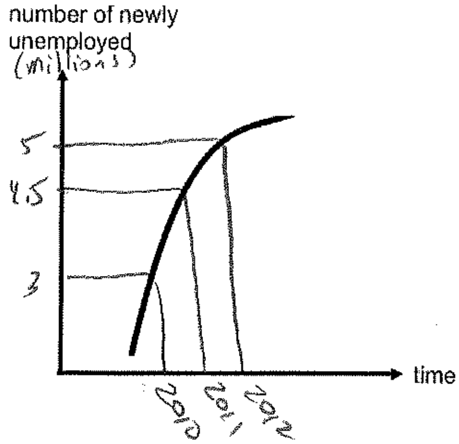
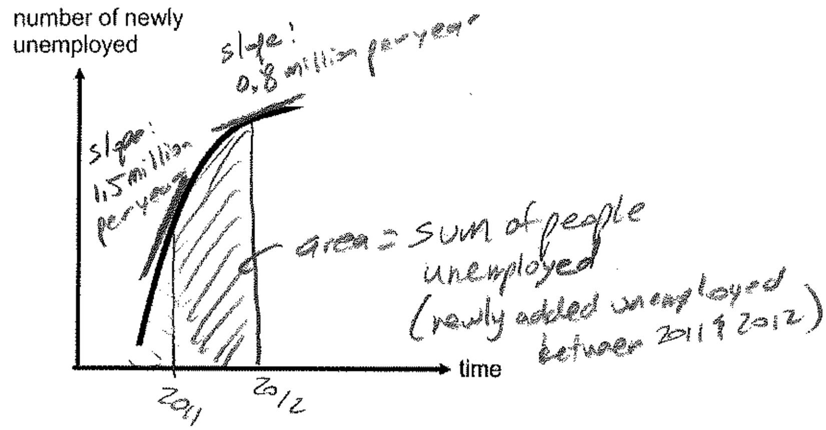
Hrs Brief Calculus – Lesson Notes: Unit 10 (Ch3) - Limits; Derivative of a Function

3.1 – *The idea of a Limit; Finding Limits using tables and graphs*

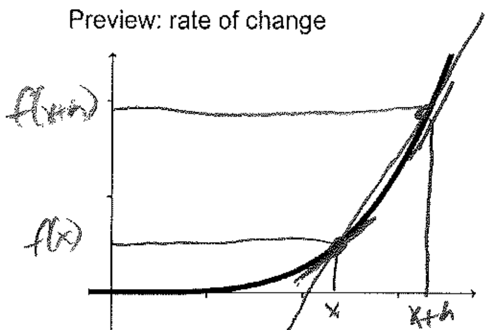
Algebra/Functions:
Focus is on functions that can model the data values of real-world scenarios.



Calculus:
Focus is on rate of change of functions and on how values 'accumulate'.



Preview: rate of change



rate of change $= \frac{\Delta y}{\Delta x} = \frac{f(x+h)-f(x)}{h}$ ← make h very small

Limits:

The concept of the limit of a function is what bridges the gap between the mathematics of algebra and the mathematics of calculus.

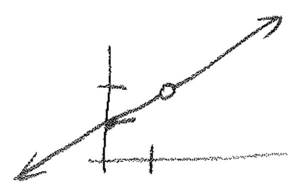
$$\lim_{x \to c} f(x) = N$$

Read:
"The limit of f of x as x approaches c equals the number N."

This means:
For all x approximately equal to c, but not equal to c, the value f(x) is approximately equal to N.

What is the domain of this function? $f(x) = \frac{x^2-1}{x-1}$

Graph the function in your calculator. D: $(-\infty, 1) \cup (1, \infty)$



Specific values of f(x) when x is close to 1:

x	f(x)	x	f(x)
0	1	2	3
0.5	1.5	1.5	2.5
0.75	1.75	1.25	2.25
0.9	1.9	1.1	2.1
0.99	1.99	1.01	2.01
0.999	1.999	1.001	2.001
$\lim_{x \to 1^-} f(x) = 2$		$\lim_{x \to 1^+} f(x) = 2$	

Criterion for $\lim_{x \to c} f(x)$ to exist: $\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x)$

The limit N of a function $y = f(x)$ as x approaches the number c does not depend on the value of f at c.

