8.1: Systems of Differential Equations

Systems of Differential Equations

For a system of equations, the solution is the values of the variables which make all the equations in the system true. For a system of differential equations, the solution is the set of solution function curves which make all the differential equations in the system true.

In this first section, we'll mainly be learning new terminology and notation, but the ideas will be very similar to what we've encountered for single differential equations.

I think the easiest way to learn this is to consider a specific example in familiar terms and use it to introduce the new notations.

Consider this system of two differential equations. Each has a different dependent variable (*x* or *y*), but both have the same independent variable (*t*)...this is true for systems of differential equations we are considering in this course: for ordinary differential equations, we can have multiple dependent variables, but only one independent variable.

$$\begin{cases} \dfrac{dx}{dt} = x + 3y + 12t - 11 \\ \dfrac{dy}{dt} = 5x + 3y - 3 \end{cases}$$

Part of the RHS involves the dependent variables, and part does not. By omitting the part which does not involve dependent variables, we form the **corresponding homogenous system of DEs**:

$$\begin{cases} \dfrac{dx}{dt} = x + 3y \\ \dfrac{dy}{dt} = 5x + 3y \end{cases}$$

Using techniques we will learn later, we may have solutions for the homogeneous equation, so let's assume we somehow know them now. There will be one solution for each differential equation in the system:

corresponding homogenous system of DEs:

$$\begin{cases} \dfrac{dx}{dt} = x + 3y \\ \dfrac{dy}{dt} = 5x + 3y \end{cases}$$

complementary solutions:

$$\begin{cases} x_C = C_1 e^{-2t} + 3C_2 e^{6t} \\ y_C = -C_1 e^{-2t} + 5C_2 e^{6t} \end{cases}$$

We might be asked to verify if this set of complementary solutions is indeed a solution to the corresponding homogenous system of DEs, which we could verify like this...

corresponding homogenous system of DEs:

$$\begin{cases} \dfrac{dx}{dt} = x + 3y \\ \dfrac{dy}{dt} = 5x + 3y \end{cases}$$

complementary solutions:

$$\begin{cases} x_C = C_1 e^{-2t} + 3C_2 e^{6t} \\ y_C = -C_1 e^{-2t} + 5C_2 e^{6t} \end{cases}$$

$$x_C = C_1 e^{-2t} + 3C_2 e^{6t} \qquad y_C = -C_1 e^{-2t} + 5C_2 e^{6t}$$

$$\frac{dx}{dt} = -2C_1 e^{-2t} + 18C_2 e^{6t} \qquad \frac{dy}{dt} = 2C_1 e^{-2t} + 30C_2 e^{6t}$$

$$\frac{dx}{dt} = x + 3y$$

$$-2C_1 e^{-2t} + 18C_2 e^{6t} \stackrel{?}{=} \left(C_1 e^{-2t} + 3C_2 e^{6t}\right) + 3\left(-C_1 e^{-2t} + 5C_2 e^{6t}\right)$$

$$\stackrel{?}{=} -2C_1 e^{-2t} + 18C_2 e^{6t} \ \ \textit{verified}$$

$$\frac{dy}{dt} = 5x + 3y$$

$$2C_1 e^{-2t} + 30C_2 e^{6t} \stackrel{?}{=} 5\left(C_1 e^{-2t} + 3C_2 e^{6t}\right) + 3\left(-C_1 e^{-2t} + 5C_2 e^{6t}\right)$$

$$\stackrel{?}{=} 2C_1 e^{-2t} + 30C_2 e^{6t} \ \ \textit{verified}$$

We might be given a set of particular solutions for the original non-homogeneous system of DEs and asked to verify if this solution is indeed a solution for the system, like this:

non-homogenous system of DEs:

$$\begin{cases} \dfrac{dx}{dt} = x + 3y + 12t - 11 \\ \dfrac{dy}{dt} = 5x + 3y - 3 \end{cases}$$

particular solutions:

$$\begin{cases} x_P = 3t - 4 \\ y_P = -5t + 6 \end{cases}$$

$$x_P = 3t - 4 \qquad y_P = -5t + 6$$

$$\frac{dx}{dt} = 3 \qquad \frac{dy}{dt} = -5$$

$$\frac{dx}{dt} = x + 3y + 12t - 11$$

$$3 \stackrel{?}{=} (3t-4) + 3(-5t+6) + 12t - 11$$

$$\stackrel{?}{=} 3t - 15t + 12t - 4 + 18 - 11$$

$$\stackrel{?}{=} 3 \ \ \textit{verified}$$

$$\frac{dy}{dt} = 5x + 3y - 3$$

$$-5 \stackrel{?}{=} 5(3t-4) + 3(-5t+6) - 3$$

$$\stackrel{?}{=} 15t - 15t - 20 + 18 - 3$$

$$\stackrel{?}{=} -5 \ \ \textit{verified}$$

Matrix Forms

Because everything is being repeated for each equation's dependent variable, a convenient way to compactly represent everything is using **matrices** which can also be thought of as **vectors**. So there are corresponding **matrix (vector) forms** for all of the things we've just seen:

Matrix Form for the DE system:

$$\begin{cases} \dfrac{dx}{dt} = x + 3y + 12t - 11 \\ \dfrac{dy}{dt} = 5x + 3y - 3 \end{cases} \qquad \frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 5 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 12t - 11 \\ -3 \end{bmatrix}$$

$$\frac{d}{dt}\vec{X} = \begin{bmatrix} 1 & 3 \\ 5 & 3 \end{bmatrix}\vec{X} + \vec{F}$$

$$\boxed{\vec{X}' = \begin{bmatrix} 1 & 3 \\ 5 & 3 \end{bmatrix}\vec{X} + \vec{F}}$$

$$\text{where } \vec{X} = \begin{bmatrix} x \\ y \end{bmatrix} \text{ and } \vec{F} = \begin{bmatrix} 12t - 11 \\ -3 \end{bmatrix}$$

...and sometimes the non-dependent variable part is written as separate column vectors like this...

$$\boxed{\vec{X}' = \begin{bmatrix} 1 & 3 \\ 5 & 3 \end{bmatrix}\vec{X} + \begin{bmatrix} 12 \\ 0 \end{bmatrix} t + \begin{bmatrix} -11 \\ -3 \end{bmatrix}}$$

The solutions can also be expressed in matrix/vector form:

Matrix Form for the complementary solutions:

$$\begin{cases} x_C = C_1 e^{-2t} + 3C_2 e^{6t} \\ y_C = -C_1 e^{-2t} + 5C_2 e^{6t} \end{cases} \qquad \boxed{\vec{X}_C = C_1\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-2t} + C_2\begin{bmatrix} 3 \\ 5 \end{bmatrix} e^{6t}}$$

Matrix Form for the particular solutions:

$$\begin{cases} x_P = 3t - 4 \\ y_P = -5t + 6 \end{cases} \qquad \boxed{\vec{X}_P = \begin{bmatrix} 3t - 4 \\ -5t + 6 \end{bmatrix}}$$

Superposition Principle for Systems

Just as with individual DEs, the general solution is the linear combination by the superposition principle of the complementary and particular solutions:

$$\vec{X} = \vec{X}_C + \vec{X}_P$$

Separate equation form of the general solution:

$$\begin{cases} x = C_1 e^{-2t} + 3C_2 e^{6t} + 3t - 4 \\ y = -C_1 e^{-2t} + 5C_2 e^{6t} - 5t + 6 \end{cases}$$

Matrix Form of the general solution:

$$\vec{X} = C_1\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-2t} + C_2\begin{bmatrix} 3 \\ 5 \end{bmatrix} e^{6t} + \begin{bmatrix} 3t - 4 \\ -5t + 6 \end{bmatrix}$$

Solution Curves

The solution to a single differential equation is a function which has a solution curve (with the constants undetermined, it is a family of functions, if we have initial conditions specified, and know the values of the constants, one particular curve).

