Honors Brief Calculus – Lesson Notes: Trig Limits and Derivatives; L'Hospital's Rule

Trig Limits

Remember:

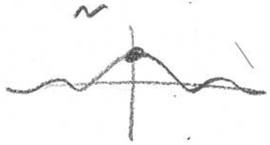
$$\lim_{x\to 0} c = c$$

$$\lim_{x\to c}\left[f(x)\cdot g(x)\right] = \lim_{x\to c} f(x)\cdot \lim_{x\to c} g(x)$$

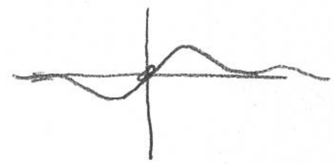
$$\lim_{x\to c}\left[f(x)\right]^n = \left[\lim_{x\to c} f(x)\right]^n$$

Graph to find the following limits. (Window x: −10 to 10, y: −3 to 3 (radian mode))

$$\lim_{x\to 0}\frac{\sin x}{x} = \boxed{1}$$



$$\lim_{x\to 0}\frac{1-\cos x}{x} = \boxed{0}$$



Use the previous limits to find the following.

1. $$\lim_{x\to 0}\frac{\sin\frac{1}{2}x}{x} = \boxed{\frac{1}{2}}$$

$$\lim_{x\to 0}\frac{1}{2}\,\frac{\sin\left(\frac{1}{2}x\right)}{\left(\frac{1}{2}x\right)}$$

$$\frac{1}{2}\cdot 1$$

$$\frac{1}{2}$$

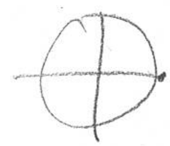
5. $$\lim_{x\to 0}\frac{\tan x}{x} = \boxed{1}$$

$$\lim_{x\to 0}\frac{\sin x}{\cos x}\cdot\frac{1}{x}$$

$$\lim_{x\to 0}\frac{\sin x}{x}\cdot\frac{1}{\cos x}$$

$$1 \cdot 1$$

$$1$$



$\cos 0 = 1$

6. $$\lim_{x\to 0}\frac{\sin 3x}{\sin 2x} = \boxed{\frac{3}{2}}$$

$$\frac{2x}{\sin 2x} = \frac{1}{\left(\frac{\sin 2x}{2x}\right)}$$

$$\lim_{x\to 0}\frac{\sin 3x}{3x}\cdot\frac{2x}{\sin 2x}\cdot\frac{3}{2}$$

$$1\cdot 1\cdot\frac{3}{2}$$

10. $$\lim_{x\to 0}\frac{1-\cos x}{x^2} = \boxed{\frac{1}{2}}$$

$$\lim_{x\to 0}\frac{(1-\cos x)}{x^2}\frac{(1+\cos x)}{(1+\cos x)}$$

$$\lim_{x\to 0}\frac{1-\cos^2 x}{x^2(1+\cos x)}$$

$$\lim_{x\to 0}\frac{\sin^2 x}{x^2(1+\cos x)}$$

$$\lim_{x\to 0}\frac{\sin x}{x}\cdot\frac{\sin x}{x}\cdot\frac{1}{1+\cos x}$$

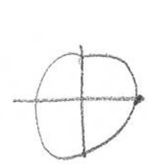
$$1\cdot 1\cdot\frac{1}{1+1}$$

$$1\cdot 1\cdot\frac{1}{2}$$

15. $$\lim_{\theta\to 0}\frac{\sin^2\theta}{\theta} = \boxed{0}$$

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\cdot\sin\theta$$

$$1\cdot 0$$



$\sin 0 = 0$

Trig Derivatives

$$\frac{d}{dx}\sin x = \lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h}$$

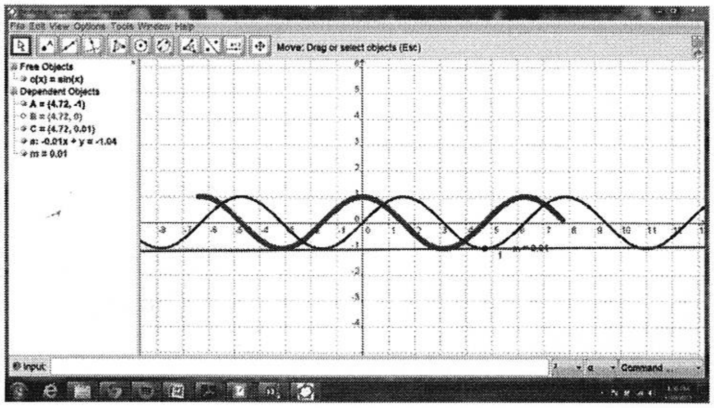
$$\frac{d}{dx}\sin x = \lim_{h\to 0}\frac{\sin x\cos h+\cos x\sin h-\sin(x)}{h}$$

$$\frac{d}{dx}\sin x = \lim_{h\to 0}[-\sin x]\left[\frac{1-\cos h}{h}\right]+\lim_{h\to 0}[\cos x]\left[\frac{\sin h}{h}\right]$$

