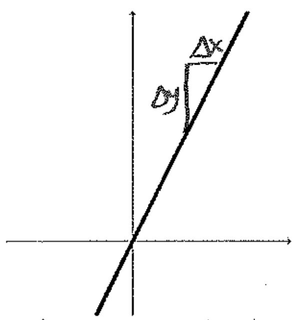
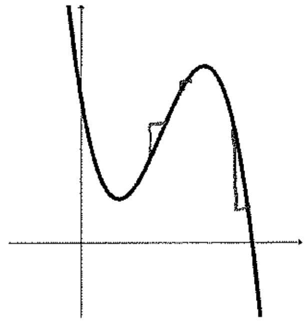
Hrs Brief Calculus – Lesson Notes: Unit 10 (Ch3,4) - Limits; Derivative of a Function

4.1 day 1 – Tangent to a curve, The Derivative

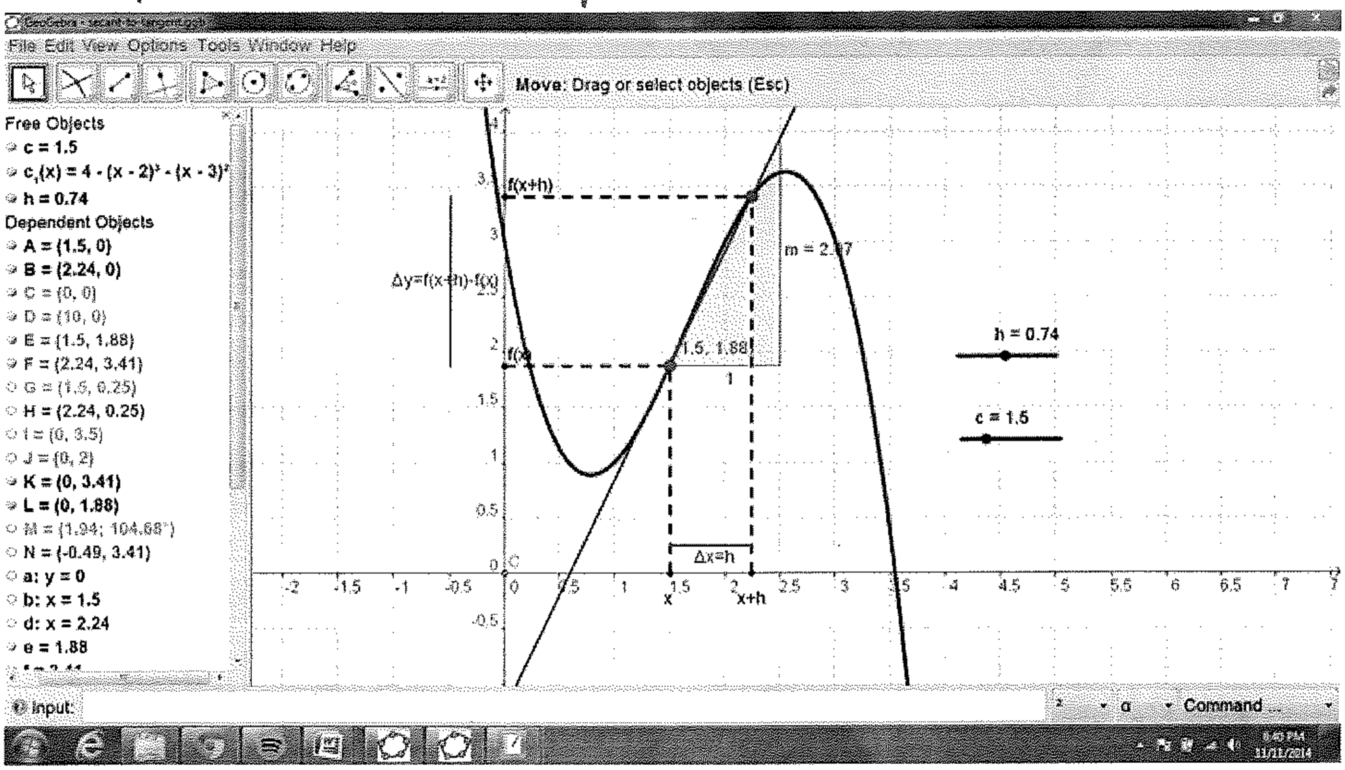
What is the slope (rate of change) of these functions?