Let's say, for example, that we have been given initial conditions and were able to solve for the constants to find that $C_1 = 2 \;\; and \;\; C_2 = 3$:

$$\vec{X} = C_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-2t} + C_2 \begin{bmatrix} 3 \\ 5 \end{bmatrix} e^{6t} + \begin{bmatrix} 3t-4 \\ -5t+6 \end{bmatrix}$$

...becomes...

$$\vec{X} = 2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-2t} + 3 \begin{bmatrix} 3 \\ 5 \end{bmatrix} e^{6t} + \begin{bmatrix} 3t-4 \\ -5t+6 \end{bmatrix}$$

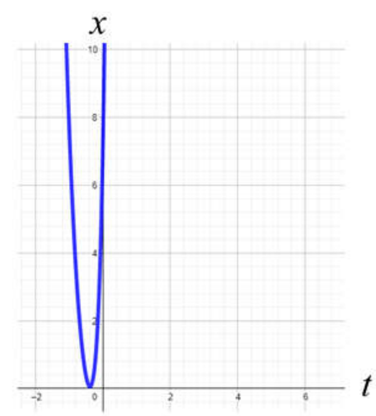
Solution Curves (the way it isn't done)

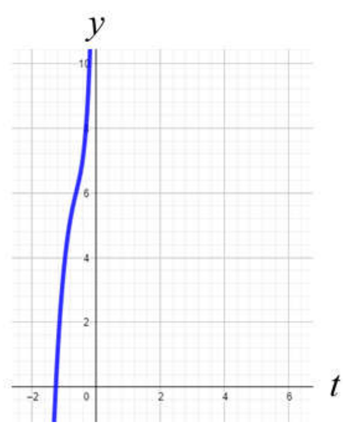
If we were considering each DE separately (which we don't do when we work with systems), then the solutions curves would be plotted individually as *x vs. t* and as *y vs. t*:

$$\vec{X} = 2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-2t} + 3 \begin{bmatrix} 3 \\ 5 \end{bmatrix} e^{6t} + \begin{bmatrix} 3t-4 \\ -5t+6 \end{bmatrix}$$

$$x(t) = 2e^{-2t} + 9e^{6t} + 3t - 4$$

$$y(t) = -e^{-2t} + 15e^{6t} - 5t + 6$$



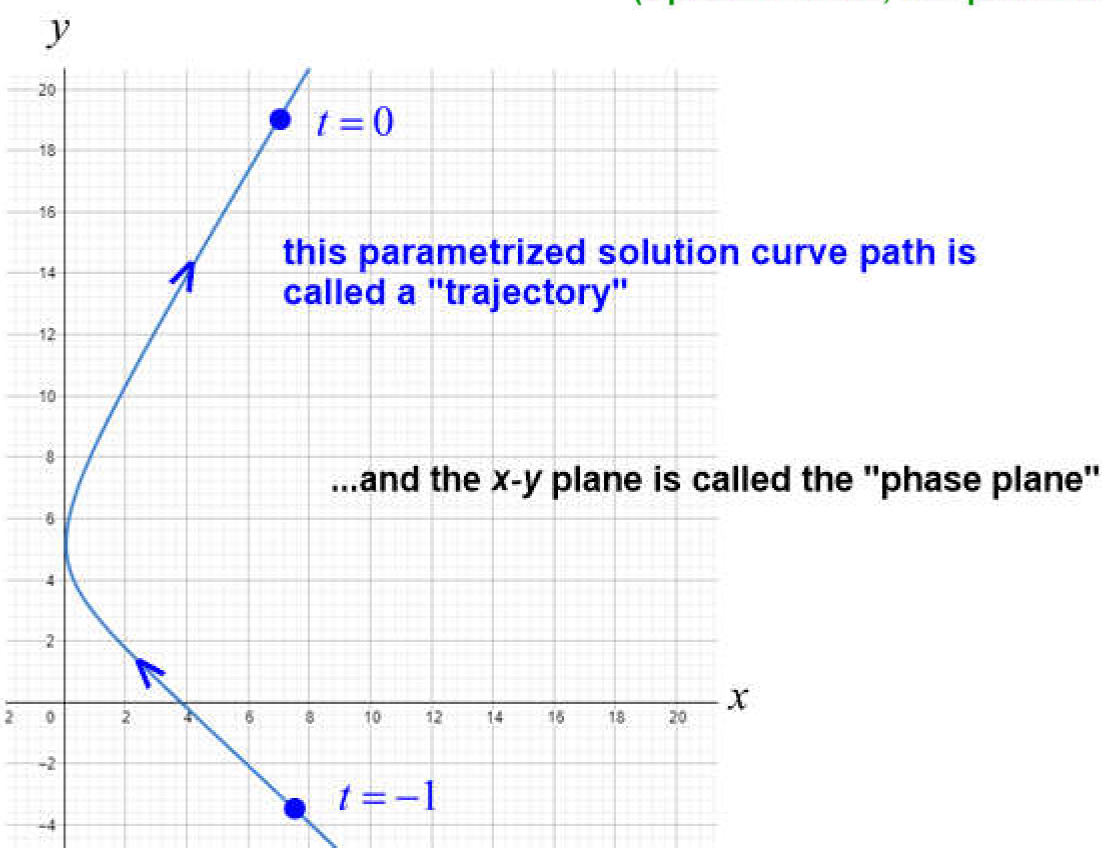


Solution Curves **(the way it _is_ done)**

Instead, because the system is considered a system because there is some relationship between x and y, we treat the two equations for *x* and *y* as **parametric equations** and consider the solution vector to be a position vector tracing out the locus of solution points in the (x,y) solution plane. The independent variable, *t*, becomes the **parameter**:

$$\vec{X} = 2\begin{bmatrix}1\\-1\end{bmatrix}e^{-2t} + 3\begin{bmatrix}3\\5\end{bmatrix}e^{6t} + \begin{bmatrix}3t-4\\-5t+6\end{bmatrix} \longrightarrow \vec{X} = \begin{bmatrix}x(t)\\y(t)\end{bmatrix} = \begin{bmatrix}2e^{-2t}+9e^{6t}+3t-4\\-e^{-2t}+15e^{6t}-5t+6\end{bmatrix}$$

(a position vector, with parameter *t*)



Fundamental Solution Sets, Linear Independence, and the Wronskian

In order to form the complete general solution for the corresponding homogenous system of DEs, the terms in the complementary solution must form a fundamental solution set, and the way we check this for systems is by using a Wronskian, but with a slightly different form:

$$Let\ \vec{X}_1 = \begin{bmatrix}x_{11}\\x_{12}\\.\\x_{n1}\end{bmatrix},\ \vec{X}_2 = \begin{bmatrix}x_{12}\\x_{22}\\.\\x_{n2}\end{bmatrix},\ ....\ \vec{X}_n = \begin{bmatrix}x_{1n}\\x_{2n}\\.\\x_{nn}\end{bmatrix}$$