$$\frac{d}{dx}\sin x = [-\sin x]\lim_{h\to 0}\left[\frac{1-\cos h}{h}\right]+[\cos x]\lim_{h\to 0}\left[\frac{\sin h}{h}\right]$$

$$\frac{d}{dx}\sin x = [-\sin x][0]+[\cos x][1]$$

$$\boxed{\frac{d}{dx}\sin x = \cos x}$$



Geogebra demonstration: Visual Derivative Sine

$$\frac{d}{dx}\cos x = \lim_{h\to 0}\frac{\cos(x+h)-\cos(x)}{h}$$

$$\frac{d}{dx}\cos x = \lim_{h\to 0}\frac{\cos x\cos h-\sin x\sin h-\cos(x)}{h}$$

$$\frac{d}{dx}\cos x = \lim_{h\to 0}[-\cos x]\left[\frac{1-\cos h}{h}\right]-\lim_{h\to 0}[\sin x]\left[\frac{\sin h}{h}\right]$$

$$\frac{d}{dx}\cos x = [-\cos x]\lim_{h\to 0}\left[\frac{1-\cos h}{h}\right]-[\sin x]\lim_{h\to 0}\left[\frac{\sin h}{h}\right]$$

$$\frac{d}{dx}\cos x = [-\cos x][0]-[\sin x][1]$$

$$\boxed{\frac{d}{dx}\cos x = -\sin x}$$

$$\frac{d}{dx}\tan x = \lim_{h\to 0}\frac{\tan(x+h)-\tan(x)}{h}$$

$$\frac{d}{dx}\tan x = \lim_{h\to 0}\frac{\dfrac{\tan x+\tan h}{1-\tan x\tan h}-\tan(x)}{h}$$

$$\frac{d}{dx}\tan x = \lim_{h\to 0}\frac{\left(\dfrac{\tan x+\tan h}{1-\tan x\tan h}-\tan(x)\right)(1-\tan x\tan h)}{h(1-\tan x\tan h)}$$

$$\frac{d}{dx}\tan x = \lim_{h\to 0}\frac{\tan x+\tan h-\tan x+\tan^2 x\tan h}{h(1-\tan x\tan h)}$$

$$\frac{d}{dx}\tan x = \lim_{h\to 0}\frac{\tan h+\tan^2 x\tan h}{h(1-\tan x\tan h)}$$

$$= \lim_{h\to 0}\frac{\tan h(1+\tan^2 x)}{h(1-\tan x\tan h)}$$

$$= \lim_{h\to 0}\frac{\tan h\sec^2 x}{h(1-\tan x\tan h)}$$

$$= \lim_{h\to 0}\frac{\tan h}{h}\frac{\sec^2 x}{(1-\tan x\tan h)}$$

$$= \left[\lim_{h\to 0}\frac{\sin h}{h}\right]\bullet\left[\lim_{h\to 0}\frac{1}{\cos h}\right]\bullet\left[\frac{\sec^2 x}{\left(1-\tan x\left[\lim_{h\to 0}\tan h\right]\right)}\right]$$

$$= [1]\bullet\left[\frac{1}{1}\right]\bullet\lim_{h\to 0}\frac{\sec^2 x}{(1-\tan x[0])}$$

$$\boxed{\frac{d}{dx}\tan x = \sec^2 x}$$

$\frac{d}{dx}\sin x = \cos x$ $\qquad$ $\frac{d}{dx}\csc x = -\csc x \cot x$ $\qquad$ (memorize)

$\frac{d}{dx}\cos x = -\sin x$ $\qquad$ $\frac{d}{dx}\sec x = \sec x \tan x$

$\frac{d}{dx}\tan x = \sec^2 x$ $\qquad$ $\frac{d}{dx}\cot x = -\csc^2 x$

17. Find the equation of the tangent line to

$y = 2\cos x - \cos 2x \quad$ at $\quad x = \frac{\pi}{3}$

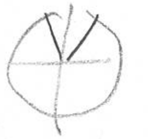
$\frac{dy}{dx} = 2(-\sin x) - (-\sin 2x)(2)$

$= -2\sin x + 2\sin 2x$

$\left.\frac{dy}{dx}\right|_{x=\frac{\pi}{3}} = -2\sin\frac{\pi}{3} + 2\sin\frac{2\pi}{3}$

$= -2\left(\frac{\sqrt{3}}{2}\right) + 2\left(\frac{\sqrt{3}}{2}\right)$

$= \boxed{0 = m}$



point: $y = 2\cos\frac{\pi}{3} - \cos\frac{2\pi}{3}$

$y = 2\left(\frac{1}{2}\right) - \left(-\frac{1}{2}\right)$

$y = 1 + \frac{1}{2}$

$y = \frac{3}{2}$

$\left(\frac{\pi}{3}, \frac{3}{2}\right)$

tangent line:

$\boxed{\left(y - \frac{3}{2}\right) = 0\left(x - \frac{\pi}{3}\right)}$

$y - \frac{3}{2} = 0$

$\boxed{y = \frac{3}{2}}$

(horiz. tangent)