$$\lim_{x \to 1} \frac{x^2-1}{x-1} = \boxed{2}$$

5. Use a calculator to complete each table and evaluate the indicated limit $\lim_{x \to 2} f(x) = \boxed{4}$

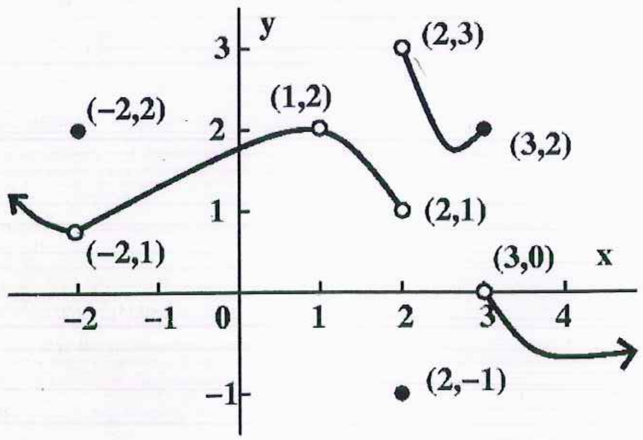
x	1.9	1.99	1.999
$f(x)=\dfrac{x^2-4}{x-2}$	3.9	3.99	3.999

$\lim_{x \to 2^-} = 4$

x	2.1	2.01	2.001
$f(x)=\dfrac{x^2-4}{x-2}$	4.1	4.01	4.001

$\lim_{x \to 2^+} = 4$

Given the graphical definition of the function, find the following:



$f(2) = -1$ $\qquad$ $\lim_{x \to -2} f(x) = 1$

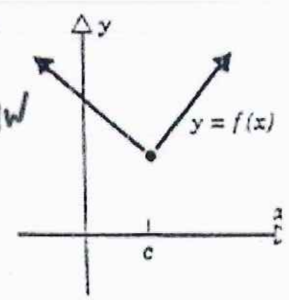
$\lim_{x \to 2^-} f(x) = 1$ $\qquad$ $\lim_{x \to 1} f(x) = 2$

$\lim_{x \to 2^+} f(x) = 3$ $\qquad$ $\lim_{x \to 2} f(x)$ does not exist

$\lim_{x \to 2} f(x)$ does not exist $\qquad$ $\lim_{x \to 3} f(x)$ does not exist

Use the graph to determine whether $\lim_{x \to c} f(x)$ exists.

9. HW



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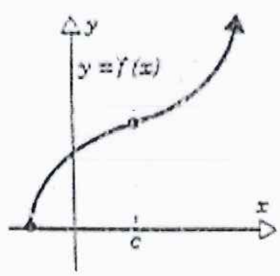

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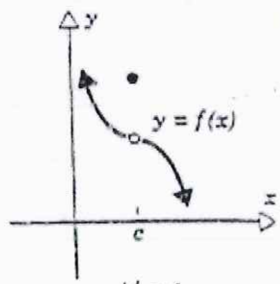

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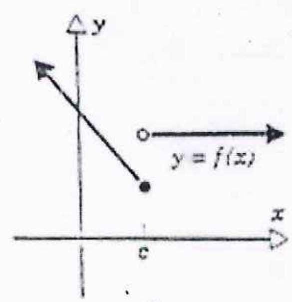

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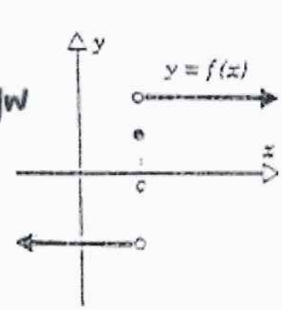
13. HW



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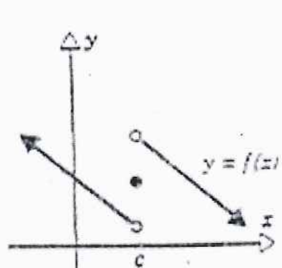

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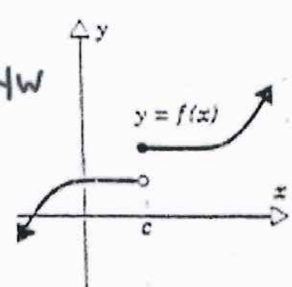
15. HW



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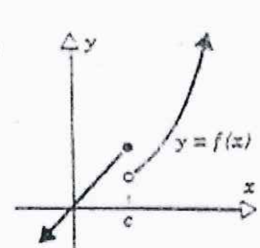
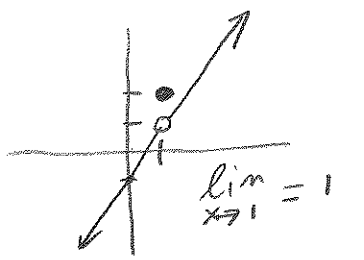


no

Determine whether $\lim\limits_{x \to c} f(x)$ exists by graphing.