slope is constant

slope is different for every x



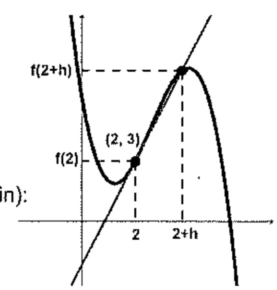
- Slope (rate of change) of a curve at a point is the slope of the line tangent to the curve at that point.
- Compute the slope: limit of difference quotient as h approaches 0.

$$f'(2)=\lim_{h\to 0}\frac{f(2+h)-f(2)}{h}$$

$f'(2)$ *is slope of* f *at* $x=2$

For any value c (x-value in the domain):

$$f'(c)=\lim_{h\to 0}\frac{f(c+h)-f(c)}{h}$$



$$f'(c)=\lim_{h\to 0}\frac{f(c+h)-f(c)}{h}$$

"f prime of x at c" means all of the following:

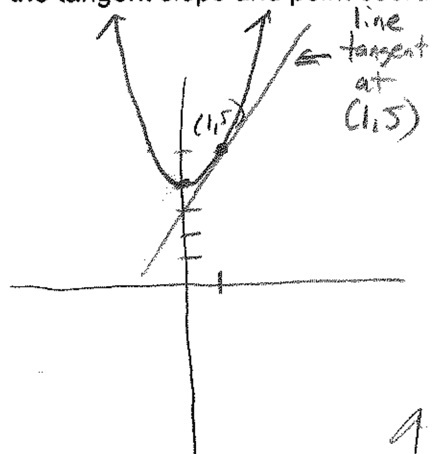
- The 'derivative' of the function f(x) at x=c.
- The rate of change of f as x changes, at x=c.
- The slope of the line tangent to f(x) at the point (c, f(c)).

Find the slope of the tangent line to the graph of f at the given point. Then find the equation of this tangent line. Graph f and the tangent line.

$$f(x) = x^2 + 4 \quad \text{at} \quad (1, 5)$$

Procedure:

- Graph the curve, locate the point, roughly sketch the tangent line.
- Compute $f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$ with c equal to the point's x coordinate.
- The result is the slope of the tangent line, m_{tan}
- Find the equation of the tangent line using point-slope form, the tangent slope and point coordinates.



$$f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h \to 0} \frac{[(1+h)^2 + 4] - [(1)^2 + 4]}{h}$$

$$= \lim_{h \to 0} \frac{(1 + 2h + h^2) + 4 - 5}{h}$$

$$= \lim_{h \to 0} \frac{2h + h^2}{h} = \lim_{h \to 0} \frac{h(2+h)}{h}$$

$$= \lim_{h \to 0} 2 + h = \boxed{2} \text{ is slope, } m_{tan}$$

so tangent line is:

$$(y - y_0) = m(x - x_0)$$

$$\boxed{(y-5) = 2(x-1)}$$

$$\text{or } y = 2x - 2 + 5$$

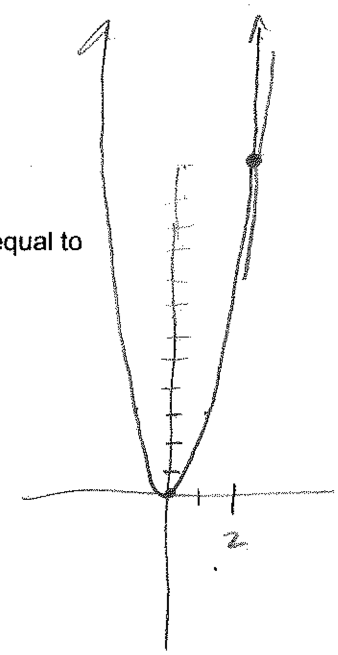
$$y = 2x + 3$$

Find the derivative of f at the given number.

$$f(x) = 3x^2 \quad at \quad 2$$

Procedure:

- Compute $f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$ with c equal to the given number.
- The result is the derivative of f at the value.



$$f'(2) = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$$

$$= \lim_{h \to 0} \frac{[3(2+h)^2] - [3(2)^2]}{h}$$

$$= \lim_{h \to 0} \frac{3(4 + 4h + h^2) - 12}{h}$$

$$= \lim_{h \to 0} \frac{12 + 12h + 3h^2 - 12}{h}$$

$$= \lim_{h \to 0} \frac{12h + 3h^2}{h}$$

$$= \lim_{h \to 0} \frac{h(12 + 3h)}{h(1)} = \lim_{h \to 0} 12 + 3h = \boxed{12}$$

Find the derivative of f at the given number.

