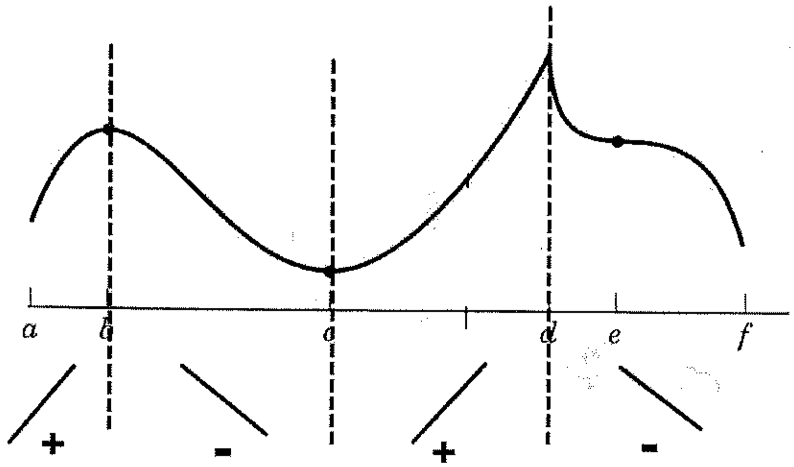
# Honors Brief Calculus – Lesson Notes: Unit 12 – Derivative Applications

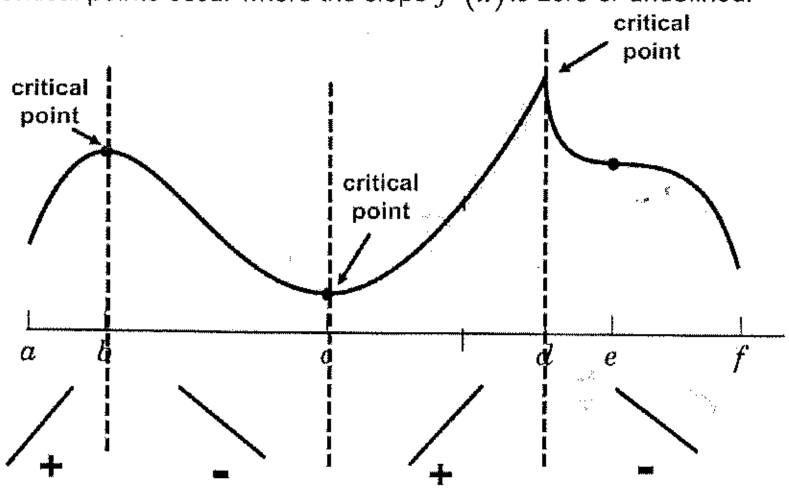
## *5.1/5.2/5.3 Curve Sketching*

The **first derivative** of a function is the instantaneous rate of change, or **slope**, of the function at each x value:




*sign of* $f'(x)$
***'slope map'***

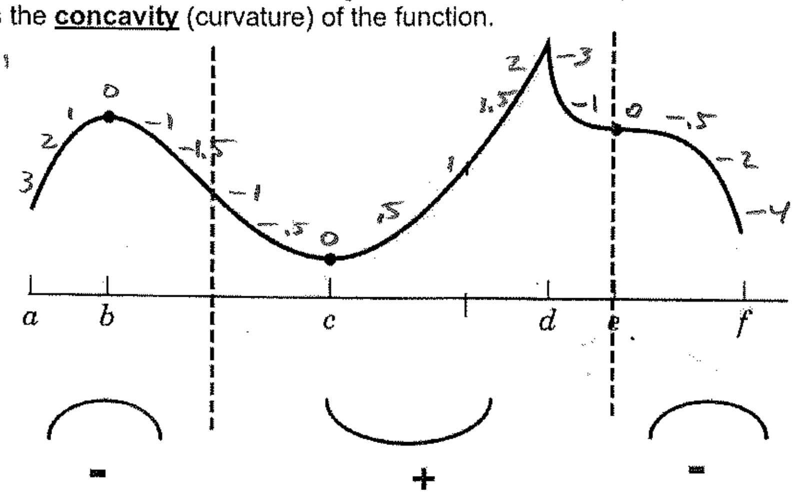
The places where slope changes sign are called 'critical points' (or critical values is there is an x, but no defined point). Critical points occur where the slope $f'(x)$ is zero or undefined.




*sign of* $f'(x)$
***'slope map'***

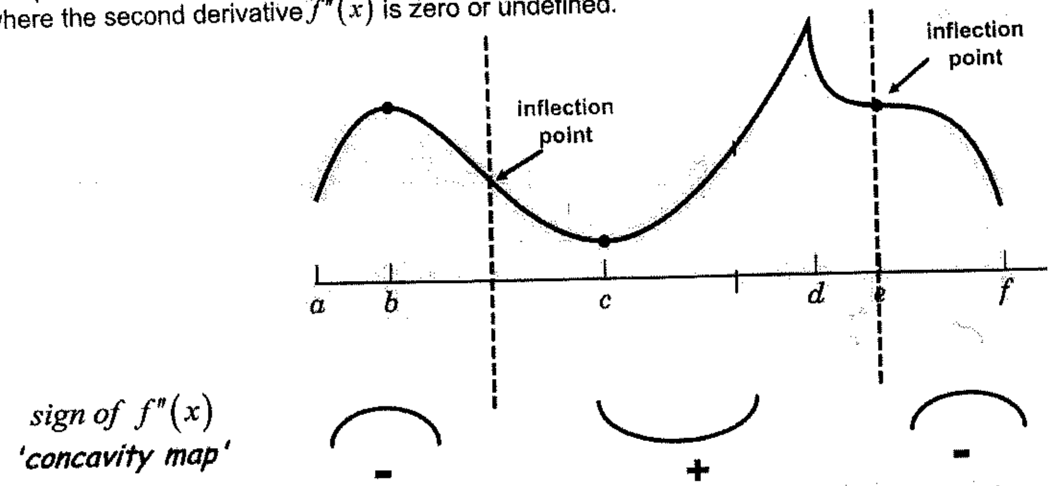
The second derivative of a function also has a meaning and relates to the shape of a function. **Second derivative** gives the **concavity** (curvature) of the function.

add "slopes"