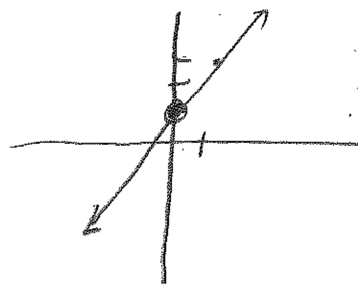
If it exists, find $\lim\limits_{x \to c} f(x)$

32. $f(x) = \begin{cases} 2x-1 & \text{if } x \neq 1 \\ 2 & \text{if } x = 1 \end{cases}$



$\lim\limits_{x \to 1} = 1$

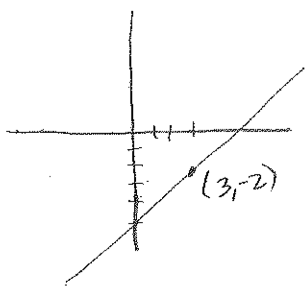
30. $f(x) = \begin{cases} 2x+1 & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$



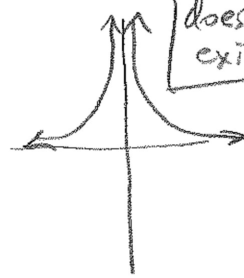
$\lim\limits_{x \to 0} = 1$

Find the limit.

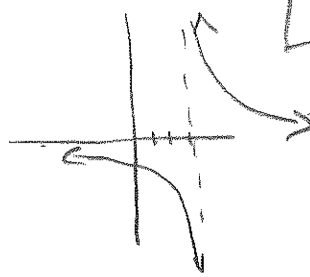
39. $\lim\limits_{x \to 3^+} (x-5) = \boxed{-2}$



42. $\lim\limits_{x \to 0^+} \dfrac{3x}{x^3}$ $\boxed{\text{does not exist}}$



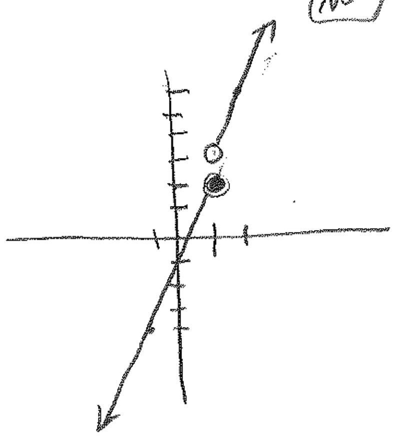
44. $\lim\limits_{x \to 3^+} \dfrac{6}{x-3}$ $\boxed{\text{does not exist}}$



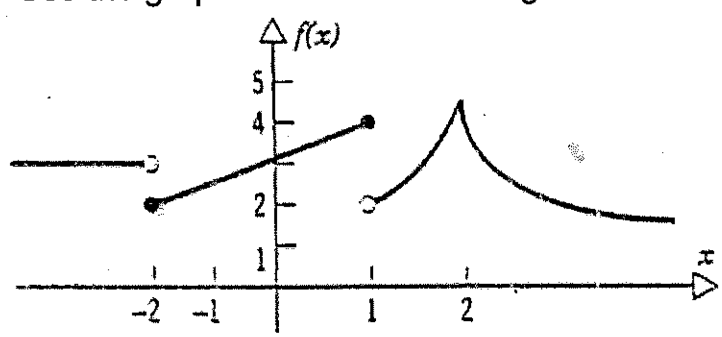
50. Find the limit $\lim\limits_{x \to 1^-} f(x)$ and $\lim\limits_{x \to 1^+} f(x)$ for the function

$$f(x) = \begin{cases} 3x-1 & \text{if } x < 1 \\ 2 & \text{if } x = 1 \\ 3x & \text{if } x > 1 \end{cases}$$

Does $\lim\limits_{x \to 1} f(x)$ exist? $\boxed{\text{NO}}$



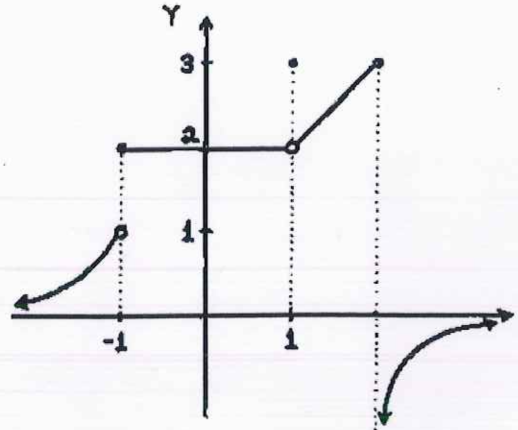
Use the graph to find the following limits:



(a) $\lim\limits_{x \to -2^+} f(x) = 2$ (b) $\lim\limits_{x \to -2^-} f(x) = 3$ (c) $\lim\limits_{x \to -2} f(x)$ Does not exist

(d) $\lim\limits_{x \to 1^+} f(x) = 2$ (e) $\lim\limits_{x \to 1^-} f(x) = 4$ (f) $\lim\limits_{x \to 1} f(x)$ Does not exist

(g) $\lim\limits_{x \to 2^+} f(x) = 4.5$ (h) $\lim\limits_{x \to 2^-} f(x) = 4.5$ (i) $\lim\limits_{x \to 2} f(x) = 4.5$



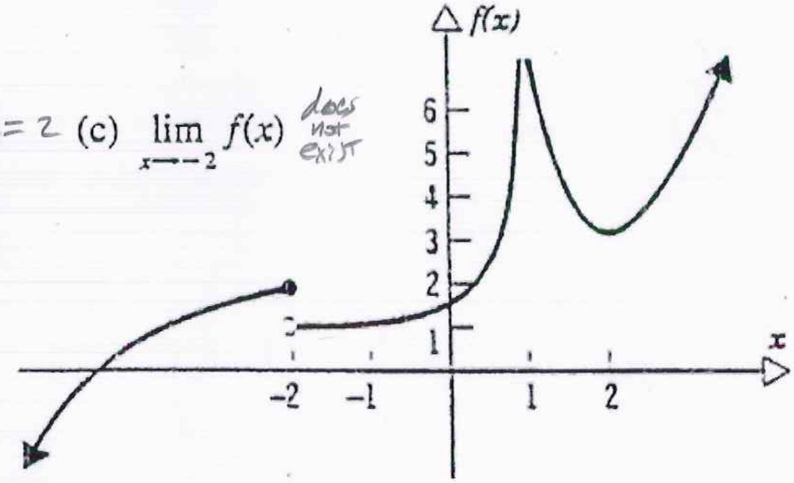
$\lim_{x\to -1^-} f(x) = -1$ $\quad$ $\lim_{x\to 1^-} f(x) = 2$ $\quad$ $\lim_{x\to 2^-} f(x) = 3$

$\lim_{x\to -1^+} f(x) = 2$ $\quad$ $\lim_{x\to 1^+} f(x) = 2$ $\quad$ $\lim_{x\to 2^+} f(x)$ does not exist

$\lim_{x\to -1} f(x)$ does not exist $\quad$ $\lim_{x\to 1} f(x) = 2$ $\quad$ $\lim_{x\to 2} f(x)$ does not exist

$f(-1) = 2$ $\quad$ $f(1) = 3$ $\quad$ $f(2) = 3$

37.



(a) $\lim_{x\to -2^+} f(x) = -2$ (b) $\lim_{x\to -2^-} f(x) = 2$ (c) $\lim_{x\to -2} f(x)$ does not exist

(d) $\lim_{x\to 1^+} f(x)$ DNE (e) $\lim_{x\to 1^-} f(x)$ DNE (f) $\lim_{x\to 1} f(x)$ DNE

(g) $\lim_{x\to 2^+} f(x) = 3$ (h) $\lim_{x\to 2^-} f(x) = 3$ (i) $\lim_{x\to 2} f(x) = 3$

3.2 – Algebraic Techniques for Finding Limits

Groups: Find the indicated limit.

#1) $\lim\limits_{x \to 1} 4$

$= \boxed{4}$

#2) $\lim\limits_{x \to -2} (3x+2)$

$= 3(-2)+2$

$= -6+2$

$= \boxed{-4}$

#3) $\lim\limits_{x \to 1} \sqrt{3x^2+1}$

$= \sqrt{3(1)^2+1}$

$= \sqrt{3+1}$

$= \sqrt{4}$

$= \boxed{2}$

$= \sqrt{\lim\limits_{x\to 1} (3x^2+1)}$

$= \sqrt{\lim\limits_{x\to 1} (3x^2) + \lim\limits_{x\to 1} (1)}$

$= \sqrt{\lim\limits_{x\to 1} (3) \cdot \lim\limits_{x\to 1} (x^2) + \lim\limits_{x\to 1} (1)}$

$= \sqrt{\lim\limits_{x\to 1} (3) \cdot \left[\lim\limits_{x\to 1} x\right]^2 + \lim\limits_{x\to 1} (1)}$