...be *n* solutions of a homogeneous system of DEs, then the set of solution vectors is linearly independent and forms a fundamental solution set if and only if the Wronskian

$$W\left(\vec{X}_1, \vec{X}_2 \ldots \vec{X}_n\right) = \begin{vmatrix}x_{11} & x_{12} & \cdots & x_{1n}\\x_{21} & x_{22} & \cdots & x_{2n}\\. & . & \cdots & .\\x_{n1} & x_{n2} & \cdots & x_{nn}\end{vmatrix} \neq 0$$

Fundamental Solution Sets, Linear Independence, and the Wronskian

For the example we are considering here...

$$\begin{cases} x_C = \ \ C_1e^{-2t} + 3C_2e^{6t} \\ y_C = -C_1e^{-2t} + 5C_2e^{6t} \end{cases}$$

...the Wronskian is formed and computed like this...

$$\begin{aligned} W &= \begin{vmatrix} e^{-2t} & 3e^{6t} \\ -e^{-2t} & 5e^{6t} \end{vmatrix} \\ &= \left(e^{-2t}\right)\left(5e^{6t}\right) - \left(3e^{6t}\right)\left(-e^{-2t}\right) \\ &= 5e^{4t} + 3e^{4t} \\ &= 8e^{4t} \neq 0 \end{aligned}$$

so these form a fundamental solution set for the homogeneous system

Note that, for a system, the Wronskian is simply populated with the complementary solution terms (without the C constants), rather than formed using derivatives directly.

Higher dimensions

Everything we've discussed is extendable to any number of *dependent* variables (we can still have only one independent variable, otherwise we are dealing with partial differential equations).

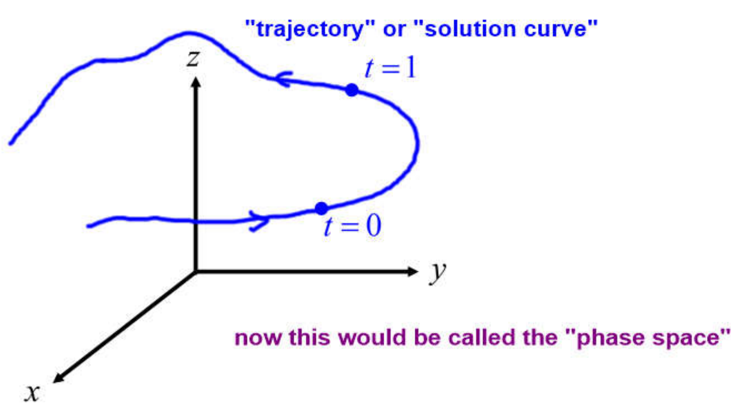
If we add a 3rd dependent variable, *z*:

$$\begin{cases} \dfrac{dx}{dt} = 6x + y + z + t \\ \dfrac{dy}{dt} = 3x + 7y - z + 10t - 3 \\ \dfrac{dz}{dt} = 2x + 9y - z + 6t \end{cases}$$

The matrix form would be:

$$\vec{X}' = \begin{bmatrix} 6 & 1 & 1 \\ 3 & 7 & -1 \\ 2 & 9 & -1 \end{bmatrix} \vec{X} + \begin{bmatrix} t \\ 10t - 3 \\ 6t \end{bmatrix} \qquad \text{with} \qquad \vec{X} = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$$

...and the solution curve would now be a parmetrized position vector in a 3-D solution space:



8.2 day 1: Eigenvalues, Eigenvectors, and DE system solutions

Finding the solutions for a System of Differential Equations

Now that we know how to express systems of differential equations and their solutions in matrix form, we can borrow some ideas from Linear Algebra to learn methods of finding the solutions for systems of DEs.

We will use something called **eigenvalues** and **eigenvectors**, and to help us see what these are, we'll first review some things about systems of regular equations...

(This assumes you have knowledge of matrix arithmetic like adding/subtracting matrices, multiplying a matrix by a scalar, or multiplying two matrices together, finding determinants, and knowing about the identity matrix. If you need a refresher, please see me in tutoring :)

(There is a lot more to know about eigenvalues and eigenvectors than what we are going to be using in this class, and if you have taken Honors Linear Algebra and already know about these, and have other ways of finding them, you are welcome to use those methods - same thing, we are just going to present and explain here how to find them and use them in solving systems of DEs)

Regular equation systems: single solution, or infinitely many solutions?

Consider these two systems of regular equations in 3 unknowns:

$$\begin{cases} x+\ y-\ \ z=6 \\ 2x-3y+2z=1 \\ -x+2y-3z=-1 \end{cases} \qquad \begin{cases} x+\ y-\ \ z=6 \\ 2x-3y+2z=1 \\ 3x+3y-3z=18 \end{cases}$$

We could use the augmented matrix / rref method to find the solutions to these systems...

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 6 \\ 2 & -3 & 2 & 1 \\ -1 & 2 & -3 & -1 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & 1 & -1 & 6 \\ 2 & -3 & 2 & 1 \\ 3 & 3 & -3 & 18 \end{array}\right]$$

rref   rref

$$\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & -\frac{1}{5} & \frac{19}{5} \\ 0 & 1 & -\frac{4}{5} & \frac{4}{5} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

If we write out the equations after rref...

$$\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad \begin{matrix} x=4 \\ y=3 \\ z=1 \end{matrix} \qquad \begin{bmatrix} 1 & 0 & -\frac{1}{5} & \frac{19}{5} \\ 0 & 1 & -\frac{4}{5} & \frac{4}{5} \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} x-\frac{1}{5}z=\frac{19}{5} \\ y-\frac{4}{5}z=\frac{4}{5} \\ 0=0 \end{matrix}$$

This system has a single solution:

$$(4,3,1)$$

This system has infinitely many solutions of the form:

$$\left(\frac{1}{5}z+\frac{19}{5},\ \frac{4}{5}z+\frac{4}{5},\ z\right)$$

...and here are a few specific solutions:

$$\left(\frac{19}{5},\ \frac{4}{5},\ 0\right) \quad \left(4,\ \frac{8}{5},\ 1\right) \quad \left(\frac{21}{5},\ \frac{12}{5},\ 2\right)$$

Regular equation systems: using inverse matrices instead of rref

We could try to solve these same two system using the matrix equation/inverse matrix method:

$$\begin{cases} x+\ y-\ \ z=6 \\ 2x-3y+2z=1 \\ -x+2y-3z=-1 \end{cases} \qquad \begin{cases} x+\ y-\ \ z=6 \\ 2x-3y+2z=1 \\ 3x+3y-3z=18 \end{cases}$$

First, we would express the system as a matrix or vector equation:

$$\begin{bmatrix} 1 & 1 & -1 \\ 2 & -3 & 2 \\ -1 & 2 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 6 \\ 1 \\ -1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & -1 \\ 2 & -3 & 2 \\ 3 & 3 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 6 \\ 1 \\ 18 \end{bmatrix}$$

$$\vec{A} \qquad \vec{X} = \vec{B} \qquad\qquad \vec{A} \qquad \vec{X} = \vec{B}$$