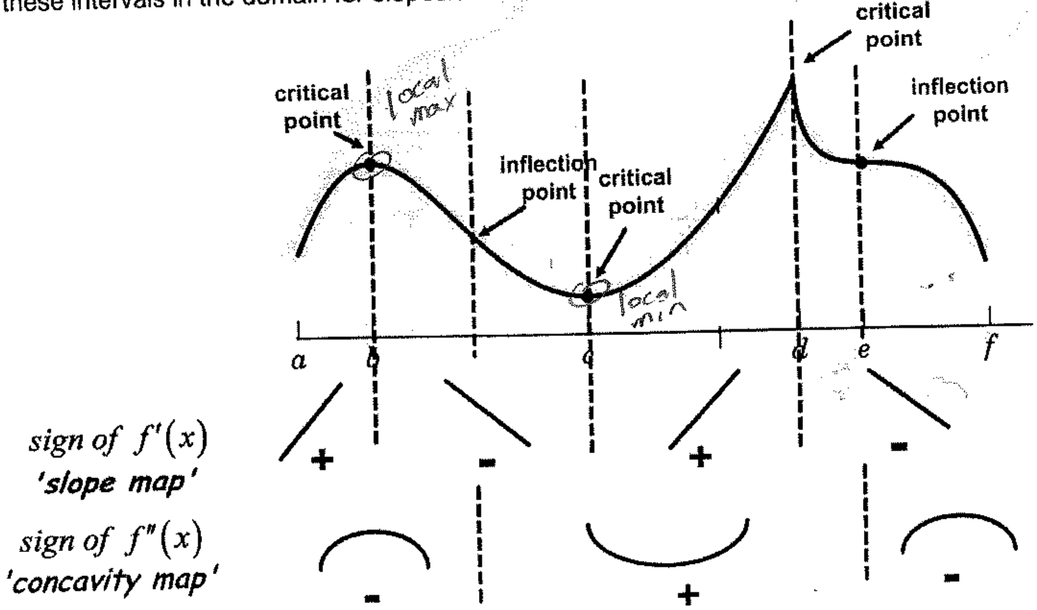



*sign of* $f''(x)$
***'concavity map'***

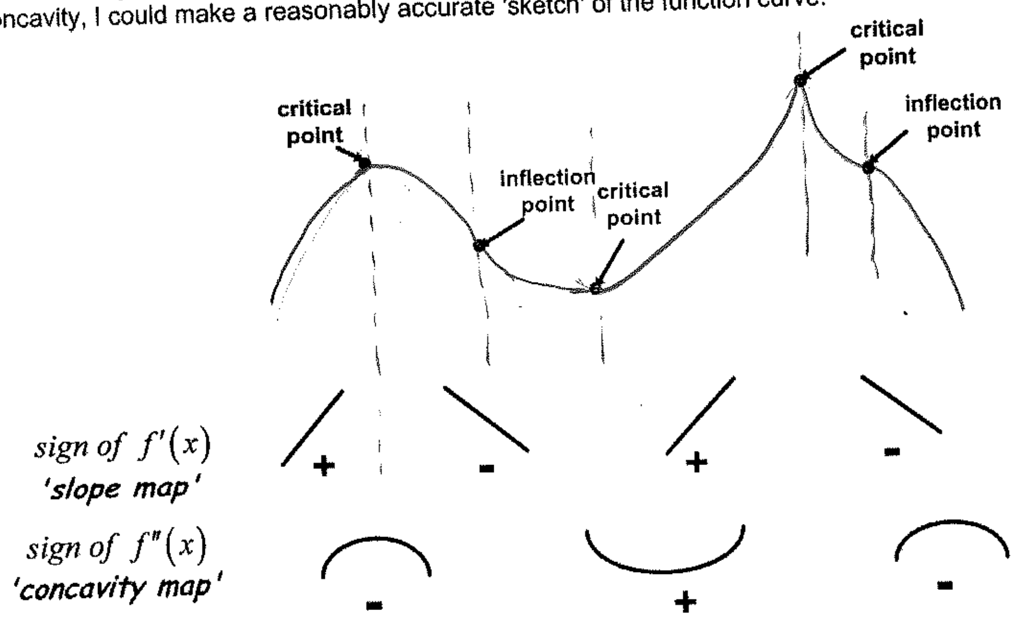
The places where concavity changes sign are called 'inflection points'. Inflection points occur where the second derivative $f''(x)$ is zero or undefined.




Note that the inflection points and the critical points are not necessarily at the same x values, so these intervals in the domain for slopes and for concavities do not always 'line up':



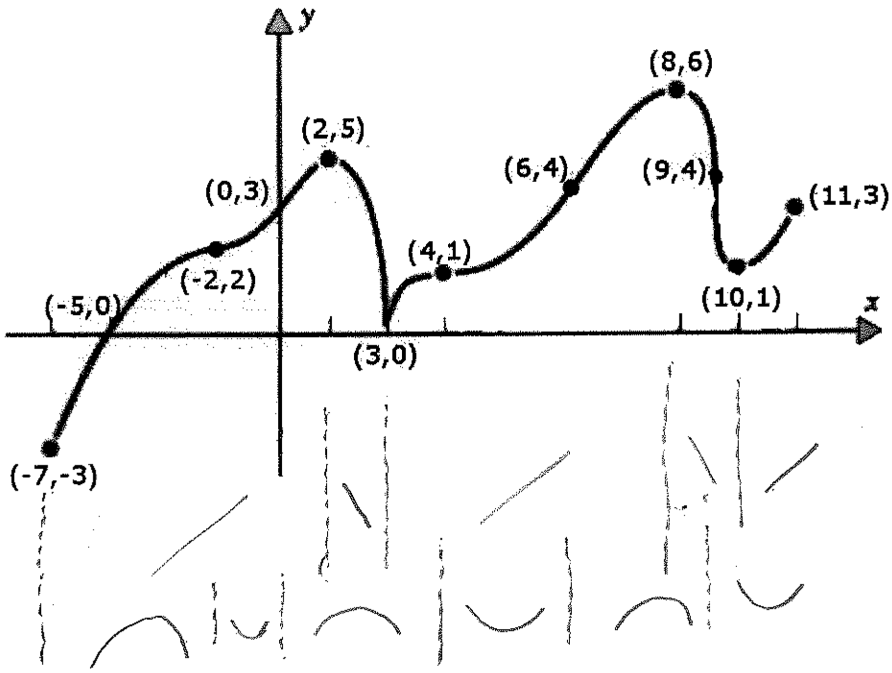

But if I were given just the critical and inflection points and the maps of the slopes and concavity, I could make a reasonably accurate 'sketch' of the function curve:



Tomorrow, we will learn a step-by-step procedure we will use to identify critical points, inflection points and slope and concavity maps from a function given in equation form.

Today, we're going to practice identifying slope and concavity maps, and critical and inflection points from a given sketch and sketching curves given these data.