$= \sqrt{3 \cdot (1)^2 + 1}$

$= \sqrt{4} = \boxed{2}$

#4) $\lim\limits_{x \to -2} \dfrac{x+2}{3x-5}$

$= \dfrac{(-2)+2}{3(-2)-5}$

$= \dfrac{0}{-4}$

$= \boxed{0}$

These illustrate general properties of limits….

$$\lim_{x\to c} b = b$$

$$\lim_{x\to c} x = c$$

$$\lim_{x\to c}\left[f(x) \pm g(x)\right] = \lim_{x\to c} f(x) \pm \lim_{x\to c} g(x)$$

$$\lim_{x\to c}\left[f(x) \cdot g(x)\right] = \lim_{x\to c} f(x) \cdot \lim_{x\to c} g(x)$$

$$\lim_{x\to c}\left[f(x)\right]^n = \left[\lim_{x\to c} f(x)\right]^n$$

$$\lim_{x\to c} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to c} f(x)}$$

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)} \quad \left(\text{if } \lim_{x\to c} g(x) \neq 0\right)$$

#5) $\lim\limits_{x \to -1} \dfrac{x^2-1}{x+1}$

This limit does exist. The function can have a limit at an x-value even if the function is undefined at this value.

$\lim\limits_{x \to -1} \dfrac{(x+1)(x-1)}{(x+1)(1)}$

$\lim\limits_{x \to -1} (x-1) = \boxed{0}$

Try graphing this function with your calculator. Is the graph shape surprising? Does this suggest a way to handle cases like this?

If $f(x) = 2x^2 + x$

a) Find:

$\lim\limits_{x \to 4} \dfrac{f(x)-f(1)}{x-4}$

$\lim\limits_{x\to 4} \dfrac{\left[2(x)^2+(x)\right] - \left[2(1)^2+(1)\right]}{x-4}$

$\lim\limits_{x\to 4} \dfrac{2x^2+x-2-1}{x-4}$

$\lim\limits_{x\to 4} \dfrac{2x^2+x-3}{x-4}$

$\lim\limits_{x\to 4} \dfrac{(2x+3)(x-1)}{(x-4)}$

Does Not Exist

$\boxed{DNE}$

b) Find:

$\lim\limits_{x \to 4} \dfrac{f(x)-f(4)}{x-4}$

$\lim\limits_{x\to 4} \dfrac{\left[2(x)^2+(x)\right] - \left[2(4)^2+(4)\right]}{x-4}$

$\lim\limits_{x\to 4} \dfrac{2x^2+x-32-4}{x-4}$

$\lim\limits_{x\to 4} \dfrac{2x^2+x-36}{x-4}$

$\lim\limits_{x\to 4} \dfrac{(2x+9)(x-4)}{(x-4)}$

$\lim\limits_{x\to 4} (2x+9)$

$\boxed{17}$

If $f(x) = 3-4x$

"average rate of change"

find $\lim\limits_{x \to 2} \dfrac{f(x)-f(2)}{x-2}$

$\lim\limits_{x\to 2} \dfrac{\left[3-4(x)\right] - \left[3-4(2)\right]}{x-2}$

$\lim\limits_{x\to 2} \dfrac{3-4x-3+8}{x-2}$

$\lim\limits_{x\to 2} \dfrac{-4x+8}{x-2}$

$\lim\limits_{x\to 2} \dfrac{-4(x-2)}{x-2}$

$\lim\limits_{x\to 2} -4 = \boxed{-4}$

#6) $\lim_{x \to 4} \frac{2x^2-32}{x^3-4x^2}$

$\lim_{x \to 4} \frac{2(x^2-16)}{x^2(x-4)}$

$\lim_{x \to 4} \frac{2(x+4)(x-4)}{x^2(x-4)}$

$\lim_{x \to 4} \frac{2(x+4)}{x^2}$

$\frac{2(4+4)}{4^2} = \frac{16}{16} = \boxed{1}$

#7) $\lim_{x \to 3} \left(\frac{x^3-27}{x-3} \right)$

$x-3 \overline{)x^3+0x^2+0x-27}$

synthetic division:

$$\begin{array}{c|cccc} 3 & 1 & 0 & 0 & -27 \\ & & 3 & 9 & 27 \\ \hline & 1 & 3 & 9 & \boxed{0} \end{array}$$