Solve by multiplying on the left by the inverse of matrix *A*:

$$\vec{A}^{-1} \bullet \vec{A}\ \vec{X} = \vec{A}^{-1} \bullet \vec{B} \qquad \vec{A}^{-1} \bullet \vec{A}\ \vec{X} = \vec{A}^{-1} \bullet \vec{B}$$

$$\vec{I}\ \vec{X} = \vec{A}^{-1} \bullet \vec{B} \qquad \vec{I}\ \vec{X} = \vec{A}^{-1} \bullet \vec{B}$$

$$\vec{X} = \vec{A}^{-1} \bullet \vec{B} \qquad \vec{X} = \vec{A}^{-1} \bullet \vec{B}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 5/8 & 1/8 & -1/8 \\ 1/2 & -1/2 & -1/2 \\ 1/8 & -3/8 & -5/8 \end{bmatrix}\begin{bmatrix} 6 \\ 1 \\ -1 \end{bmatrix} \qquad \begin{bmatrix} x \\ y \\ z \end{bmatrix}=[error]\begin{bmatrix} 6 \\ 1 \\ 18 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix}$$

For a system with a single solution, the inverse of the *A* matrix will exist, and matrix *A* is called a **non-singular** matrix.

$$\begin{vmatrix} 1 & 1 & -1 \\ 2 & -3 & 2 \\ -1 & 2 & -3 \end{vmatrix}=8$$

The determinant of a non-singular matrix is non-zero, so determinant not equal to zero means there is a single solution.

For a system with infinitely many solutions, the inverse of the *A* matrix will **<u>not</u>** exist, and matrix *A* is called a **singular** matrix.

$$\begin{vmatrix} 1 & 1 & -1 \\ 2 & -3 & 2 \\ 3 & 3 & -3 \end{vmatrix}=0$$

The determinant of a singular matrix=0, so determinant equal to zero means there is more than one solution to the system.

Regular equation systems with RHS zero: non-trivial solutions only if determinant = 0

If we did these same steps with systems with zero RHS...

$$\begin{cases} -x + y - z = 0 \\ -x + 2y + 3z = 0 \\ 3x - y + 4z = 0 \end{cases}$$

Matrix equation form:

$$\begin{bmatrix} -1 & 1 & -1 \\ -1 & 2 & 3 \\ 3 & -1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\vec{A} \quad \vec{X} = \vec{B}$$

$$\vec{X} = \vec{A}^{-1} \bullet \vec{B}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 11/29 & -3/29 & 5/29 \\ 13/29 & 7/29 & -2/29 \\ -5/29 & 4/29 & 3/29 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$ The single solution is (0,0,0).

$$\begin{vmatrix} -1 & 1 & -1 \\ -1 & 2 & 3 \\ 3 & -1 & 4 \end{vmatrix} = 29$$

...and the fact that the determinant is non-zero tells us that the only solution for this system is the 'trivial solution' (0,0,0).

$$\begin{cases} 2x - y - z = 0 \\ x + y - 5z = 0 \\ 3x + y - 9z = 0 \end{cases}$$

Matrix equation form:

$$\begin{bmatrix} 2 & -1 & -1 \\ 1 & 1 & -5 \\ 3 & 1 & -9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\vec{A} \quad \vec{X} = \vec{B}$$

$$\vec{X} = \vec{A}^{-1} \bullet \vec{B} \qquad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = [error] \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Using rref method instead...

$$\begin{bmatrix} 2 & -1 & -1 & 0 \\ 1 & 1 & -5 & 0 \\ 3 & 1 & -9 & 0 \end{bmatrix} rref \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} x - 2z = 0 \\ y - 3z = 0 \end{matrix}$$

Infinitely many solutions of form: $(2z,\ 3z,\ z)$

$$\begin{vmatrix} 2 & -1 & -1 \\ 1 & 1 & -5 \\ 3 & 1 & -9 \end{vmatrix} = 0$$

...and the fact that the determinant = 0 tells us that there are multiple solutions to this system, not just the trivial solution.

So if you have a system with RHS zero, the only way to have non-trivial solutions is if the determinant of the coefficient matrix is zero.

Eigenvalues and Eigenvectors

Now we'll define something new from Linear Algebra, **eigenvalues** and **eigenvectors**.

For a coefficient matrix, *A*:

$$\begin{bmatrix} 2 & -1 & -1 \\ 1 & 1 & -5 \\ 3 & 1 & -9 \end{bmatrix}$$

If you can find a scalar λ and a vector $\vec{K}$ such that: $\vec{A}\vec{K} = \lambda \vec{K}$

then $\lambda = an\ eigenvalue\ of\ \vec{A}$

$\vec{K} = an\ eigenvector\ of\ \vec{A}$

Find the Eigenvalues of a matrix

To find the eigenvalues of a matrix, we'll do some linear algebra manipulation of the eigenvalue definition equation:

$$\vec{A}\vec{K} = \lambda \vec{K}$$

$$\vec{A}\vec{K} - \lambda \vec{K} = \vec{0}$$

$$\vec{A}\vec{K} - \lambda \vec{I}\vec{K} = \vec{0}$$

$$\left(\vec{A} - \lambda \vec{I}\right)\vec{K} = \vec{0}$$

This form is similar to a matrix equation for a system, it will only have non-trivial solutions for the eigenvectors $\vec{K}$ if the matrix inside the parentheses has a determinant which equals zero:

$$\det\left(\vec{A} - \lambda \vec{I}\right) = \left|\vec{A} - \lambda \vec{I}\right| = 0$$

The eigenvalues are the values of λ which make this determinant zero.

Ex) Find the eigenvalues of *A* if $A = \begin{bmatrix} 2 & -1 & -1 \\ 1 & 1 & -5 \\ 3 & 1 & -9 \end{bmatrix}$

This equation is used to find the eigenvalues: $\boxed{\left|\vec{A} - \lambda \vec{I}\right| = 0}$

$$\left|\vec{A} - \lambda \vec{I}\right| = 0$$

$$\left|\begin{bmatrix} 2 & -1 & -1 \\ 1 & 1 & -5 \\ 3 & 1 & -9 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right| = 0$$

$$\begin{vmatrix} 2-\lambda & -1 & -1 \\ 1 & 1-\lambda & -5 \\ 3 & 1 & -9-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)\left[(1-\lambda)(-9-\lambda)-(-5)(1)\right]-(-1)\left[(1)(-9-\lambda)-(-5)(3)\right]+(-1)\left[(1)(1)-(1-\lambda)(3)\right]=0$$

$$-\lambda^3 - 6\lambda^2 + 16\lambda = 0$$ ← This is called the **characteristic equation** of the matrix.

$$\lambda\left(\lambda^2 + 6\lambda - 16\right) = 0$$

$$\lambda(\lambda+8)(\lambda-2) = 0$$

$$\lambda = 0,\ \lambda = -3,\ \lambda = 2$$ ← This matrix has 3 **eigenvalues**.