#1) Add slope and concavity maps




slope map:

concavity map:

Find the following:

a) domain $[-7, 11]$

b) intercepts $(-5,0)$ $(0,3)$ $(3,0)$

c) increasing intervals $(-7,2) \cup (3,8) \cup (10,11)$

d) decreasing intervals $(2,3) \cup (8,10)$

e) for what numbers does $f'(x)$ not exist $(3,0)$ (possibly $(9,4)$)

f) local max points $(2,5)$ $(8,6)$ $(11,3)$

g) local min points $(-7,-3)$ $(3,0)$ $(10,1)$

h) critical points $(2,5)$ $(3,0)$ $(8,6)$ $(10,1)$

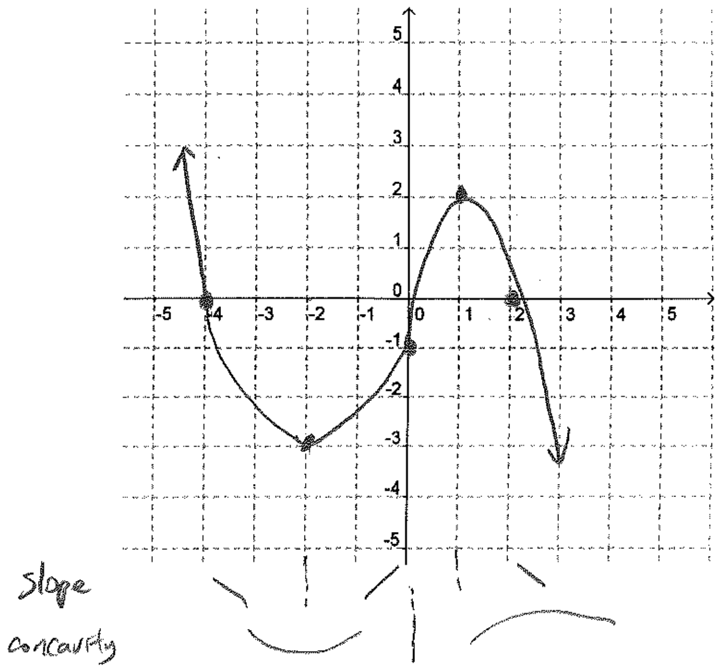
i) intervals where concave up $(-2,0) \cup (4,6) \cup (9,11)$

j) intervals where concave down $(-7,-2) \cup (2,4) \cup (6,9)$

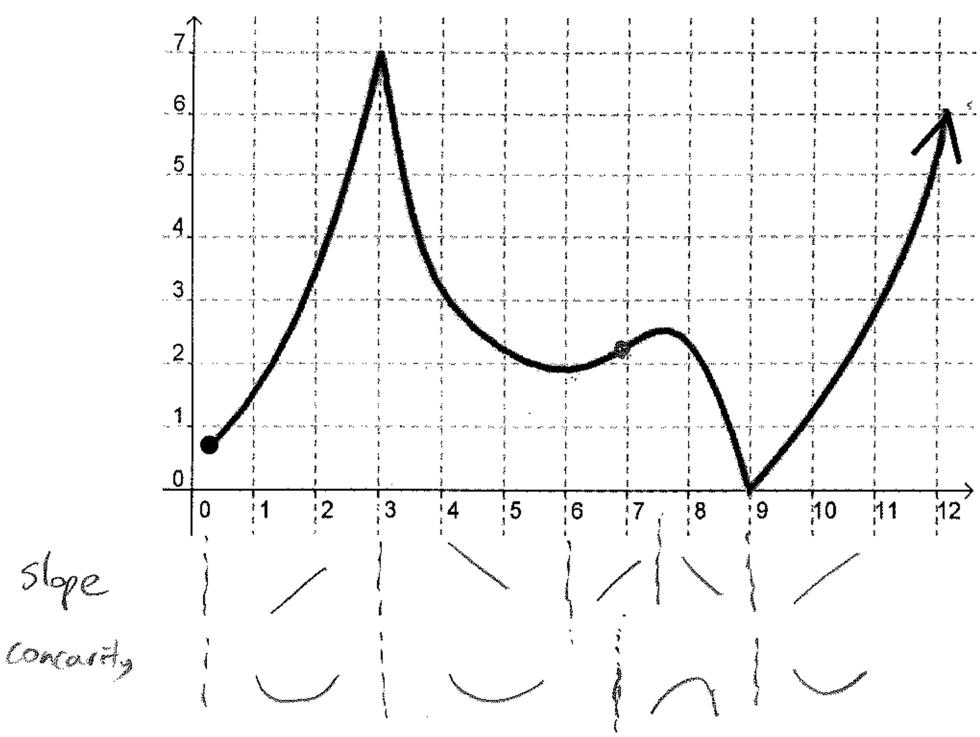
points: $(3,0)$
intervals: $(2,3)$
look the same
(must know the difference based on context)

#2) Given the following info, draw axes, add slope and concavity maps below, and sketch the function curve.

intercepts at (-4,0) (2,0) (0,-1)
local max at (1,2)
local min at (-2,-3)
inflection point at (0,-1)
increasing: (-2,1)
decreasing: $(-\infty,-2) \cup (1,\infty)$
concave up: $(-\infty,0)$
concave down: $(0,\infty)$



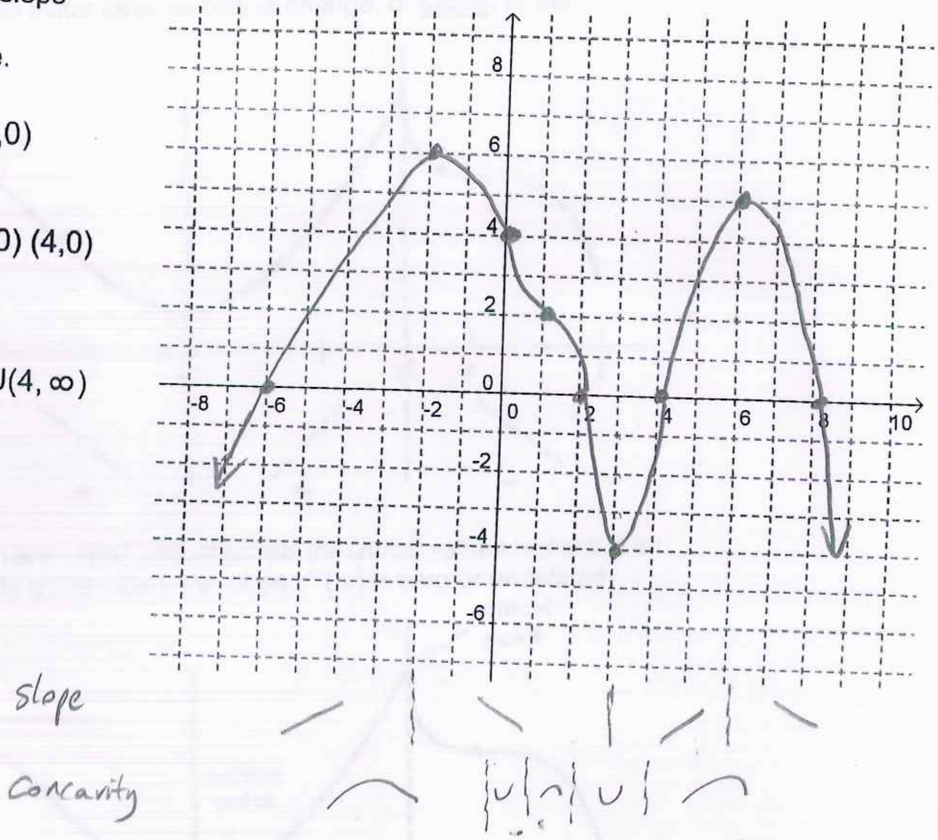

#3)
a) Give the approximate coordinates of all critical points, local maxima and minima, and inflection points.
b) Give the approximate x-intervals where the function is increasing, decreasing, concave up, concave down.