Procedure:
- Compute $f'(c)=\lim_{h\to 0}\frac{f(c+h)-f(c)}{h}$ with c equal to the given number.
- The result is the derivative of f at the value.

$f(x)=-x^2+2x-1$ at - 1

$f'(-1)=\lim_{h\to 0}\frac{f(-1+h)-f(h)}{h}$

$=\lim_{h\to 0}\frac{[-(-1+h)^2+2(-1+h)-1]-[-(-1)^2+2(-1)-1]}{h}$

$=\lim_{h\to 0}\frac{-(1-2h+h^2)-2+2h-1-(-1-2-1)}{h}$

$=\lim_{h\to 0}\frac{-1+2h-h^2-2+2h-1+4}{h}$

$=\lim_{h\to 0}\frac{4h+h^2}{h}$

$=\lim_{h\to 0} 4+h = \boxed{4}$

Find the derivative of f at the given number.

Procedure:
- Compute $f'(c)=\lim_{h\to 0}\frac{f(c+h)-f(c)}{h}$ with c equal to the given number.
- The result is the derivative of f at the value.

$f(x)=x^5-3x+1$ at 0

$f'(0)=\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$

$=\lim_{h\to 0}\frac{[(0+h)^5-3(0+h)+1]-[(0)^5-3(0)+1]}{h}$

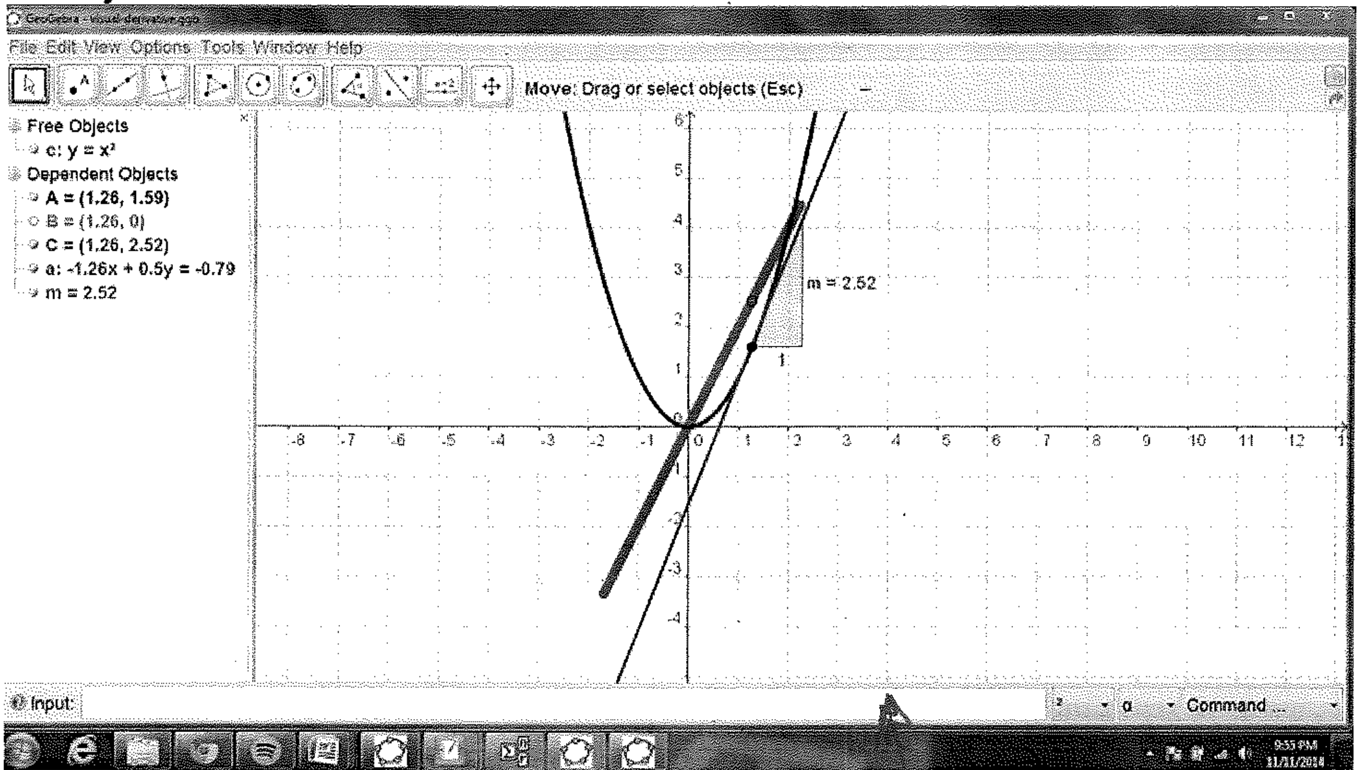
$=\lim_{h\to 0}\frac{h^5-3h+1-1}{h}$

$=\lim_{h\to 0}\frac{h^5-3h}{h}$

$=\lim_{h\to 0} h^4-3$

$=\boxed{-3}$

4.1 day 2 – The Derivative as a Function



Geogebra - Visual Derivative

this example: $f(x)=x^2$

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
$$=\lim_{h\to 0}\frac{[(x+h)^2]-[(x)^2]}{h}$$
$$=\lim_{h\to 0}\frac{x^2+2xh+h^2-x^2}{h}$$
$$=\lim_{h\to 0}\frac{2xh+h^2}{h}$$
$$=\lim_{h\to 0} 2x+h=\boxed{2x}$$

Since we can use f(x) to compute the value of the derivative, f '(c) at any point x=c, that means the derivative is also a function of x.

Derivative function: $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