Find the corresponding Eigenvectors for each Eigenvalue of a matrix

Ex) Find the eigenvectors of *A* if $A=\begin{bmatrix}2 & -1 & -1\\ 1 & 1 & -5\\ 3 & 1 & -9\end{bmatrix}$

Once we have the eigenvalues for the matrix:

$$\lambda=0,\ \lambda=-3,\ \lambda=2$$

...we use this equation to find each eigenvalue's corresponding eigenvector:

$$\left(\vec{A}-\lambda\vec{I}\right)\vec{K}=\vec{0}$$

$\underline{for\ \lambda=0}$:

$$\left(\vec{A}-\lambda\vec{I}\right)\vec{K}=\vec{0}$$

$$\left(\begin{bmatrix}2 & -1 & -1\\ 1 & 1 & -5\\ 3 & 1 & -9\end{bmatrix}-(0)\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right)\begin{bmatrix}k_1\\ k_2\\ k_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$$\begin{bmatrix}2 & -1 & -1\\ 1 & 1 & -5\\ 3 & 1 & -9\end{bmatrix}\begin{bmatrix}k_1\\ k_2\\ k_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

use rref to solve this system of equations:

$$\begin{bmatrix}2 & -1 & -1 & 0\\ 1 & 1 & -5 & 0\\ 3 & 1 & -9 & 0\end{bmatrix} rref \begin{bmatrix}1 & 0 & -2 & 0\\ 0 & 1 & -3 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$$

$$k_1-2k_3=0$$
$$k_2-3k_3=0$$
$$\left(2k_3,\ 3k_3,\ k_3\right)$$

The eigenvector which corresponds to $\lambda=0$ is:

$$\begin{bmatrix}2\\ 3\\ 1\end{bmatrix}$$

choose any k_3: (except zero) $\left(2k_3,\ 3k_3,\ k_3\right)\rightarrow(2,\ 3,\ 1)$

We can then find eigenvectors for the other two eigenvalues...

$$\left(\vec{A}-\lambda\vec{I}\right)\vec{K}=\vec{0}$$

$\underline{for\ \lambda=-8}$:

$$\left(\vec{A}-\lambda\vec{I}\right)\vec{K}=\vec{0}$$

$$\left(\begin{bmatrix}2 & -1 & -1\\ 1 & 1 & -5\\ 3 & 1 & -9\end{bmatrix}-(-8)\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right)\begin{bmatrix}k_1\\ k_2\\ k_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$$\begin{bmatrix}2+8 & -1 & -1\\ 1 & 1+8 & -5\\ 3 & 1 & -9+8\end{bmatrix}\begin{bmatrix}k_1\\ k_2\\ k_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

use rref to solve this system of equations:

$$\begin{bmatrix}10 & -1 & -1 & 0\\ 1 & 9 & -5 & 0\\ 3 & 1 & -1 & 0\end{bmatrix} rref \begin{bmatrix}1 & 0 & -2/13 & 0\\ 0 & 1 & -7/13 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$$

$$k_1-\frac{2}{13}k_3=0$$
$$k_2-\frac{7}{13}k_3=0$$
$$\left(\frac{2}{13}k_3,\ \frac{7}{13}k_3,\ k_3\right)$$

$\underline{for\ \lambda=2}$:

$$\left(\vec{A}-\lambda\vec{I}\right)\vec{K}=\vec{0}$$

$$\left(\begin{bmatrix}2 & -1 & -1\\ 1 & 1 & -5\\ 3 & 1 & -9\end{bmatrix}-(2)\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 00\\ 0 & & 1\end{bmatrix}\right)\begin{bmatrix}k_1\\ k_2\\ k_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$$\begin{bmatrix}2-2 & -1 & -1\\ 1 & 1-2 & -5\\ 3 & 1 & -9-2\end{bmatrix}\begin{bmatrix}k_1\\ k_2\\ k_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

use rref to solve this system of equations:

$$\begin{bmatrix}0 & -1 & -1 & 0\\ 1 & -1 & -5 & 0\\ 3 & 1 & -11 & 0\end{bmatrix} rref \begin{bmatrix}1 & 0 & -4 & 0\\ 0 & 1 & 1 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$$

$$k_1-4k_3=0$$
$$k_2+k_3=0$$
$$\left(4k_3,\ -k_3,\ k_3\right)$$

choose any k_3: (except zero) $\left(\frac{2}{13}k_3,\ \frac{7}{13}k_3,\ k_3\right) \to (2,\ 7,\ 13)$

The eigenvector which corresponds to $\lambda = -8$ is:

$$\begin{bmatrix} 2 \\ 7 \\ 13 \end{bmatrix}$$

choose any k_3: (except zero) $(4k_3,\ -k_3,\ k_3) \to (4,\ -1,\ 1)$

The eigenvector which corresponds to $\lambda = 2$ is:

$$\begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix}$$

Solving a System of DEs using Eigenvalues and Eigenvectors

Okay, now that we know how to find the eigenvalues and eigenvectors for a matrix, we can learn how to use them to solve systems of differential equations.

In the last section, we were given a system like this:

$$\begin{cases} \frac{dx}{dt} = x + 3y \\ \frac{dy}{dt} = 5x + 3y \end{cases}$$

...and we converted it into matrix form like this:

$$\vec{X}' = \begin{bmatrix} 1 & 3 \\ 5 & 3 \end{bmatrix} \vec{X}$$

Then they gave us a solution like this:

$$\vec{X} = C_1 \vec{X}_1 + C_2 \vec{X}_2$$

$$\vec{X} = C_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-2t} + C_2 \begin{bmatrix} 3 \\ 5 \end{bmatrix} e^{6t}$$

...and we just verified that the solution was valid.

But many of the given solutions all had this same form: $\vec{X} = C_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-2t} + C_2 \begin{bmatrix} 3 \\ 5 \end{bmatrix} e^{6t}$

...which suggests that maybe this is a general form for all, or at least many, solutions to systems of DEs.

We could postulate the existence of a general solution to a system of DEs where (ignoring the *C* constant for now) the general form would be:

$$\vec{X} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} e^{\lambda t} \qquad or \qquad \vec{X} = \vec{K} e^{\lambda t}$$

If we took the derivative of both sides...

$$\vec{X}' = \lambda \vec{K} e^{\lambda t}$$ (the *K* vector is filled with only constants)

So for a system of DEs of the form: $\vec{X}' = \vec{A} \vec{X}$

If the solutions were of the form: $\vec{X} = \vec{K} e^{\lambda t}$ with derivative $\vec{X}' = \lambda \vec{K} e^{\lambda t}$

then substituting these into the system DE equation... $\vec{X}' = \vec{A} \vec{X}$

$$\lambda \vec{K} e^{\lambda t} = \vec{A} \vec{K} e^{\lambda t}$$

Dividing out the exponential factor and rearranging we get:

$$\lambda \vec{K} = \vec{A} \vec{K}$$

$$\vec{A} \vec{K} - \lambda \vec{K} = 0$$

$$\vec{A} \vec{K} - \lambda \vec{I} \vec{K} = 0$$

$$\left(\vec{A} - \lambda \vec{I}\right) \vec{K} = 0$$

Which means we will have non-trivial $\vec{K}$ only if the determinant of what's in the parentheses is zero.

That makes λ and $\vec{K}$ an **eigenvalue** and **eigenvector** of the coefficient matrix for the system of DEs.