critical points (slope = 0 or undef): (3,7) (6,2) (7.5,2.5) (9,0)
local max: (3,7) (7.5,2.5)
local min: (0.2,0.8) (6,2) (9,0)
inflection points (concavity = 0 or undef): (3,7) (7,2.2) (9,0)
increasing: $(0.2,3) \cup (6,7.5) \cup (9,\infty)$
decreasing: $(3,6) \cup (7.5,9)$
concave up: $(0.2,3) \cup (3,7) \cup (9,\infty)$
concave down: $(7,9)$

#4) Given the following info, add slope and concavity maps below, and sketch the function curve.

intercept: (0,4) (2,0) (8,0) (-6,0)
local max: (-2,6) (6,5)
local min: (3,-4)
inflection point: (0,4) (1,2) (2,0) (4,0)
increasing: (-∞,-2)U(3,6)
decreasing: (-2,3)U(6,∞)
concave up: (0,1)U(2,4)
concave down: (-∞,0)U(1,2)U(4,∞)



## 5.1/5.2/5.3 Curve Sketching – The First and Second Derivative Tests

We can sketch a function curve if we can obtain intercepts, critical points, inflection points, and intervals where we know the curve is increasing, decreasing, concave up, and concave down.

If we have the equation of a function, we can obtain these things using the first and second derivatives of the function. This is referred to as using the **first and second derivative tests**.

**First derivative test (slope)**

If $f'(x) > 0$, then $f$ is increasing in that interval.

If $f'(x) < 0$, then $f$ is decreasing in that interval.

**Critical points** (where slope changes sign) occur where:

$f'(x) = 0$ *or* $f'(x)$ *is undefined*

**Second derivative test (concavity)**

If $f''(x) > 0$, then $f$ is concave up in that interval.

If $f''(x) < 0$, then $f$ is concave down in that interval.

**Inflection points** (where concavity changes sign) occur where:

$f''(x) = 0$ *or* $f''(x)$ *is undefined*

We'll follow a step-by-step procedure to use these derivative tests to obtain the info we need to sketch a function given it's equation:

Precalc

1) State the domain.
2) Find the x- and y-intercepts.

1st derivative

3) Compute $f'(x)$ and find critical values where $f'(x) = 0$ *or* $f'(x)$ *is undefined*
4) Divide the domain into intervals split at the critical values and test the sign of $f'(x)$ in each interval to make a slope map.
5) Plug critical value x's into orig. function and list local max and min points.

2nd derivative

6) Compute $f''(x)$ and find inflection point x-values where $f''(x) = 0$ *or* $f''(x)$ *is undefined*
7) Divide the domain into intervals split at the inflection points and test the sign of $f''(x)$ in each interval to make a concavity map.
8) Plug inflection x's into orig. function to get inflection points, then sketch.

Sketch the function curve for $f(x) = x^3 - 3x^2 + 4$

1) D: $(-\infty, \infty)$

2) y int (x=0)

$f(0) = 4$

$\boxed{(0,4)}$

x int (y=0)

$x^3 - 3x^2 + 4 = 0$

$(x+1)(x-2)^2 = 0$

$\boxed{(-1,0)\ (2,0)}$

from zero
testing/
synth div

$f(0) = 4$

$f(2) = (2)^3 - 3(2)^2 + 4$

$= 8 - 12 + 4$

$= 0$

3) $f'(x) = 3x^2 - 6x$

critical values:

$f'(x) = 0$ $\quad$ $f'(x)$ undef

$3x^2 - 6x = 0$

$(3x)(x-2) = 0$

$\left.\begin{array}{l} x = 0 \\ x = 2 \end{array}\right\}$ critical values

4) $(-\infty, 0)(0,2)(2,\infty)$

test: $f'(x) = (3x)(x-2)$

at: $-1 \quad 1 \quad 3$

$(-)(-) \quad (+)(-) \quad (+)(+)$

$+ \quad - \quad +$

slope: / \ /

$(-\infty,0)\ (0,2)\ (2,\infty)$

$(0,4)\ (2,0)$

5) $(0,4)$ is local max
$(2,0)$ is local min

6) $f''(x) = 6x - 6$

inflection values:

$f''(x) = 0$ $\quad$ $f''(x)$ undef

$6x - 6 = 0$

$6x = 6$

$\left.x = 1\right\}$ inflection value

7) $(-\infty, 1)(1, \infty)$

test: $f''(x) = 6x - 6$

at: $0 \quad 2$

$(-) \quad (+)$

concavity

$(-\infty, 1)\ (1,\infty)$

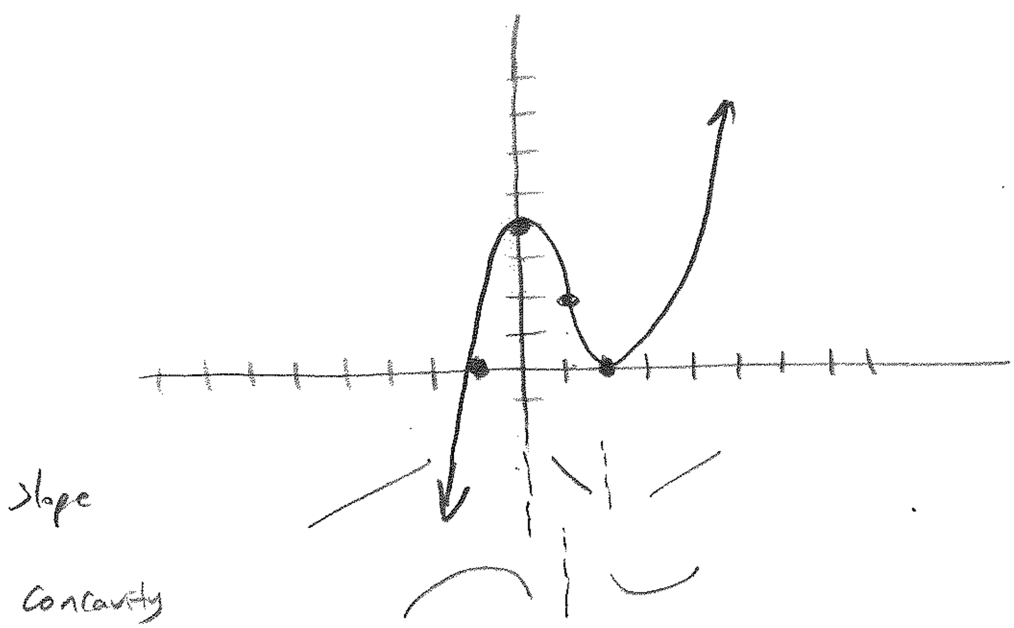
$(1,2)$

$f(1) = (1)^3 - 3(1)^2 + 4$

$= 1 - 3 + 4$

$= 2$

8) $\boxed{(1,2) \text{ is inflection point}}$



Sketch the function curve for $f(x) = 4x^3 + 6x^2$

1) D: $(-\infty, \infty)$

2) $\underline{\text{y int } (x=0)}$

$f(0) = 0$

$\boxed{(0,0)}$

$\underline{\text{x int } (y=0)}$

$4x^3 + 6x^2 = 0$

$(2x^2)(2x+3) = 0$

$x = 0 \quad 2x+3 = 0$

$x = \frac{-3}{2}$

$\boxed{(0,0) \ (-\tfrac{3}{2}, 0)}$

$f(-1) = 4(-1)^3 + 6(-1)^2$

$= -4 + 6$

$= 2$

$f(0) = 0$

3) $f'(x) = 12x^2 + 12x$

critical values:

$\underline{f'(x) = 0} \qquad \underline{f'(x) \text{ undef}}$

$12x^2 + 12x = 0$

$(12x)(x+1) = 0$

$\left.\begin{array}{l} x = 0 \\ x = -1 \end{array}\right\}$ critical values

4) $(-\infty, -1)(-1, 0)(0, \infty)$

test $f'(x) = (12x)(x+1)$

at: $-2 \quad -\frac{1}{2} \quad 1$

$(-)(-) \quad (-)(+) \quad (+)(+)$

$+ \quad - \quad +$

$\boxed{\text{slope: } \nearrow \ \searrow \ \nearrow \quad (-\infty, -1)\ (-1, 0)\ (0, \infty)}$

$(-1, 2) \quad (0, 0)$

5) $\boxed{\begin{array}{l} (-1, 2) \text{ is local max} \\ (0, 0) \text{ is local min} \end{array}}$

6) $f''(x) = 24x + 12$

inflection values:

$\underline{f''(x) = 0} \qquad \underline{f''(x) \text{ undef}}$

$24x + 12 = 0$

$24x = -12$

$\left. x = -\frac{1}{2} \right\}$ inflection value

7) $(-\infty, -\frac{1}{2})(-\frac{1}{2}, \infty)$

test $f''(x) = 24x + 12$

at $-1 \quad 0$

$(-) \quad (+)$

$\boxed{\text{concavity: } \frown \ \smile \quad (-\infty, -\tfrac{1}{2})\ (-\tfrac{1}{2}, \infty)}$

$(-\frac{1}{2}, 1)$ $\quad f(-\frac{1}{2}) = 4(-\frac{1}{2})^3 + 6(-\frac{1}{2})^2$

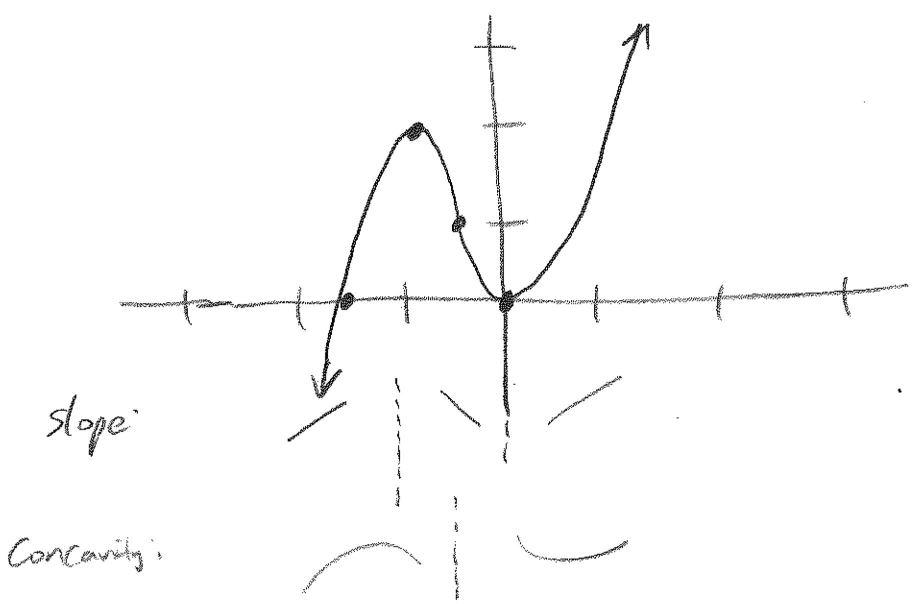
$= 4(-\frac{1}{8}) + \frac{6}{4}$

$= -\frac{1}{2} + \frac{6}{4}$

$= -\frac{2}{4} + \frac{6}{4}$

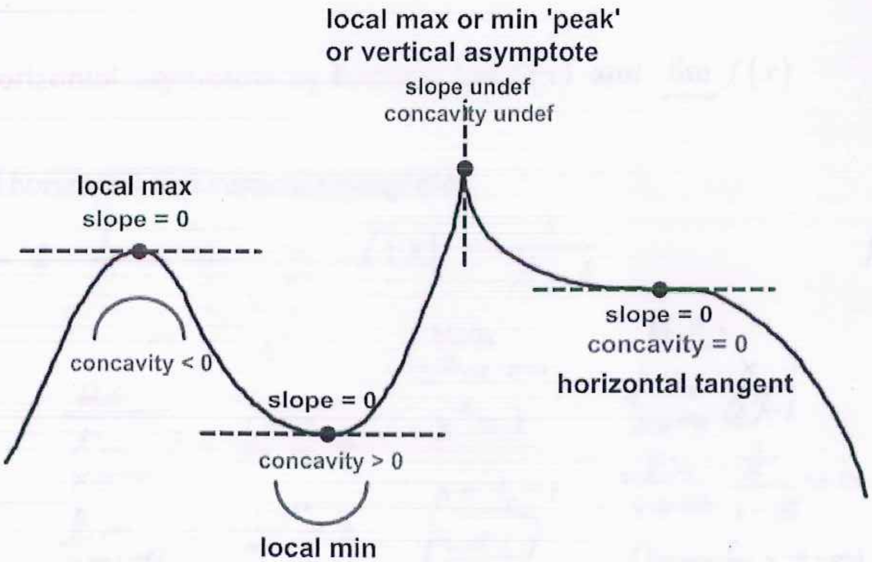
$= \frac{4}{4} = 1$

8) $\boxed{(-\tfrac{1}{2}, 1) \text{ is an inflection point}}$



## *5.1/5.2/5.3 Finding Extreme, Applications*

We we sketch functions, 'point of interest' always occur at critical points where the first derivative is either zero or undefined. If we take the look at the second derivative at these x=values we can quickly see what the curve is doing at this x-value:




<u>To investigate points of interest</u>:

1) Compute $f'(x)$ and find critical x-values where $f'(x)=0$ *or is undefined*

2) Plug these x-values into the original equation to find matching y values.

3) Compute $f''(x)$ and find the concavity at each critical x-value.

4) Use first and second derivative results to determine what curve is doing:

$f'(x)=0$ and $f''(x)<0$ = local maximum

$f'(x)=0$ and $f''(x)>0$ = local maximum

$f'(x)=0$ and $f''(x)=0$ = horizontal tangent

$f'(x)$ *is undefined* = local max or min 'peak' or vertical asymptote

Determine where $f'(x)=0$. Use the Second Derivative Test to determine the local maxima and local minima of each function.