x^2+3x+9

$\lim_{x \to 3} (x^2+3x+9)$

$(3)^2+3(3)+9$

$9+9+9$

$\boxed{27}$

#8) $\lim_{x \to 4} \frac{\sqrt{x}-2}{x-4} \frac{(\sqrt{x}+2)}{(\sqrt{x}+2)}$

$\lim_{x \to 4} \frac{(x-4)1}{(x-4)(\sqrt{x}+2)}$

$\lim_{x \to 4} \frac{1}{\sqrt{x}+2}$

$\frac{1}{\sqrt{4}+2} = \boxed{\frac{1}{4}}$

#9) $\lim_{x \to 3} \left[\frac{3}{x-3} - \frac{x}{x-3} \right]$

$\lim_{x \to 3} \frac{3-x}{x-3}$

$\lim_{x \to 3} \frac{-(x-3)}{(x-3)}$

$\lim_{x \to 3} (-1) = \boxed{-1}$

#10) Find $\lim_{x \to 2} f(x)$ and $f(2)$,

when $f(x) = \begin{cases} 4x^3+x & \text{if } x \neq 2 \\ 8 & \text{if } x = 2 \end{cases}$

LH $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (4x^3+x) = 4(2)^3+(2) = 34$

RH $\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (4x^3+x) = 4(2)^3+(2) = 34$

so $\lim_{x \to 2} f(x) = \boxed{34}$

$f(2) = \boxed{8}$

#11) Find $\lim_{x \to 1} f(x)$ and $f(1)$,

when $f(x) = \begin{cases} \frac{4x^3+x-5}{x-1} & \text{if } x \neq 1 \\ 8 & \text{if } x = 1 \end{cases}$

LH, RH both $\lim_{x \to 1} \frac{4x^3+x-5}{x-1}$

$$\begin{array}{c|cccc} 1 & 4 & 0 & 1 & -5 \\ & & 4 & 4 & 5 \\ \hline & 4 & 4 & 5 & 0 \end{array}$$

$\lim_{x \to 1} (4x^2+4x+5) = \boxed{13}$

$4(1)^2+4(1)+5$

$f(1) = \boxed{8}$

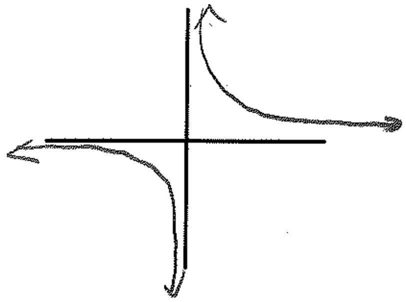
#12) Assume that $\lim_{x \to c} f(x) = 5$ and $\lim_{x \to c} g(x) = 2$ to find each limit

$\lim_{x \to c} \frac{f(x)}{g(x)-f(x)} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x) - \lim_{x \to c} f(x)} = \frac{5}{2-5} = \boxed{\frac{5}{-3}}$

3.4 – Limits at Infinity, Infinite Limits

Limits at Infinity

Graph: $f(x) = \dfrac{1}{x}$



a. $\lim\limits_{x\to\infty} \dfrac{1}{x} = 0$ b. $\lim\limits_{x\to -\infty} \dfrac{1}{x} = 0$

c. $\lim\limits_{x\to\infty} \dfrac{1}{x^n} = 0$ d. $\lim\limits_{x\to -\infty} \dfrac{1}{x^n} = 0$

What about this one? (Try graphing)

$$\lim_{x\to\infty} \frac{3x-2}{4x-1} = 0.75 \left(\frac{3}{4}\right)$$

Finding LIMITS at INFINITY of a RATIONAL FUNCTION

1. DIVIDE each term of BOTH the numerator and the denominator by the highest power of x that appears in the denominator.
2. Take the limit of the new numerator and denominator.

$$\lim_{x\to\infty} \frac{3x-2}{4x-1} = \lim_{x\to\infty} \frac{\frac{3x}{x} - \frac{2}{x}}{\frac{4x}{x} - \frac{1}{x}} = \lim_{x\to\infty} \frac{3 - \cancel{\frac{2}{x}}^{\to 0}}{4 - \cancel{\frac{1}{x}}} = \boxed{\frac{3}{4}}$$

$$\lim_{x\to\infty} \frac{2x^2-5x+2}{5x^2+7x-1}$$

$$\lim_{x\to\infty} \frac{\frac{2x^2}{x^2} - \frac{5x}{x^2} + \frac{2}{x^2}}{\frac{5x^2}{x^2} + \frac{7x}{x^2} - \frac{1}{x^2}}$$

$$\lim_{x\to\infty} \frac{2 - \frac{5}{x} + \frac{2}{x^2}}{5 + \frac{7}{x} - \frac{1}{x^2}} = \boxed{\frac{2}{5}}$$