- The function $f'(x)$ is 'the derivative of f at x'.
- 'Differentiate f' means to 'find the derivative of f'.
- For a given x, the derivative function output is the slope of f(x) at the given x.

$f(x)=x^2+4$

Find the slope at x=1:

$$f'(1)=\lim_{h\to 0}\frac{f(1+h)-f(1)}{h}$$
$$=\lim_{h\to 0}\frac{[(1+h)^2+4]-[(1)^2+4]}{h}$$
$$=\lim_{h\to 0}\frac{1+2h+h^2+4-5}{h}$$
$$=\lim_{h\to 0}\frac{2h+h^2}{h}$$
$$=\lim_{h\to 0}\frac{h(2+h)}{h(1)}$$
$$=\lim_{h\to 0} 2+h$$
$$f'(1)=2+(0)=\boxed{2}$$

Find $f'(x)$, the derivative function:

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
$$=\lim_{h\to 0}\frac{[(x+h)^2+4]-[(x)^2+4]}{h}$$
$$=\lim_{h\to 0}\frac{x^2+2xh+h^2+4-x^2-4}{h}$$
$$=\lim_{h\to 0}\frac{2xh+h^2}{h}$$
$$=\lim_{h\to 0}\frac{h(2x+h)}{h(1)}$$
$$=\lim_{h\to 0} 2x+h$$
$$=2x+(0)$$
$$f'(x)=\boxed{2x}$$

$f'(x)=2x$ gives slope at any x

so slope at x=1,

$f'(1)=2(1)=\boxed{2}$

Find $f'(x)$, the derivative function:

#26. $f(x) = 2x^2 + x + 1$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{[2(x+h)^2 + (x+h) + 1] - [2(x)^2 + (x) + 1]}{h}$$

$$= \lim_{h \to 0} \frac{2(x^2 + 2xh + h^2) + x + h + 1 - 2x^2 - x - 1}{h}$$

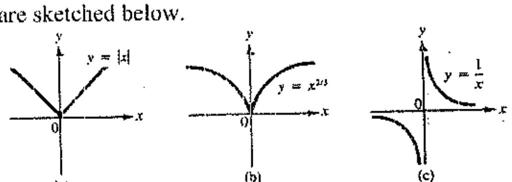
$$= \lim_{h \to 0} \frac{2x^2 + 4xh + 2h^2 + x + h + 1 - 2x^2 - x - 1}{h}$$

$$= \lim_{h \to 0} \frac{4xh + 2h^2 + h}{h}$$

$$= \lim_{h \to 0} 4x + 2h + 1 = \boxed{4x + 1}$$

Differentiability and Continuity

Not all functions have a derivative for every value of x. Three functions that do not have derivatives when $x = 0$ are sketched below.