48. $f(x)=x^3-12x-4$

$f'(x)=3x^2-12$

$3x^2-12=0$

$3x^2=12$

$x^2=4$

$x=+2,-2$

(critical values)

$f(2)=(2)^3-12(2)-4$
$=8-24-4$
$=-20$

$f(-2)=(-2)^3-12(-2)-4$
$=-8+24-4$
$=12$

$(2,-20)$ $(-2,12)$

$f''(x)=6x$

$f''(2)=6(2)$
$=12$
$+$
concave up so

$\boxed{(2,-20) \text{ is a local minimum}}$

$f''(-2)=6(-2)$
$=-12$
$-$
concave down so

$\boxed{(-2,12) \text{ is a local maximum}}$

Rational functions often have **asymptotes**.
These can also help when we sketch.

**Vertical Asymptotes occur at uncancelled zero in denominator.**

**Find horizontal asymptote by finding** $\lim_{x\to\infty} f(x)$ **and** $\lim_{x\to-\infty} f(x)$

Find all horizontal and vertical asymptotes.

$f(x) = 2 - \frac{1}{x^2} = \frac{2x^2}{x^2} - \frac{1}{x^2} = \frac{2x^2-1}{x^2}$

V.A.

$x^2=0$

$\boxed{x=0}$

H.A.

$\lim_{x\to\infty} 2 - \frac{1}{x^2} = 2$

$\lim_{x\to-\infty} 2 - \frac{1}{x^2} = 2$

$\boxed{y=2}$

$f(x) = \frac{x}{x^2-1}$

V.A.

$x^2-1=0$

$x^2=1$

$x=1,-1$

$\boxed{x=1}$

$\boxed{x=-1}$

H.A.

$\lim_{x\to\infty} \frac{x}{x^2-1}$

$=\lim_{x\to\infty} \frac{\frac{1}{x}}{1-\frac{1}{x^2}} = 0$

(same for $x\to-\infty$)

$\boxed{y=0}$

$f(x)=\frac{5}{1+e^{-x}}$

V.A.

$1+e^{-x}=0$

$e^{-x}=-1$

$\ln(e^{-x})=\ln(-1)$

$-x=$ undef

$\boxed{\text{no V.A.}}$

H.A.

$\lim_{x\to\infty} \frac{5}{1+e^{-x}}$

$\frac{5}{1+e^{-\infty}}$

$=\frac{5}{1+\frac{1}{e^{\infty}}} = 5$

$\lim_{x\to-\infty} \frac{5}{1+e^{-x}}$

$\frac{5}{1+e^{\infty}} = 0$

$\boxed{y=5}$ on RH

$\boxed{y=0}$ on LH

If we have a rational function, we find asymptotes in the 'precalc' part of the sketching procedure:

| Precalc | 1st derivative | 2nd derivative |
|---|---|---|
| 1) State the domain.<br>2) Find the x- and y-intercepts.<br>**2b) Find vertical and horizontal asymptotes** | 3) Compute $f'(x)$ and find critical values where $f'(x)=0$ *or* $f'(x)$ *is undefined*<br>4) Divide the domain into intervals split at the critical values and test the sign of $f'(x)$ in each interval to make a slope map.<br>5) Plug critical value x's into orig. function and list local max and min points. | 6) Compute $f''(x)$ and find inflection point x-values where $f''(x)=0$ *or* $f''(x)$ *is undefined*<br>7) Divide the domain into intervals split at the inflection points and test the sign of $f''(x)$ in each interval to make a concavity map.<br>8) Plug inflection x's into orig. function to get inflection points, then sketch. |

64. **Demand Function** A certain item can be produced at a cost of \$10 per unit. The demand equation for this item is

$$p = \cancel{d(x)} = 90 - .02x$$

where p is the price in dollars and x is the number of units. Find

a. The revenue function  b. The maximum revenue

c. the profit function  d. the maximum profit.

a) $R(x) = px = \boxed{90x - .02x^2}$

b) $R'(x) = 90 - .04x = 0$

$.04x = 90$

$x = 2250$

$R''(x) = -.04$ ∩ local max

max Revenue occurs at $X = 2250$ units

$R(2250) = 90(2250) - .02(2250)^2 = \boxed{\$101250}$

c) $P(x) = R(x) - C(x)$

$= 90x - .02x^2 - 10x$

$= \boxed{80x - .02x^2}$

d) $P'(x) = 80 - .04x = 0$

$.04x = 80$

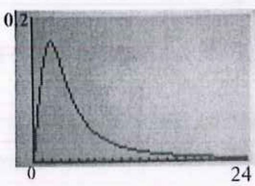
$x = 2000$

$P''(x) = -.04$ ∩ (max)

$P(2000) = 80(2000) - .02(2000)^2 = \boxed{\$80000}$

45. The concentration $C$ of a certain drug in the bloodstream $t$ hours after an injection into muscle tissue is given by

$$C(t) = \frac{2t}{16+t^3}$$




When is the concentration greatest?

(quotient rule)

$$C'(t) = \frac{(16+t^3)\frac{d}{dt}[2t] - (2t)\frac{d}{dt}[16+t^3]}{(16+t^3)^2}$$

$$= \frac{(16+t^3)2 - (2t)(3t^2)}{(16+t^3)^2}$$

$$= \frac{32 + 2t^3 - 6t^3}{(16+t^3)^2}$$

$$= \boxed{\frac{32 - 4t^3}{(16+t^3)^2}} = 0$$

$32 - 4t^3 = 0$

$4t^3 = 32$

$t^3 = 8$

$t = 2$ (point of interest)

$$C(t) = \frac{2(2)}{16+(2)^3} = \boxed{.16667 \text{ max concentration at } t = 2 \text{ hours}}$$

if you really want to be sure $t=2$ is a max, could check 2nd deriv.

$$C'(t) = \frac{32 - 4t^3}{(16+t^3)^2}$$

$$C''(t) = \frac{(16+t^3)^2(-12t^2) - (32-4t^3)(2(16+t^3)(3t^2))}{(16+t^3)^4}$$

$$C''(2) = \frac{(576)(-48) - (0)(-)}{331776} = -.083$$

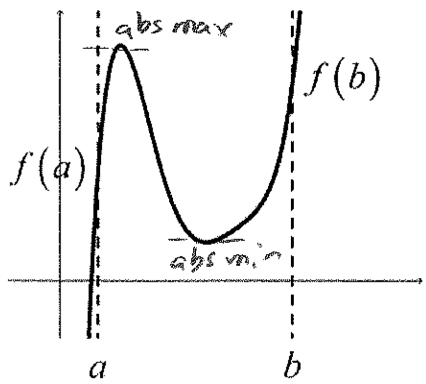
(plugging in without simplifying)

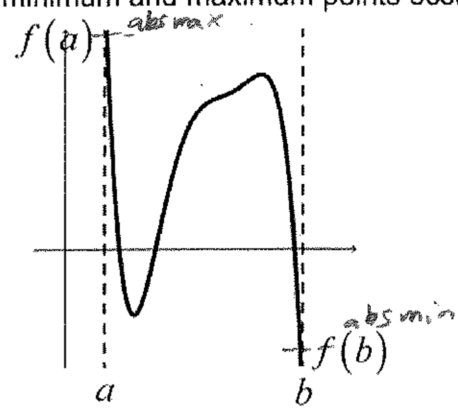
∩

so $t = 2$ is a max

## 5.4 – Optimization

Within an x-interval, where can the absolute minimum and maximum points occur?




