2$, $y = 4$, $x = 0$
revolved about the x-axis.



Find these volumes using the Disc Method: $\longrightarrow$ "Washer" Method



We find the volume of the solid 'washer' by finding the volume of the entire disc, and then subtracting the volume in the hole.

These volumes have the same 'width' but different 'heights'.

$$V = V_{entire\ cylinder} - V_{hole}$$

$$V = \int_a^b \pi y_{rect}^{\ 2} dx - \int_a^b \pi y_{hole}^{\ 2} dx$$

We can update our procedure to include 'no hole' and 'hole cases:

1) Draw a sketch (and show 3-D rotation into a solid).
2) Select method ('disc')
3) Draw the rectangle (disc: rectangle is $\perp$ to axis of rotation)
4) Rotate the rectangle to make the cylinder shape and label r, Δh
5) Make an integral using the volume equation:

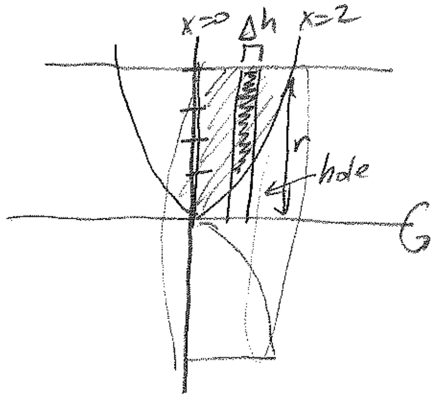
disc method (no hole): $V = \int_a^b \pi r^2 \Delta h$

disc method (hole): $V = \int_a^b \pi r_{rect}^{\ 2} \Delta h - \int_a^b \pi r_{hole}^{\ 2} \Delta h$

6) Use geometry to change r, Δh to x's and y's.
7) The dx or dy sets the integration limits.
8) Substitute to get everything in terms of the integration variable.
9) Use a calculator to evaluate the integral to find the volume.

Find these volumes using the Disc Method: $\longrightarrow$ "Washer" Method

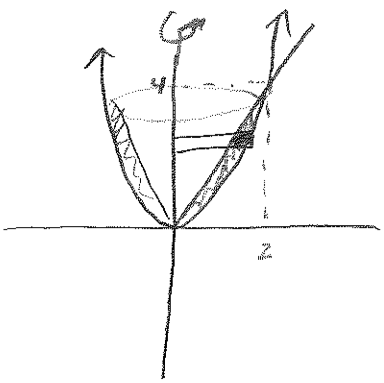
3. Bounded by $y = x^2, \quad y = 4, \quad x = 0$
revolved about the x-axis.



$$V = V_{rect} - V_{hole}$$
$$= \int \pi r_{rect}^2 \Delta h - \int \pi r_{hole}^2 \Delta h$$
$$= \int \pi y_{rect}^2 dx - \int \pi y_{hole}^2 \Delta h \qquad (y = x^2)$$
$$= \int_0^2 \pi (4)^2 dx - \int_0^2 \pi (x^2)^2 dx$$
$$= \int_0^2 (16\pi - \pi x^4) dx$$
$$= \boxed{80.4248}$$

4. Bounded by $y = x^2$ *and* $y = 2x$
revolved about the y-axis.

$x^2 = 2x$
$x^2 - 2x = 0$
$x(x-2) = 0$
$x = 0 \quad x = 2$



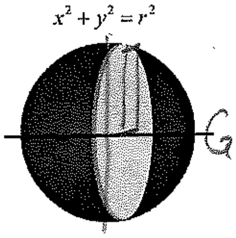
$$V = V_{rect} - V_{hole}$$
$$= \int \pi r_{rect}^2 \Delta h - \int \pi r_{hole}^2 \Delta h$$
$$= \int \pi x_{rect}^2 dy - \int \pi x_{hole}^2 dy$$
$$= \int_0^4 \pi (\sqrt{y})^2 dy - \int_0^4 \pi \left(\tfrac{1}{2}y\right)^2 dy$$
$$= \int_0^4 \left(\pi y - \tfrac{1}{4}\pi y^2\right) dy$$
$$= \boxed{8.3776}$$

rect boundary set by $y = x^2$ so $x = \sqrt{y}$

hole boundary set by line $y = 2x$ so $x = \frac{1}{2}y$

Calculus and solids of revolution were used to derive the equations for things like the volume of a sphere:

$x^2 + y^2 = r^2$



half sphere:

$x^2 + y^2 = r^2$ so $y^2 = r^2 - x^2$

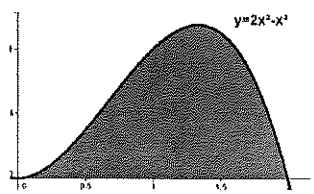
$$V = \int \pi r^2 \Delta h = \int \pi y^2 dx$$
$$= \int_0^r \pi (r^2 - x^2) dx \qquad \text{(no numbers, can't use calculator)}$$
$$= \int_0^r (\pi r^2 - \pi x^2) dx \qquad r \text{ is a constant}$$
$$= \left[\pi r^2 x\right]_0^r - \left[\pi \tfrac{1}{3} x^3\right]_0^r$$
$$= \left(\pi r^2(r) - \pi r^2(0)\right) - \left(\pi \tfrac{1}{3}(r)^3 - \pi \tfrac{1}{3}(0)^3\right)$$
$$= \pi r^3 - \tfrac{1}{3}\pi r^3 = \tfrac{2}{3}\pi r^3$$

full sphere $= 2 \times \frac{2}{3}\pi r^3 = \boxed{\frac{4}{3}\pi r^3}$ (the geometry formula)

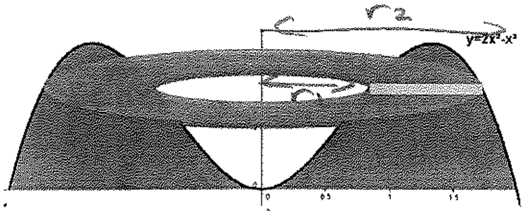
Volumes of Solids of Revolution – Shell Method

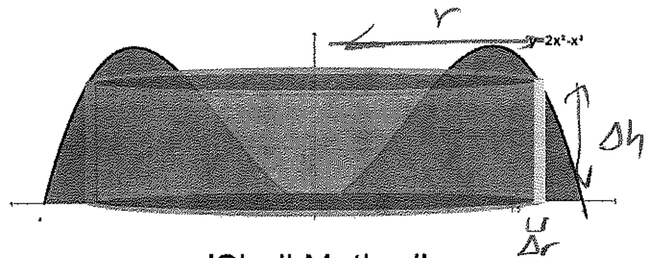
Some problems are difficult to solve using the disc/washer method: (day2)

Find volume of solid obtained by rotating about the y-axis



$(y = 2x^2 - x^3)$



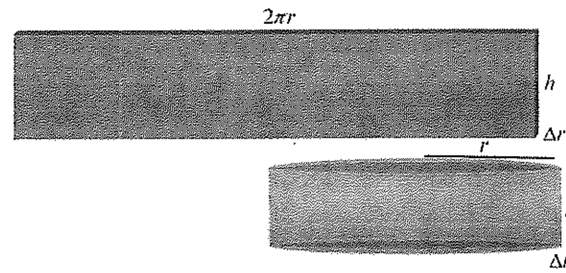


$V = \int \pi r_2^2 \Delta h - \int \pi r_1^2 \Delta h$

$= \int \pi x_2^2 \, dy - \int \pi x_1^2 \, dy$

(requires solving $y = 2x^2 - x^3$ for x in terms of y)

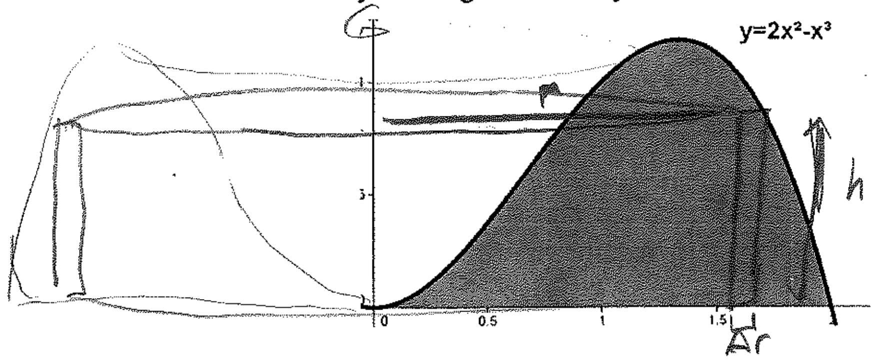
'Shell Method'