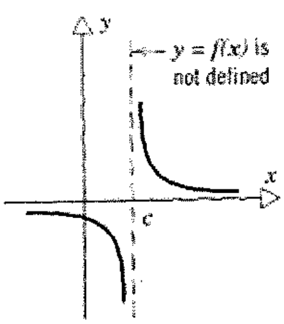
"Corner" no tangent at $x = 0$ | "Cusp" vertical tangent at $x = 0$ | Function undefined at $x = 0$

Continuity does not imply differentiability

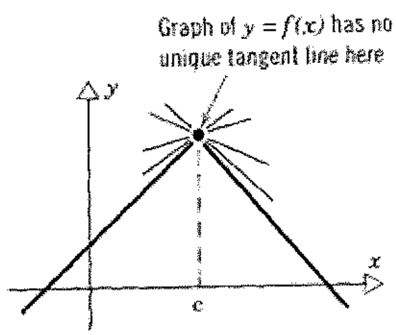
If the derivative of a function exists at $x = c$, then $f(x)$ is continuous at $x = c$.

A function is said to be **differentiable at** $x = c$, if it has a derivative when $x = c$.

These functions do not have tangent lines at $x = c$, and, hence, the derivative does not exist at $x = c$.



(a)



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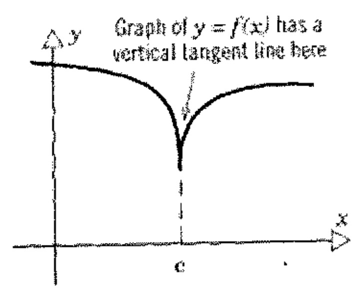

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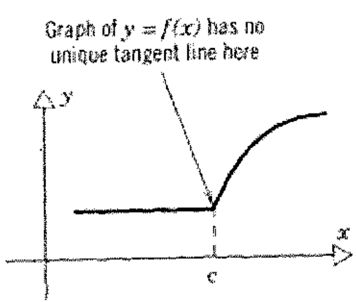

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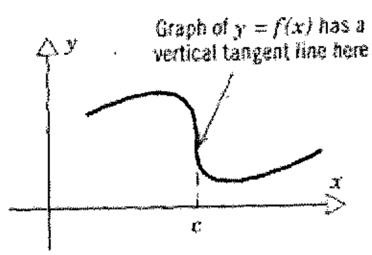

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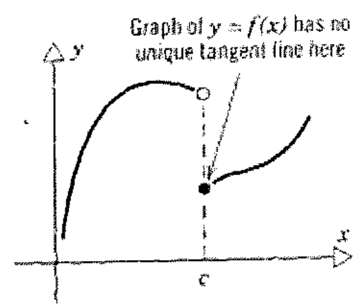