Solving a System of DEs using Eigenvalues and Eigenvectors

This is easiest to see with a specific example:

Ex) Find the general solution to this system:

$$\begin{cases} \dfrac{dx}{dt} = x + 3y \\ \dfrac{dy}{dt} = 5x + 3y \end{cases}$$

Write in matrix form:

$$\vec{X}' = \vec{A}\,\vec{X}$$

$$\vec{X}' = \begin{bmatrix} 1 & 3 \\ 5 & 3 \end{bmatrix}\vec{X}$$

Find the eigenvalues of matrix *A*:

$$\left|\vec{A} - \lambda\,\vec{I}\right| = 0$$

$$\left\|\begin{bmatrix} 1 & 3 \\ 5 & 3 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right\| = 0$$

$$\begin{vmatrix} 1-\lambda & 3 \\ 5 & 3-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(3-\lambda) - (3)(5) = 0$$

$$\lambda^2 - 4\lambda - 12 = 0$$

$$(\lambda + 2)(\lambda - 6) = 0$$

$$\lambda = -2,\ \lambda = 6$$

Find the corresponding eigenvectors:

$\underline{\lambda = -2}$

$$\left(\vec{A} - \lambda\,\vec{I}\right)\vec{K} = \vec{0}$$

$$\left(\begin{bmatrix} 1 & 3 \\ 5 & 3 \end{bmatrix} - (-2)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\vec{K} = \vec{0}$$

$$\begin{bmatrix} 1+2 & 3 \\ 5 & 3+2 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 3 & 0 \\ 5 & 5 & 0 \end{bmatrix} rref \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$k_1 + k_2 = 0$$

$$(-k_2,\ k_2) \underset{choose\ k_2}{\longrightarrow} (-1,\ 1)$$

$$\begin{bmatrix} -1 \\ 1 \end{bmatrix} \ for\ \lambda = -2$$

$\underline{\lambda = 6}$

$$\left(\vec{A} - \lambda\,\vec{I}\right)\vec{K} = \vec{0}$$

$$\left(\begin{bmatrix} 1 & 3 \\ 5 & 3 \end{bmatrix} - (6)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\vec{K} = \vec{0}$$

$$\begin{bmatrix} 1-6 & 3 \\ 5 & 3-6 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -5 & 3 & 0 \\ 5 & -3 & 0 \end{bmatrix} rref \begin{bmatrix} 1 & -3/5 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$k_1 - \frac{3}{5}k_2 = 0$$

$$\left(\frac{3}{5}k_2,\ k_2\right) \underset{choose\ k_2}{\longrightarrow} (3,\ 5)$$

$$\begin{bmatrix} 3 \\ 5 \end{bmatrix} \ for\ \lambda = 6$$

Use each eigenvalue/eigenvector pair to form one term of the solution for the system:

$$\begin{bmatrix} -1 \\ 1 \end{bmatrix} \ for\ \lambda = -2 \qquad \begin{bmatrix} 3 \\ 5 \end{bmatrix} \ for\ \lambda = 6$$

$$\vec{X} = C_1\begin{bmatrix} -1 \\ 1 \end{bmatrix}e^{-2t} + C_2\begin{bmatrix} 3 \\ 5 \end{bmatrix}e^{6t}$$

Finding the constants with an initial condition

Then if we are given an initial condition, we can find the constants:

$$\vec{X}' = \begin{bmatrix} 1 & 3 \\ 5 & 3 \end{bmatrix} \vec{X} \qquad \vec{X}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

← This is equivalent to:

$$x(0) = 1$$
$$y(0) = 2$$

Once we solve, we write our solution for each dependent variable separately:

$$\vec{X} = C_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t} + C_2 \begin{bmatrix} 3 \\ 5 \end{bmatrix} e^{6t}$$

$$x(t) = -C_1 e^{-2t} + 3C_2 e^{6t} \qquad y(t) = C_1 e^{-2t} + 5C_2 e^{6t}$$

...plug in our initial conditions to get a system for the constants:

$$1 = -C_1 e^{-2(0)} + 3C_2 e^{6(0)} \qquad 2 = C_1 e^{-2(0)} + 5C_2 e^{6(0)}$$

$$1 = -C_1 + 3C_2 \qquad 2 = C_1 + 5C_2$$

$$\begin{cases} -C_1 + 3C_2 = 1 \\ C_1 + 5C_2 = 2 \end{cases}$$

...solve the system: $\begin{bmatrix} -1 & 3 & 1 \\ 1 & 5 & 2 \end{bmatrix} rref \begin{bmatrix} 1 & 0 & 1/8 \\ 0 & 1 & 3/8 \end{bmatrix}$

...and fill in the constants to form the general solution:

$$\boxed{\vec{X} = \frac{1}{8} \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t} + \frac{3}{8} \begin{bmatrix} 3 \\ 5 \end{bmatrix} e^{6t}}$$

8.2 day 2: Repeated Eigenvalues

Just as with auxiliary equations encountered earlier, there may be repeated roots

In the last section we saw that for a system of DEs expressed in matrix form:

$$\vec{X}' = \vec{A}\vec{X}$$

If matrix A is n x n, in the examples we've seen so far, there were n distinct real eigenvalues, and there was one corresponding eigenvector for each eigenvalue. Furthermore, these eigenvectors were linearly independent, so the general solution of the system of DEs could be written as:

$$\vec{X} = C_1\vec{K}_1 e^{\lambda_1 t} + C_2\vec{K}_2 e^{\lambda_2 t} + \ldots + C_n\vec{K}_n e^{\lambda_n t}$$

This is similar to what we saw earlier in the course where we found that higher order differential equations had multiple solutions combined together using the superposition principle, and we were often using an auxiliary equation to identify the individual term's solution.

Back the, sometimes the auxiliary equation had separate distinct roots, which resulted in combining multiple terms each of the same form, but sometimes we had repeated real roots and sometimes complex roots. The same can happen with systems...sometimes, there will be repeated eigenvalues or complex eigenvalues. In this section, we explore how to handle the repeated eigenvalues case.

Sometimes, a single eigenvalue can have more than one corresponding eigenvector

For some systems, when we have a repeated eigenvalue, it will turn out that we can find multiple linearly independent eigenvectors corresponding to this same eigenvalue. The following example shows how this can occur and how we handle this case:

Ex) Find the general solution for $\vec{X}' = \begin{bmatrix} 1 & -2 & 2 \\ -2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}\vec{X}$

We start by finding eigenvalues: $\begin{vmatrix} 1-\lambda & -2 & 2 \\ -2 & 1-\lambda & -2 \\ 2 & -2 & 1-\lambda \end{vmatrix} = 0$

$$(1-\lambda)\left[(1-\lambda)(1-\lambda)-(-2)(-2)\right]-(-2)\left[(-2)(1)-(-2)(2)\right]+2\left[(-2)(-2)-(1)(2)\right]$$

$$-\lambda^3+3\lambda^2+9\lambda+5=0$$

$$-\left(\lambda^3-3\lambda^2-9\lambda-5\right)=0$$

Using synthetic division to try values, we find that -1 works, so this factors into:

$$-(\lambda+1)\left(\lambda^2-4\lambda-5\right)=0$$

$$-(\lambda+1)(\lambda+1)(\lambda-5)=0$$

$$-(\lambda+1)^2(\lambda-5)=0$$

So we have one distinct eigenvalue: $\lambda = 5$

and a repeated eigenvalue of multiplicity 2 at: $\lambda = -1$

Sometimes, a single eigenvalue can have more than one corresponding eigenvector

So we have one distinct eigenvalue: $\lambda = 5$

and a repeated eigenvalue of multiplicity 2 at: $\lambda = -1$

For the single eigenvalue at $\lambda = 5$, we proceed normally to find the corresponding eigenvector:

$$\left(\vec{A} - \lambda \vec{I}\right)\vec{K} = \vec{0}$$

$$\left(\begin{bmatrix} 1 & -2 & 2 \\ -2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1-\lambda & -2 & 2 \\ -2 & 1-\lambda & -2 \\ 2 & -2 & 1-\lambda \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

which results in solving this augmented matrix system using rref and getting these equations:

$$\begin{bmatrix} -4 & -2 & 2 & 0 \\ -2 & -4 & -2 & 0 \\ 2 & -2 & -4 & 0 \end{bmatrix} rref \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} k_1 - k_3 = 0 \quad or \quad k_1 = k_3 \\ k_2 + k_3 = 0 \quad or \quad k_2 = -k_3 \end{matrix}$$

which results *k* values of the form: $\left(k_3, \ -k_3, \ k_3\right)$

We can choose anything we want for k_3 (except zero) in order to get an eigenvector for $\lambda = 5$

choose $k_3 = 1$ or $k_3 = 2$ or $k_3 = -4$

$$\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix} \qquad \begin{bmatrix} -4 \\ 4 \\ -4 \end{bmatrix}$$

But regardless of what value we choose for k_3 we are still dealing with the same eigenvector.