Find the derivative, *dy/dx*

18. $y = \sec 3x$

$\frac{dy}{dx} = \sec 3x \tan 3x (3)$

$= \boxed{3\sec 3x \tan 3x}$

20. $y = \cos^6 x = (\cos x)^6$

$\frac{dy}{dx} = 6(\cos x)^5(-\sin x)$

$= \boxed{-6\sin x \cos^5 x}$

24. $y = \left(\ln\left(\cos e^{3x}\right)\right)^4$

$\frac{dy}{dx} = 4\left[\ln(\cos(e^{3x}))\right]^3 \frac{1}{\cos(e^{3x})}(-\sin(e^{3x}))(e^{3x})(3)$

$= \frac{-12\sin(e^{3x})\left[\ln(\cos(e^{3x}))\right]^3}{\cos(e^{3x})}$

$= \boxed{-12\tan(e^{3x})\left[\ln(\cos(e^{3x}))\right]^3}$

L'Hospital's Rule

$\lim_{x \to 1} \frac{\ln x}{x-1}$ The 'divide everything by largest degree x from denominator' trick doesn't work here.

Cases like this are called 'Indeterminate forms':

$\frac{0}{0}$ and $\frac{\infty}{\infty}$

For indeterminate forms, we can use **L'Hospital's Rule**

Pronounced "L-oh-pe-tal" (French)

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

In other words, find the derivative of the numerator and the derivative of the denominator and try again.

Find the limit using L'Hospital's Rule.

$\lim_{x \to \infty} \frac{2x^2 - 2}{x+1}$ $\left(\frac{\infty}{\infty}\right)$

$\lim_{x \to \infty} \frac{4x}{1} = \frac{\infty}{1} = \boxed{\infty}$

25. $\lim_{x \to 1} \frac{\ln x}{x-1}$ $\left(\frac{0}{0}\right)$

$\lim_{x \to 1} \frac{\frac{1}{x}}{1} = \frac{1}{1} = \boxed{1}$

27. $\lim_{\theta \to 0} \frac{\tan \theta}{\theta}$ $\left(\frac{0}{0}\right)$

$\lim_{\theta \to 0} \frac{\sec^2 \theta}{1} = \frac{\frac{1}{\cos^2 \theta}}{1} = \frac{1}{1} = \boxed{1}$

28. $\lim_{x \to 0} \frac{e^x - 1}{\sin x}$ $\left(\frac{0}{0}\right)$

$\lim_{x \to 0} \frac{e^x}{\cos x} = \frac{1}{1} = \boxed{1}$

29. $\lim_{x\to 3} \frac{x-3}{3x^2-13x+12}$ $\frac{(0)}{(0)}$

$\lim_{x\to 3} \frac{1}{6x-13} = \boxed{\frac{1}{5}}$

30. $\lim_{t\to 0} \frac{te^t}{1-e^t}$ $\left(\frac{0}{0}\right)$

$\lim_{t\to 0} \frac{te^t+e^t(1)}{-e^t}$

$\lim_{t\to 0} \frac{e^t(t+1)}{e^t(-1)}$

$\lim_{t\to 0} -(t+1) = \boxed{-1}$

34. $\lim_{x\to 0} \frac{x-\ln(x+1)}{1-\cos 2x}$ $\left(\frac{0}{0}\right)$

$\lim_{x\to 0} \frac{1-\frac{1}{x+1}}{2\sin 2x}$ $\left(\frac{0}{0}\right)$ $= \frac{1-(x+1)^{-1}}{2\sin 2x}$

(again) $\lim_{x\to 0} \frac{\frac{1}{(x+1)^2}}{4\cos 2x} = \frac{\frac{1}{(0+1)^2}}{4(1)} = \boxed{\frac{1}{4}}$

31. $\lim_{x\to 0^+} \frac{e^x+e^{-x}-2}{1-\cos 2x}$ $\left(\frac{0}{0}\right)$

$\lim_{x\to 0^+} \frac{e^x-e^{-x}}{2\sin 2x}$ $\left(\frac{0}{0}\right)$

(again) $\lim_{x\to 0^+} \frac{e^x+e^{-x}}{4\cos 2x} = \frac{2}{4(1)} = \boxed{\frac{1}{2}}$

35. $\lim_{x\to 0} \frac{2-x^2-2\cos x}{x^4}$ $\left(\frac{0}{0}\right)$

$\lim_{x\to 0} \frac{-2x+2\sin x}{4x^3}$ $\left(\frac{0}{0}\right)$

(again) $\lim_{x\to 0} \frac{-2+2\cos x}{12x^2}$ $\left(\frac{0}{0}\right)$

(again) $\lim_{x\to 0} \frac{-2\sin x}{24x}$ $\left(\frac{0}{0}\right)$

(again) $\lim_{x\to 0} \frac{-2\cos x}{24} = \frac{-2(1)}{24} = \boxed{-\frac{1}{12}}$