$$V = 2\pi r h \Delta r$$

A thin-walled cylindrical shell 'folds out' to become a rectangular box

Find volume of solid obtained by rotating about the y-axis



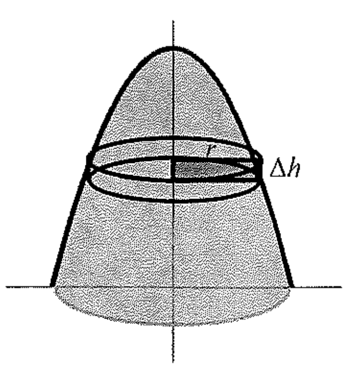
$V = \int 2\pi r h \Delta r$

$= \int_0^2 2\pi x y \, dx$

$= \int_0^2 2\pi x (2x^2 - x^3) \, dx$

$= \boxed{10.0531}$

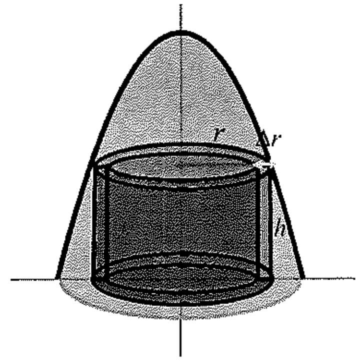
'Disc Method'



$V = \int \pi r^2 \Delta h$

rect ⊥ axis of rotation

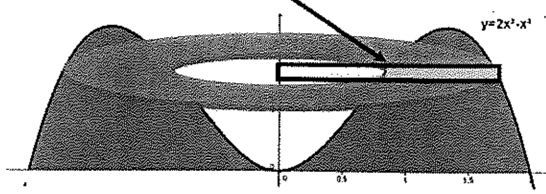
'Shell Method'



$V = \int 2\pi r h \Delta r$

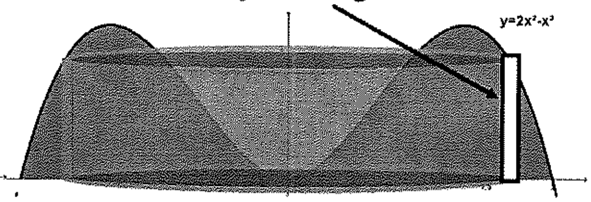
rect || axis of rotation

rectangle doesn't go all the way to axis
there is a 'hole' so we need two integrals



Disc/washer

rectangle goes all the way to axis
no 'hole' so only one integral



Shell

$$V = \int \pi r_2^2 \Delta h - \int \pi r_1^2 \Delta h$$

$$V = \int_y \pi x_2^2 dy - \int_y \pi x_1^2 dy$$

$$V = \int_x \pi y_2^2 dx - \int_x \pi y_1^2 dx$$

$$V = \int 2\pi h \Delta r$$

$$V = \int_x 2\pi x \left[f(x) \right] dx$$

$$V = \int_y 2\pi y \left[f(y) \right] dy$$

Here is the procedure for all the cases:

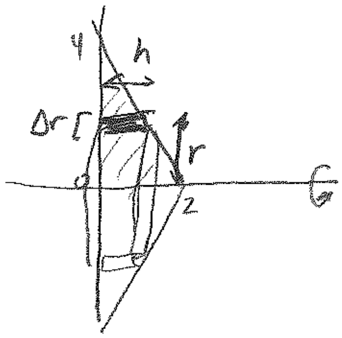
1) Draw a sketch (and show 3-D rotation into a solid).
2) Select method ('disc' or 'shell')
3) Draw the rectangle (disc: rectangle $\perp$ to axis of rotation,
   shell: rectangle $||$ to axis of rotation)
4) Rotate the rectangle to make the cylinder and label r, Δh (disc) or r h Δr (shell).
5) Make an integral using the volume equation:

	disc method	shell method
No hole:	$V = \int_a^b \pi r^2 \Delta h$	$V = \int_a^b 2\pi r h \Delta r$
Hole:	$V = \int_a^b \pi r_{rect}{}^2 \Delta h - \int_a^b \pi r_{hole}{}^2 \Delta h$	$V = \int_a^b 2\pi r h_{rect} \Delta r - \int_a^b 2\pi r h_{hole} \Delta r$

6) Use geometry to change r, h, Δr, Δh to x's and y's.
7) The dx or dy sets the integration limits.
8) Substitute to get everything in terms of the integration variable.
9) Use a calculator to evaluate the integral to find the volume.