These three eigenvectors are not linearly independent, because they are just constant multiples of each other. That is why it is fine for us to choose any of them to be the eigenvector for $\lambda = 5$

Sometimes, a single eigenvalue can have more than one corresponding eigenvector

So we have one distinct eigenvalue: $\lambda = 5$

and a repeated eigenvalue of multiplicity 2 at: $\lambda = -1$

We can do the same steps for the single eigenvalue at $\lambda = -1$:

$$\left(\vec{A} - \lambda \vec{I}\right)\vec{K} = \vec{0}$$

$$\left(\begin{bmatrix} 1 & -2 & 2 \\ -2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1-\lambda & -2 & 2 \\ -2 & 1-\lambda & -2 \\ 2 & -2 & 1-\lambda \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

which results in solving this augmented matrix system using rref and getting these equations:

$$\begin{bmatrix} 2 & -2 & 2 & 0 \\ -2 & 2 & -2 & 0 \\ 2 & -2 & 2 & 0 \end{bmatrix} rref \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad k_1 - k_2 + k_3 = 0 \quad or \quad k_1 = k_2 - k_3$$

which results *k* values of the form: $\left(k_2 - k_3,\ k_2,\ \ k_3\right)$

Sometimes (but not always) we will get a system with a single equation like this where to establish k_1, we need to pick both k_2 and k_3 and we can pick two different combinations which result in linearly independent eigenvectors both associated with the same eigenvalue, like this:

choose $k_2 = 1,\ k_3 = 0$

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

choose $k_2 = 1,\ k_3 = 1$

$$\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

These two eigenvectors are not constant multiples of each other so each forms a separate, linearly independent solution term. We would then combine each of these with the eigenvector for the other eigenvalue into the general solution like this:

$$\vec{X} = C_1\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}e^{5t} + C_2\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}e^{-t} + C_3\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}e^{-t}$$

More commonly, there is only a single eigenvector for a repeated eigenvalue

So sometimes we get lucky and can form multiple eigenvectors for the repeated eigenvalue, one for each 'multiplicity' of that distinct real root. But usually, this is not the case; usually, there will only be a single eigenvector even for a repeated eigenvalue of multiplicity 2 or higher. We need a method to find a second solution in this case.

When this happened with standard auxiliary equations before, we added a second term multiplied by an extra *x*:

$$y = C_1 e^{m_1 x} + C_2 x e^{m_1 x}$$

...and something fairly similar works here. The second solution will include multiplying by an extra *t* but also include a term of the original form, but with a 2nd, different eigenvector.

...and something fairly similar works here. The second solution will include multiplying by an extra *t* but also include a term of the original form, but with a 2nd, different eigenvector.

Here is the form for a solution with a single eigenvalue of multiplicity 2:

$$\vec{X} = C_1 \vec{K} e^{\lambda_1 t} + C_2 \left(\vec{K} t e^{\lambda_1 t} + \vec{P} e^{\lambda_1 t} \right)$$

To see how we compute this new 2nd eigenvector P let's assume this is the correct solution form and look at just the new, 2nd, solution (ignoring the constant for now):

For a 2nd-order system: $\vec{X}' = \vec{A}\vec{X}$

There are two terms in the solution: $\vec{X} = C_1 \vec{X}_1 + C_2 \vec{X}_2$

and there is a single eigenvalue λ with multiplicity 2

first solution is: $\vec{X}_1 = \vec{K} e^{\lambda_1 t}$

assume second solution is of the form: $\vec{X}_2 = \vec{K} t e^{\lambda_1 t} + \vec{P} e^{\lambda_1 t}$

Taking the derivative of this 2nd solution:

$$\vec{X}_2' = \vec{K} t\left(\lambda_1 e^{\lambda_1 t}\right) + \vec{K} e^{\lambda_1 t} + \vec{P} \lambda_1 e^{\lambda_1 t}$$

$$= \lambda_1 \vec{K} t e^{\lambda_1 t} + \left(\vec{K} + \vec{P} \lambda_1\right) e^{\lambda_1 t}$$

Substituting X' and X into the system and rearranging:

$$\lambda_1 \vec{K} t e^{\lambda_1 t} + \left(\vec{K} + \vec{P} \lambda_1\right) e^{\lambda_1 t} = \vec{A}\left(\vec{K} t e^{\lambda_1 t} + \vec{P} e^{\lambda_1 t}\right)$$

$$\left(\vec{A}\vec{K} - \lambda_1 \vec{K}\right) t e^{\lambda_1 t} + \left(\vec{A}\vec{P} - \vec{K} - \lambda_1 \vec{P}\right) e^{\lambda_1 t} = 0$$

For this equation to be true for all t both of the expressions in the parentheses must equal zero:

$$\left(\vec{A}\vec{K} - \lambda_1 \vec{K}\right) = \vec{0} \qquad \vec{A}\vec{P} - \vec{K} - \lambda_1 \vec{P} = 0$$

$$\left(\vec{A} - \lambda_1 \vec{I}\right)\vec{K} = \vec{0} \qquad \vec{A}\vec{P} - \lambda_1 \vec{I}\vec{P} = \vec{K}$$

$$\left(\vec{A} - \lambda_1 \vec{I}\right)\vec{P} = \vec{K}$$

The left equation is saying that the K vector is the regular eigenvector for the eigenvalue λ_1

The right equation is saying that we can find the vector P by using the same procedure we would for finding and eigenvector, but with the eigenvector K for the RHS in the rref matrix instead of a zero vector, and if we follow this procedure, the resulting X_2 will also be a solution to the system.