In a given interval, absolute max and min are either at local extrema or at the ends of the interval.

Test for absolute minimum/maximum of a continuous function $f$ on a closed interval $[a,b]$

**Step 1:** Find all critical numbers in $(a,b)$ (where $f'(x)=0$ *or DNE*).

**Step 2:** Compute the values of $f$ at each of these critical points of f and compute $f(a)$, $f(b)$.

**Step 3:** Select the largest and smallest values of $f$ obtained in step 2.

$f(x)=x^2-8x \quad on\,[-1,10]$

1) $f'(x)=2x-8$

$2x-8=0$

$2x=8$

$x=4$

$f(4)=(4)^2-8(4)$
$=16-32$
$=-16$

$(4,-16)$

2) at ends of interval

$f(-1)=(-1)^2-8(-1)$
$=1+8$
$=9$

$(-1,9)$

$f(10)=(10)^2-8(10)$
$=100-80$
$=20$

$(10,20)$

| x | y | |
|---|---|---|
| 4 | -16 | ← abs min at $(4,-16)$ |
| -1 | 9 | |
| 10 | 20 | ← abs max at $(10,20)$ |

## Optimization application problems

**Step 1:** Identify the quantity to be maximized or minimized.

**Step 2:** Assign symbols to represent other variables in the problem.
*Use an illustration to assist you.*

**Step 3:** Determine the relationships among these variables.

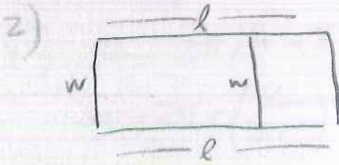
**Step 4:** Express the quantity to be optimized as a function of <u>**one**</u> of these variables. (Be sure to state the domain).

**Step 5:** Apply the test for absolute maximum and absolute minimum to this function on the interval of the domain.

30. A farmer wants to enclose 6000 **square meters** of land in a rectangular plot and then divide it into two plots with a fence parallel to one of the sides. What are the dimensions of the rectangular plot that require the **least** amount of **fence**?

1) minimize fencing

2)




3) $A = l \cdot w$ $\quad F = 3w + 2l$

$6000 = l \cdot w$

$l = \frac{6000}{w}$

4) $F = 3w + 2l$

$F(w) = 3w + 2\left(\frac{6000}{w}\right)$

domain $F$: $(0, \infty)$

5) $F(w) = 3w + 12000w^{-1}$

$F'(w) = 3 - 12000w^{-2}$

$F'(w) = 3 - \frac{12000}{w^2}$

critical points:

$F'(w) = 0$

$\frac{12000}{w^2} = 3$

$3w^2 = 12000$

$w^2 = 4000$

$w = \sqrt{4000}$

$F'(w)$ DNE

$w^2 = 0$

$w = 0$

values at critical points and interval ends

| $w$ | $F(w)$ |
|---|---|
| $0$ | $F(0) = 3(0) + 2\left(\frac{6000}{0}\right)$ undef |
| $\sqrt{4000}$ | $F(\sqrt{4000}) = 3\sqrt{4000} + 2\left(\frac{6000}{\sqrt{4000}}\right) = 379.47$ |

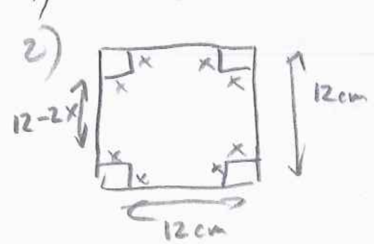
$\boxed{\text{width} = \sqrt{4000} = 63.246 \text{ m}}$

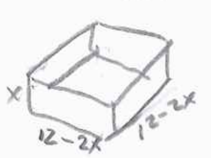
$\boxed{\text{length} = \frac{6000}{\sqrt{4000}} = 94.868 \text{ m}}$

32. An open box with a square is to be made from a square piece of cardboard 12 cm on a side by cutting out a square from each corner and turning up the sides. Find the dimensions of the box that yield the maximum volume.

1) maximize volume

2)




3) $V = l \cdot w \cdot h$

$V = x(12-2x)(12-2x)$

4) $v(x) = x(144 - 48x + 4x^2)$

$v(x) = 4x^3 - 48x^2 + 144x$

domain $v$: $(0, 6)$

5) $v'(x) = 12x^2 - 96x + 144$

critical points

$v'(x) = 0$

$12x^2 - 96x + 144 = 0$

$12(x^2 - 8x + 12) = 0$

$12(x-2)(x-6) = 0$

$x = 2, x = 6$

$v'(x)$ DNE (none)

values at critical points and interval ends

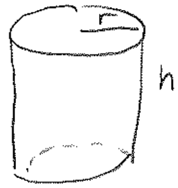
| | $x$ | $v(x)$ |
|---|---|---|
| critical | $2$ | $v(2) = 4(2)^3 - 48(2)^2 + 144(2) = 128$ |
| | $6$ | $v(6) = 4(6)^3 - 48(6)^2 + 144(6) = 0$ |
| end | $0$ | $v(0) = 4(0)^3 - 48(0)^2 + 144(0) = 0$ |

$\boxed{V_{max} = 128 \text{ cm}^3 \text{ when } x = 2 \text{ cm}}$

49. Prove that a cylindrical container of fixed volume $V$ requires the least material (minimum surface area) when its height is twice its radius.

1) minimize surface area

2)




lateral side  two ends

3) $V = \pi r^2 h \quad SA = \underbrace{2\pi r h}_{\text{lateral side}} + \underbrace{2(\pi r^2)}_{\text{two ends}}$

everything in terms of r

$V_{(\text{fixed})} = \pi r^2 h$

$h = \frac{V}{\pi r^2}$

4) $SA = 2\pi r h + 2\pi r^2$

$SA = 2\pi r\left(\frac{V}{\pi r^2}\right) + 2\pi r^2$

$SA = \frac{2\pi r V}{\pi r^2} + 2\pi r^2$

$SA = \frac{2V}{r} + 2\pi r^2$

$SA(r) = 2Vr^{-1} + 2\pi r^2$ (domain: $(0,\infty)$)

$SA'(r) = -2Vr^{-2} + 4\pi r$

$SA'(r) = \frac{-2V}{r^2} + 4\pi r$

5) critical points

$SA'(r) = 0$

$\frac{-2V}{r^2} + 4\pi r = 0$

$4\pi r = \frac{2V}{r^2}$

$r^3 = \frac{2V}{4\pi}$

$r^3 = \frac{V}{2\pi}$

but $V = \pi r^2 h$

so $r^3 = \frac{\pi r^2 h}{2\pi}$

$r^3 = \frac{1}{2} r^2 h$

$r = \frac{1}{2} h$

or $h = 2r$
height is twice radius

more detail: solving for $SA'(r) = 0$ finds a critical point but this could be a minimum or maximum.