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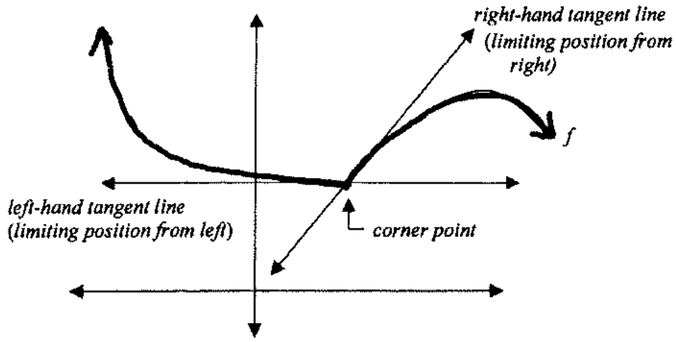
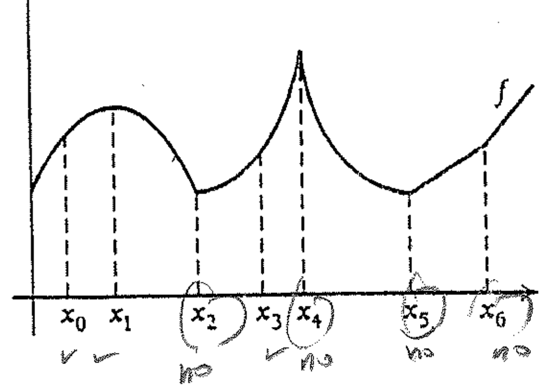
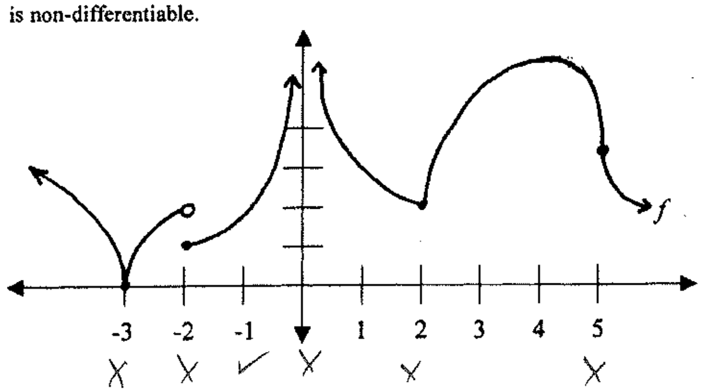
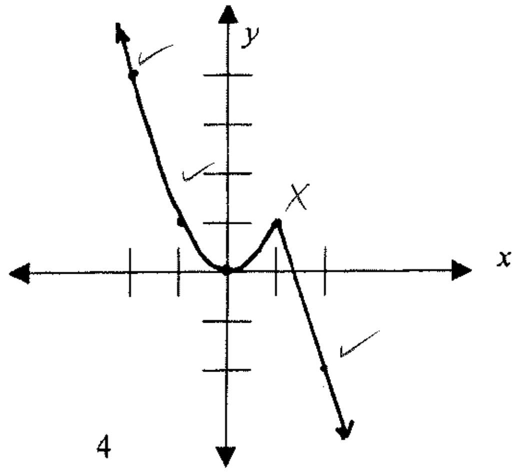


List the points at which the following function is not differentiable.



Example 1: Given below is the graph of $y = f(x)$. Determine the points where f is non-differentiable.





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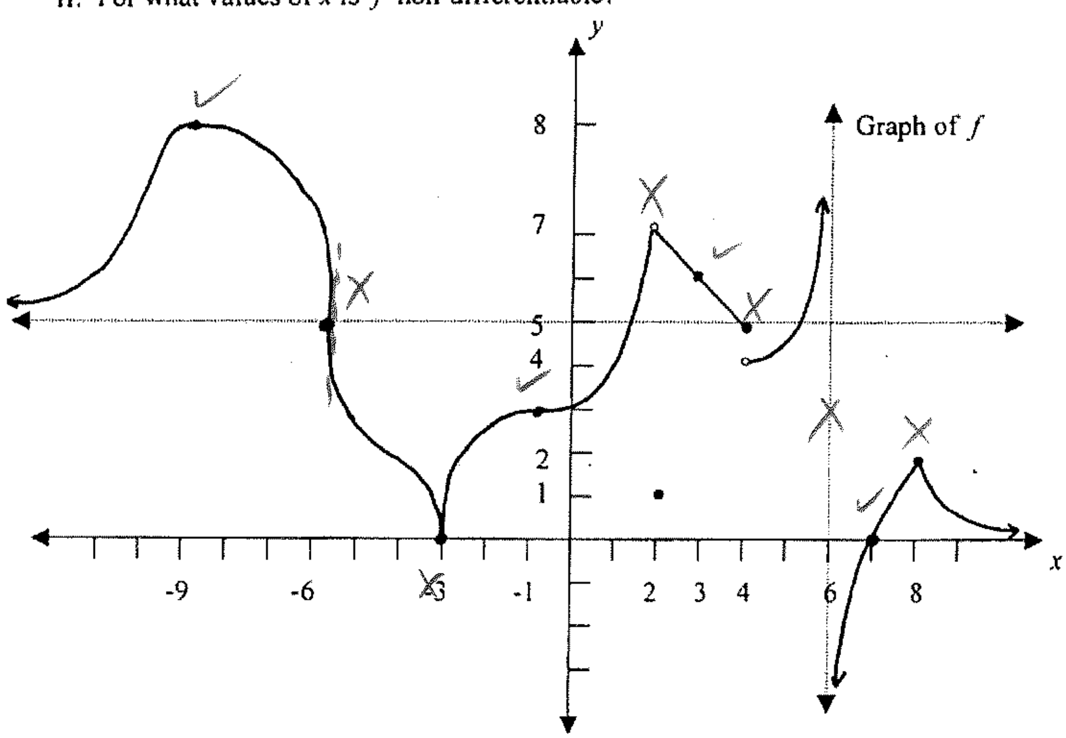
II. For what values of x is f non-differentiable?