Find these volumes using the Shell Method:

1. Bounded by $2x + y = 4, \quad y = 0, \quad x = 0$ revolved about the x-axis.



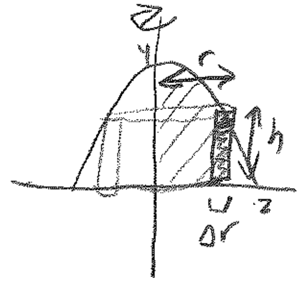
$$V = \int 2\pi r h \Delta r$$
$$= \int_0^4 2\pi y x \, dy$$
$$= \int_0^4 2\pi y \left(\tfrac{1}{2}(4-y)\right) dy$$
$$= \boxed{33.5103}$$

$2x + y = 4$

$2x = 4 - y$

$x = \tfrac{1}{2}(4-y)$

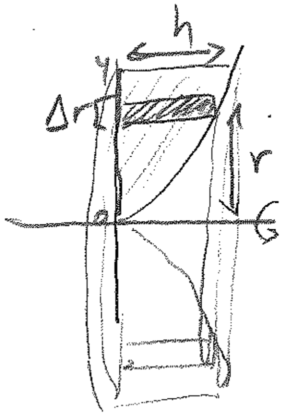
2. In the first quadrant and bounded by $y = 4 - x^2, \quad y = 0, \quad x = 0$ revolved about the y-axis.



$y = 4 - x^2$

$$V = \int 2\pi r h \Delta r$$
$$= \int 2\pi x y \, dx$$
$$= \int_0^2 2\pi x (4 - x^2) dx$$
$$= \boxed{25.1327}$$

3. Bounded by $y = x^2, \quad y = 4, \quad x = 0$ revolved about the x-axis.

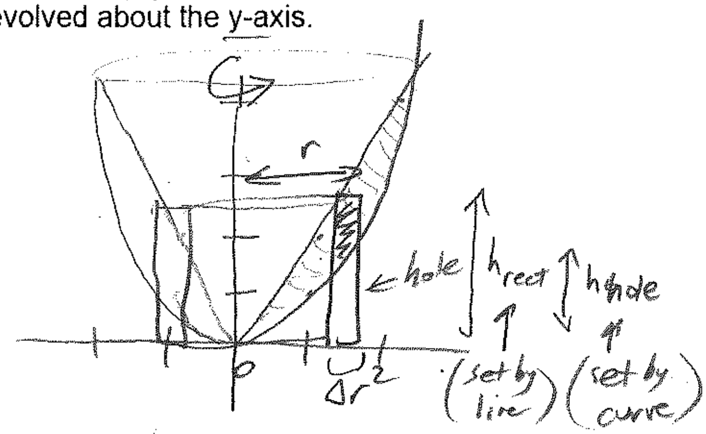


$y = x^2$

$x = \sqrt{y}$

$$V = \int 2\pi r h \Delta r$$
$$= \int 2\pi y x \, dy$$
$$= \int_0^4 2\pi y (\sqrt{y}) dy$$
$$= \boxed{80.4248}$$

4. Bounded by $y = x^2$ *and* $y = 2x$ revolved about the y-axis.



$$V = V_{rect} - V_{hole}$$
$$= \int 2\pi r h \Delta r - \int 2\pi r h \Delta r$$
$$= \int 2\pi x y_{rect} \, dx - \int 2\pi x y_{hole} \, dx$$
$$= \int_0^2 2\pi x (2x) dx - \int_0^2 2\pi x (x^2) dx$$
$$= \int_0^2 2\pi x (2x - x^2) dx$$
$$= \boxed{8.3776}$$