To show this is a minimum, could check concavity with 2nd derivative:

$SA'(r) = -2Vr^{-2} + 4\pi r$

$SA''(r) = 4Vr^{-3} + 4\pi$

$SA''(r) = \frac{4V}{r^3} + 4\pi$

since V and r are positive $SA''$ is positive concavity is concave up ∪ so this local extremum is a local minimum

## *5.6 – Related Rates*

In natural, social, and behavioral sciences, quantities are often related but vary with time. Problems involving rates of related variables are referred to as **related rate problems**.

### Solving Related Rate Problems

**Step 1:** Draw a picture.

**Step 2:** Identify the variables and assign symbols to them.

**Step 3:** Identify and interpret rates of change as derivatives.

**Step 4:** Write an equation.

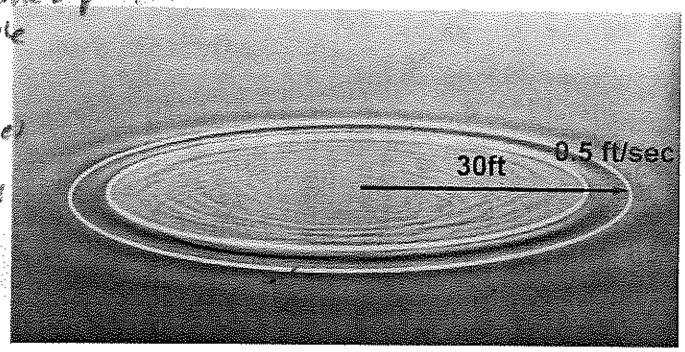
**Step 5:** Differentiate.

**Step 6:** Substitute numerical values for the variables and the derivatives, and solve for the unknown rate.

A child throws a stone into a still pond, causing a circular ripple to spread. If the radius of the circle increases at a constant rate of 0.5 feet per second, how fast is the area of the ripple increasing when the radius of the ripple is 30 feet?

2) variables: $t$ = time (sec) from when stone enters pond,
$r$ = radius (ft) of circular ripple
$A$ = area ($ft^2$) of circle

3) rates:
$\frac{dr}{dt}$ = rate at which radius increases
$\frac{dA}{dt}$ = rate at which area increases




4) equation: $A = \pi r^2$

5) differentiate: $\frac{d}{dt}[A] = \frac{d}{dt}[\pi r^2]$

$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$

6) solve: $\frac{dA}{dt} = 2\pi(30 ft)(0.5 ft/sec)$

$\boxed{\frac{dA}{dt} = 94.25 \ ft^2/sec}$

2. Assume $x$ and $y$ are differentiable functions of $t$.
Find $dx/dt$ when $x = 2$, $y = 3$, and $dy/dt = 2$.

$$x^2 - y^2 = -5$$

$\frac{d}{dt}[x^2 - y^2] = \frac{d}{dt}[-5]$

$2x\frac{dx}{dt} - 2y\frac{dy}{dt} = 0$

$2x\frac{dx}{dt} = 2y\frac{dy}{dt}$

$\frac{dx}{dt} = \frac{y}{x}\frac{dy}{dt}$.

$\frac{dx}{dt} = \frac{(3)}{(2)}(2)$

$\boxed{\frac{dx}{dt} = 3}$

10. A point is moving along the graph $y = 3x^2$. When the particle is at (2, 12), its $x$-coordinate is decreasing at the rate of 2 units per minute. How fast is the $y$-coordinate changing at that point?

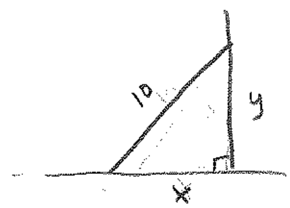
$$\frac{d}{dt}(y) = \frac{d}{dt}(3x^2)$$

$$\frac{dy}{dt} = 6x\frac{dx}{dt}$$

$$\frac{dy}{dt} = 6(2)(2)$$

$$\frac{dy}{dt} = \boxed{24 \text{ units/min}}$$

11. You are standing on top of a 10-foot ladder that is leaning against a vertical wall. The foot of the ladder is slipping away from the wall at the rate of 0.5 foot per second. At what rate is the top of the ladder coming down when it is 5 feet off the ground?




$\frac{dy}{dt} = ?$

$\frac{dx}{dt} = 0.5$

$$x^2+y^2=10^2$$

$$x^2+y^2=100$$

$$\frac{d}{dt}(x^2+y^2)=\frac{d}{dt}(100)$$

$$2x\frac{dx}{dt}+2y\frac{dy}{dt}=0$$

$$2y\frac{dy}{dt}=-2x\frac{dx}{dt}$$

$$\frac{dy}{dt}=\frac{-x}{y}\frac{dx}{dt}$$

$$\frac{dy}{dt}=-\frac{x}{y}\frac{dx}{dt}$$

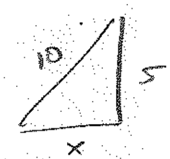
$$\frac{dy}{dt}=\frac{-(\sqrt{75})}{(5)}(0.5)$$

$$\frac{dy}{dt}=\frac{-5\sqrt{3}}{5}(0.5)$$

"down"

$$\frac{dy}{dt}=-\frac{\sqrt{3}}{2}=\boxed{-0.866 \text{ ft/sec}}$$

at this instant:




$x^2+25=100$

$x^2=75$

$x=\sqrt{75}$

#16. Air is pumped into a balloon with a spherical shape at the *rate* of *80 cubic centimeters per second.* How **fast** is the **surface of the balloon** increasing when the radius is 10 centimeters?

$$V = \frac{4}{3}\pi r^3 \quad S = 4\pi r^2$$

$\frac{dV}{dt} = 80 \text{ cm}^3/\text{sec}$

$r = 10 \text{ cm}$

$\frac{dS}{dt} = ?$

$$\frac{d}{dt}(V)=\frac{d}{dt}\left(\frac{4}{3}\pi r^3\right)$$

$$\frac{dV}{dt}=4\pi r^2\frac{dr}{dt}$$

Solve for $\frac{dr}{dt}$:

$$\frac{dr}{dt}=\frac{1}{4\pi r^2}\frac{dV}{dt}$$

substitute

$$\frac{d}{dt}(S)=\frac{d}{dt}(4\pi r^2)$$

$$\frac{dS}{dt}=8\pi r\frac{dr}{dt}$$

$$\frac{dS}{dt}=8\pi r\left(\frac{1}{4\pi r^2}\frac{dV}{dt}\right)$$

$$\frac{dS}{dt}=\frac{8\pi r}{4\pi r^2}\frac{dV}{dt}$$

$$\frac{dS}{dt}=\frac{2}{r}\frac{dV}{dt}$$

$$\frac{dS}{dt}=\frac{2}{(10)}(80)=\boxed{16 \text{ cm}^2/\text{sec}}$$