4.6 – The Derivative as an Instantaneous Velocity

You get into your car at noon and drive non-stop until 3:00pm. If you've driven a total of 150 miles, what was your average velocity (average speed)?

$$\frac{s_2 - s_1}{t_2 - t_1}$$

$$v_{avg} = \frac{\Delta s}{\Delta t} = \frac{150 \text{ miles}}{3 \text{ hrs}}$$

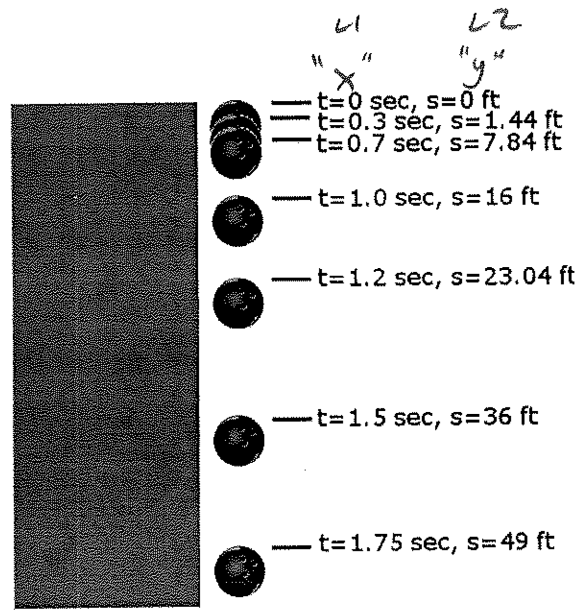
$$\boxed{v_{avg} = 50 \text{ miles/hr}}$$

Average Velocity:

the ratio of the change in position Δs to the change in time Δt $\qquad v_{avg} = \frac{\Delta s}{\Delta t}$

$\frac{\text{change in position}}{\text{elapsed time}} \qquad v_{avg} = \frac{s_2 - s_1}{t_2 - t_1}$

A bowling ball is dropped from the roof of a 5 story building (wind resistance is negligible)



1) Use regression analysis to find a quadratic equation model for distance s, as a function of time t:

$s(t) = 16t^2$ ($r^2 = 1$, perfect model)

2) Estimate the velocity (speed) of the ball at t=1.0 by finding the average velocity using the following time values:

a) average velocity from t=0.7 sec to t=1.0 sec

$$v_{avg} = \frac{\Delta s}{\Delta t} = \frac{s_2 - s_1}{t_2 - t_1} = \frac{16 - 7.84}{1.0 - 0.7} = 27.2 \text{ ft/sec}$$

$$\boxed{v_{avg} = 27.2 \text{ ft/sec}}$$

b) average velocity from t=1.0 sec to t=1.2 sec

$$v_{avg} = \frac{23.04 - 16}{1.2 - 1.0} = 35.2 \text{ ft/sec}$$

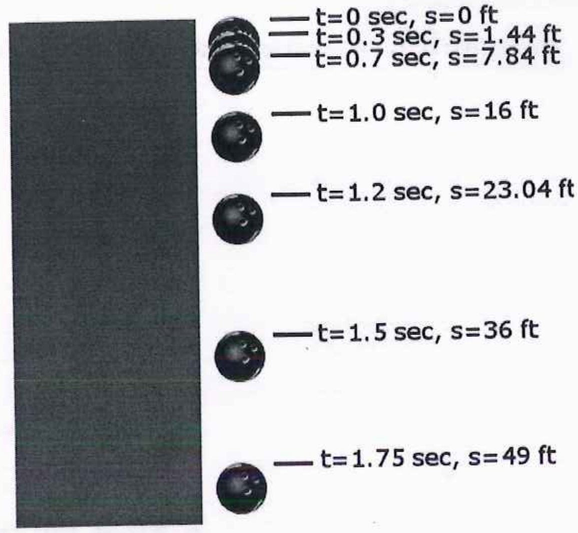
$$\boxed{v_{avg} = 35.2 \text{ ft/sec}}$$