$$\lim_{x\to\infty} \frac{x^2-2x+1}{x^3+5x+4}$$

$$\lim_{x\to\infty} \frac{\frac{1}{x} - \frac{2}{x^2} + \frac{1}{x^3}}{1 + \frac{5}{x^2} + \frac{4}{x^3}}$$

$$\frac{0-0+0}{1+0+0}$$

$$\frac{0}{1} = \boxed{0}$$

$$\lim_{x\to\infty} \frac{5x^3-1}{x^2+1}$$

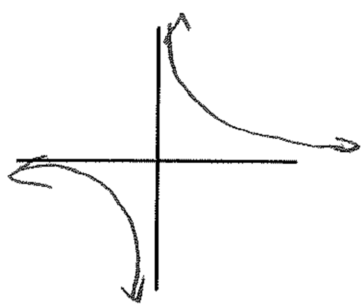
$$\lim_{x\to\infty} \frac{\frac{5x^3}{x^2} - \frac{1}{x^2}}{\frac{x^2}{x^2} + \frac{1}{x^2}}$$

$$\lim_{x\to\infty} \frac{5x - \cancel{\frac{1}{x^2}}^{\to 0}}{1 + \cancel{\frac{1}{x^2}}^{\to 0}}$$

$$\lim_{x\to\infty} 5x = \boxed{\infty}$$

Infinite Limits

Graph: $f(x) = \dfrac{1}{x}$



$$\lim_{x\to 0^-} \frac{1}{x} = \boxed{-\infty}$$

$$\lim_{x\to 0^+} \frac{1}{x} = \boxed{\infty}$$

← these are not numbers
means function is unbounded
with a positive or negative value.

Where do infinite limits occur? at uncancelled zeros in denominator

Once you detect an infinite limit will occur, you only need to determine the sign $+\infty$ or $-\infty$

$$\lim_{x\to 0^+}\frac{1}{x}\ \frac{(+)}{(+)}$$

$\boxed{\infty}$

$$\lim_{x\to 1^-}\frac{x-1}{x^2-3x+2}$$

$\lim_{x\to 1^-}\frac{(x-1)}{(x-2)(x-1)}$

$\lim_{x\to 1^-}\frac{1}{(x-2)}$

$\frac{1}{1-2}=\boxed{-1}$

but $\lim_{x\to 2^-}\frac{x-1}{(x-2)(x-1)}$

$\lim_{x\to 2^-}\frac{1}{x-2}\ \frac{(+)}{(-)}=\boxed{-\infty}$

$$\lim_{x\to 1^-}\frac{1}{x}$$

$\frac{1}{1}$

1

$$\lim_{x\to 1^-}\frac{x+1}{x^2-3x+2}$$

$\lim_{x\to 1^-}\frac{(x+1)}{(x-1)(x-2)}$

$\frac{(+)}{(-)(-)}$

$\frac{(+)}{(+)}$

$\boxed{\infty}$

$$\lim_{x\to 5^+}\frac{x+1}{5-x}$$

$\frac{(+)}{(-)}$

$\boxed{-\infty}$

$$\lim_{x\to 0^+}\frac{x\left(x^2-1\right)}{x^2}$$

$\lim_{x\to 0^+}\frac{(x+1)(x-1)}{x}$

$\frac{(+)(-)}{(+)}$

$\frac{(-)}{(+)}$

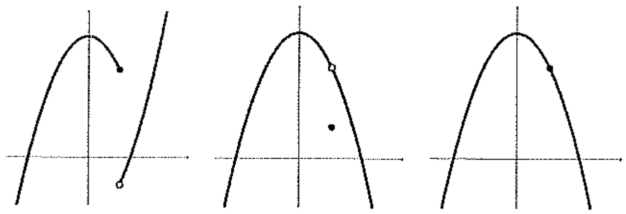
$\boxed{-\infty}$

Finding VERTICAL (infinite limits) and HORIZONTAL (limits at infinity) ASYMPTOTES

1. Graphically
2. Using LIMITS --- this will be discussed in a later section.

3.3 – Continuous Functions

One of these things is not like the other...



discontinuous discontinuous continuous

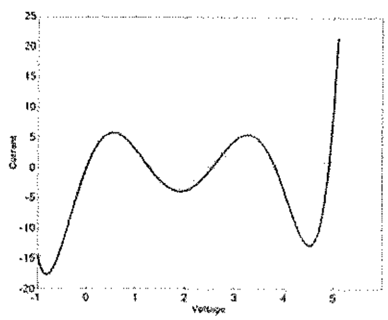
Conditions for a Function to Be Continuous at c
To summarize, a function f is continuous at c provided that three conditions are met:

Condition 1 $f(c)$ is defined;
that is, c is in the domain of the function

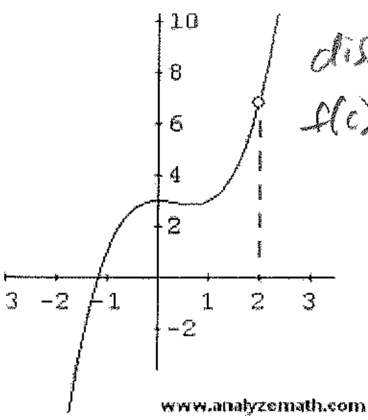
Condition 2 $\lim_{x \to c} f(x)$ exists

Condition 3 $\lim_{x \to c} f(x) = f(c)$

A **polynomial** function f is **continuous** at every real #.

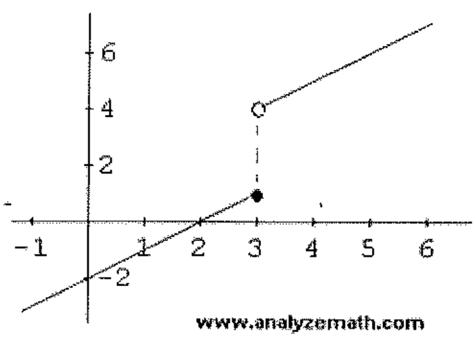


polynomials are always continuous



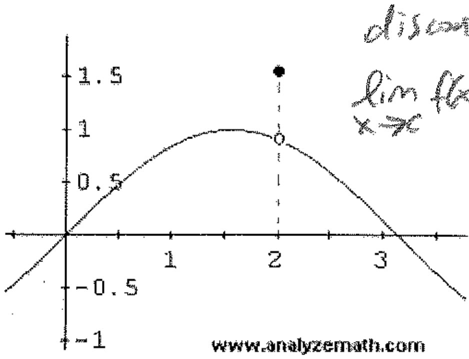
discontinuous

$f(c)$ undef



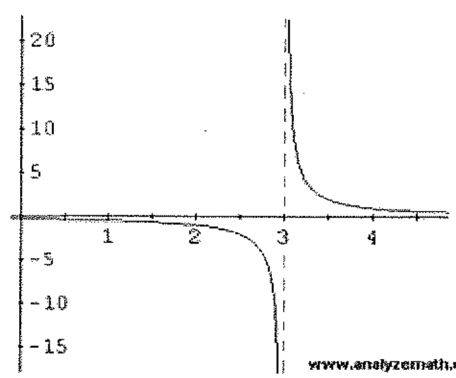
discontinuous

$\lim_{x \to c} f(x)$ DNE
(LH ≠ RH)



discontinuous

$\lim_{x \to c} f(x) \neq f(c)$



discontinuous

$f(c)$ undef

&

$\lim_{x \to c} f(x)$ DNE

Determine whether the function f is continuous at c.

#2 $f(x)=\begin{cases}1-3x^2 & \text{if } x\neq 0\\ 1 & \text{if } x=0\end{cases}$ at $c=0$

$\lim_{x\to 0} f(0) = 1-3(0)^2$

$= 1$

$= f(0)$

$\boxed{\text{Continuous}}$

#6. $f(x)=\begin{cases}2x+1 & \text{if } x\leq 0\\ 2x & \text{if } x>0\end{cases}$ at $c=0$

$\lim_{x\to 0^-} f(x) = 2(0)+1 = 1$

$\lim_{x\to 0^+} f(x) = 2(0) = 0$

$\lim_{x\to 0}$ DNE

$\boxed{\text{Not continuous}}$

Determine the value of the constant k that will make the function f continuous for all x.

$f(x)=\begin{cases}1-4x & \text{if } x<2\\ kx^2-3x+2 & \text{if } 2\leq x\end{cases}$

limits must be equal

$\lim_{x\to 2} 1-4x = \lim_{x\to 2} kx^2-3x+2$

$1-4(2) = k(2)^2-3(2)+2$

$-7 = 4k-4$

$-3 = 4k$

$\boxed{k=-\frac{3}{4}}$

Is the function f defined by $f(x)=\dfrac{x^2+x-12}{x-3}$ continuous at 3?

no, $f(3)$ undefined

If not, can f be redefined at 3 to make it continuous at 3?

$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x) = \dfrac{(x-3)(x+4)}{(x-3)}$

$= \lim_{x\to 3} (x+4) = 7$

so if $f(x) = \begin{cases}\dfrac{x^2+x-12}{x-3} & x\neq 3\\ 7 & x=3\end{cases}$

is continuous