Are these good estimates of the velocity of the ball at 1.0 second?
Why or why not? No, the velocity is constantly changing. At $t = 1.0$, v is somewhere between 27.2 and 35.2 ft/sec

How could we compute a more accurate value for the velocity at 1.0 second?

Hint: Remember, velocity is the rate of change of the distance vs. time function.

$$s = f(t) = 16t^2$$



3) Find the derivative of the distance as a function of time, and evaluate that function at t=1.0.

$$f'(t) = \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{[16(t+\Delta t)^2] - [16(t)^2]}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{16(t^2 + 2t\Delta t + \Delta t^2) - 16t^2}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{16t^2 + 32t\Delta t + 16\Delta t^2 - 16t^2}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{32t\Delta t + 16\Delta t^2}{\Delta t}$$

$$= \lim_{\Delta t \to 0} 32t + 16\Delta t = \boxed{32t}$$

$f'(t) = 32t$

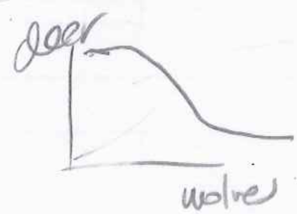
so at $t = 1$ sec:

$f'(1) = 32(1)$

$= \boxed{32 \text{ ft/sec}}$

Instantaneous velocity at $t = 1.0$ sec

finish with:



Average Velocity:

The ratio of the change in position Δs to the change in time Δt

$$\frac{\text{change in position}}{\text{elapsed time}}$$

$$v_{avg} = \frac{\Delta s}{\Delta t}$$

$$v_{avg} = \frac{s_2 - s_1}{t_2 - t_1}$$

Instantaneous Velocity

The rate of change of distance with time which is the slope of the distance curve at a point

If $s = f(t)$ then velocity is the derivative of the distance function:

$$f'(t) = \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t}$$

3. **Average Velocity** Suppose the function $s = f(t) = 16t^2$ relates the distance s (in feet) an object travels in time t (in seconds). Compute the average velocity, $\Delta s/\Delta t$, from $t = 3$ to:

a) $t = 3.5$

$s_2 = f(3.5) = 16(3.5)^2 = 196$ ft

$s_1 = f(3) = 16(3)^2 = 144$ ft

$V_{avg} = \frac{\Delta s}{\Delta t} = \frac{196 - 144}{3.5 - 3} = \boxed{104 \text{ ft/sec}}$

b) $t = 3.1$

$s_2 = f(3.1) = 16(3.1)^2 = 153.76$ ft

$s_1 = f(3) = 16(3)^2 = 144$ ft

$V_{avg} = \frac{\Delta s}{\Delta t} = \frac{153.76 - 144}{3.1 - 3} = \boxed{97.6 \text{ ft/sec}}$

6. **Velocity** The position s (in meters) of a particle in time t (in seconds) is given by $s = f(t) = t^2 - 4t$. Find the velocity at $t = 0$; at $t = 3$; at any time t.

(instantaneous)

at any t

$\boxed{f'(t) = 2t - 4}$

at $t = 0$

$f'(0) = 2(0) - 4$

$\boxed{f'(0) = -4 \text{ m/s}}$

at $t = 3$

$f'(3) = 2(3) - 4$

$= 6 - 4$

$= \boxed{2 \text{ m/s}}$

$$f'(t) = \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{[(t+\Delta t)^2 - 4(t+\Delta t)] - [(t)^2 - 4(t)]}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{t^2 + 2t\Delta t + \Delta t^2 - 4t - 4\Delta t - t^2 + 4t}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{2t\Delta t + \Delta t^2 - 4\Delta t}{\Delta t}$$

$$= \lim_{\Delta t \to 0} 2t + \Delta t - 4$$

$f'(t) = 2t - 4$