Example: finding the general solution with repeated eigenvalue with only one associated eigenvector

Ex) Find the general solution for $\vec{X}' = \begin{bmatrix} 3 & -18 \\ 2 & -9 \end{bmatrix} \vec{X}$

First find eigenvectors for the matrix... $\begin{vmatrix} 3-\lambda & -18 \\ 2 & -9-\lambda \end{vmatrix} = 0$

$$(3-\lambda)(-9-\lambda)-(-18)(2)=0$$
$$\lambda^2+6\lambda+9=0$$
$$(\lambda+3)^2=0$$

One eigenvalue with mulitplicity two: $\lambda = -3$

Find corresponding eigenvector(s): $\begin{bmatrix} 3-(-3) & -18 & 0 \\ 2 & -9-(-3) & 0 \end{bmatrix}$

$$\begin{bmatrix} 6 & -18 & 0 \\ 2 & -6 & 0 \end{bmatrix} rref \begin{bmatrix} 1 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$k_1 - 3k_2 = 0 \quad or \quad k_1 = 3k_2 \qquad (3k_2,\ k_2)$ choose k_2=*1* $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$

Now we need a 2nd, linearly independent eigenvector, so redo the procedure, but this time using the eigenvector we just found as the RHS instead of zeros:

$$\begin{bmatrix} 3-(-3) & -18 & 3 \\ 2 & -9-(-3) & 1 \end{bmatrix}$$

$$\begin{bmatrix} 6 & -18 & 3 \\ 2 & -6 & 1 \end{bmatrix} rref \begin{bmatrix} 1 & -3 & 1/2 \\ 0 & 0 & 0 \end{bmatrix}$$

$p_1 - 3p_2 = \frac{1}{2} \quad or \quad p_1 = 3p_2 + \frac{1}{2} \qquad \left(3p_2 + \frac{1}{2},\ p_2\right)$ choose p_2=*0* $\begin{bmatrix} 1/2 \\ 0 \end{bmatrix}$

Now write the general solution using the form:

$$\vec{X} = C_1 \vec{X}_1 + C_2 \vec{X}_2$$
$$\vec{X} = C_1 \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} e^{\lambda_1 t} + C_2 \left(\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} t e^{\lambda_1 t} + \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} e^{\lambda_1 t} \right)$$

$$\boxed{\vec{X} = C_1 \begin{bmatrix} 3 \\ 1 \end{bmatrix} e^{-3t} + C_2 \left(\begin{bmatrix} 3 \\ 1 \end{bmatrix} t e^{-3t} + \begin{bmatrix} 1/2 \\ 0 \end{bmatrix} e^{-3t} \right)}$$

Eigenvalue with multiplicity 3

What if we have a single eigenvalue with multiplicity of 3? It turns out that you can extend this procedure by making including a 3rd solution with the following form:

$$\vec{X}_3 = \vec{K}\frac{t^2}{2}e^{\lambda_1 t} + \vec{P}\,te^{\lambda_1 t} + \vec{Q}\,e^{\lambda_1 t}$$

Where...

$$\left(\vec{A} - \lambda_1 \vec{I}\right)\vec{K} = \vec{0}$$

$$\left(\vec{A} - \lambda_1 \vec{I}\right)\vec{P} = \vec{K}$$

$$\left(\vec{A} - \lambda_1 \vec{I}\right)\vec{Q} = \vec{P}$$

So K is the original eigenvector for the eigenvalue, P is created using K on the RHS, and then Q is created using P on the RHS.

In fact, there are ways to extend this concept to higher multiplicities but multiplicity 3 is as high as we will see in this course.

A multiplicity of 3 example...

Ex) Find the general solution for $\vec{X}' = \begin{bmatrix} 2 & 1 & 6 \\ 0 & 2 & 5 \\ 0 & 0 & 2 \end{bmatrix} \vec{X}$

First find eigenvectors for the matrix... $\begin{vmatrix} 2-\lambda & 1 & 6 \\ 0 & 2-\lambda & 5 \\ 0 & 0 & 2-\lambda \end{vmatrix} = 0$

$$(2-\lambda)\left[(2-\lambda)(2-\lambda)\right] - 0 + 0 = 0$$

$$(2-\lambda)^3 = 0$$

One eigenvalue with mulitplicity three: $\lambda = 2$

Find corresponding eigenvector(s): $\begin{bmatrix} 2-2 & 1 & 6 & 0 \\ 0 & 2-2 & 5 & 0 \\ 0 & 0 & 2-2 & 0 \end{bmatrix}$

$$\begin{bmatrix} 0 & 1 & 6 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} rref \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$k_2 = 0 \quad and \quad k_3 = 0 \qquad (k_1,\ 0,\ 0)$ choose $k_1=1$ $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Now we need the 2nd vector...use the result from last step for the RHS:

$$\begin{bmatrix} 0 & 1 & 6 & 1 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} rref \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$p_2 = 1 \quad and \quad p_3 = 0 \qquad (p_1,\ 1,\ 0)$ choose $p_1=0$ $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

Next, we need the 3rd vector...use the result from last step for the RHS:

$$\begin{bmatrix} 0 & 1 & 6 & 0 \\ 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} rref \begin{bmatrix} 0 & 1 & 0 & -6/5 \\ 0 & 0 & 1 & 1/5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$q_2 = -6/5 \quad and \quad q_3 = 1/5 \qquad (q_1,\ -6/5,\ 1/5)$ choose $q_1=0$ $\begin{bmatrix} 0 \\ -6/5 \\ 1/5 \end{bmatrix}$

Finally, we write the general solution using the form:

$$\vec{X} = C_1 \vec{X}_1 + C_2 \vec{X}_2 + C_3 \vec{X}_3$$

$$\vec{X} = C_1 \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} e^{\lambda_1 t} + C_2 \left(\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} te^{\lambda_1 t} + \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} e^{\lambda_1 t} \right) + C_3 \left(\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} \frac{t^2}{2} e^{\lambda_1 t} + \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} te^{\lambda_1 t} + \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} e^{\lambda_1 t} \right)$$

$$\vec{X} = C_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{2t} + C_2 \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} te^{2t} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^{2t} \right) + C_3 \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \frac{t^2}{2} e^{2t} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} te^{2t} + \begin{bmatrix} 0 \\ -6/5 \\ 1/5 \end{bmatrix} e^{2t} \right)$$

Which is really this system of equations...

$$\begin{cases} x(t) = C_1 e^{2t} + C_2 te^{2t} + C_3 \frac{t^2}{2} e^{2t} \\ y(t) = C_2 e^{2t} + C_3 te^{2t} - \frac{6}{5} e^{2t} \\ z(t) = \frac{1}{5} C_3 e^{2t} \end{cases}$$

8.2 day 3: Complex Eigenvalues

Eigenvalues may be complex conjugate pairs

We've seen how to handle the cases when we have distinct, real eigenvalues, and also repeated real eigenvalues. The third possibility is that the eigenvalues may occur in complex conjugate pairs, so we need a procedure for handling this case.

We'll present this here as a procedure without derivation. There is a separate PDF which works the example we'll show in class but showing all intermediate steps that develop the procedure we are using if you are interested to have a more intuitive, conceptual understanding of what is happening as we use this procedure.

Procedure for finding a solution to a DE system with complex eigenvalues

1) Express the system in matrix form and find the eigenvalues using $\left|\vec{A}-\lambda\vec{I}\right|=0$

2) For any eigenvalues which appear as complex conjugate pairs, write these in the form $\lambda=\alpha\pm\beta i$

3) Using the positive case eigenvalue $\lambda=\alpha+\beta i$ use $\left[\vec{A}-\lambda\vec{I}\right]=\vec{0}$ to find the system and write out the equations (you won't be able to solve using calculator rref because of the imaginary values).

4) This will result in a system of equations for the eigenvector constants k_1, k_2, ... Use the equations to express all of the constants in terms of one (parameter) constant, then choose a convenient value for this constant to form the eigenvector $\vec{K}$

5) Find vectors for the real and imaginary parts of the eigenvector: $\vec{B}_1=\mathrm{Re}\left\{\vec{K}\right\},\ \ \vec{B}_2=\mathrm{Im}\left\{\vec{K}\right\}$

6) The system solution is then given by:

$$\vec{X}=C_1\vec{X}_1+C_2\vec{X}_2$$

$$\vec{X}_1=\left(\vec{B}_1\cos\beta t-\vec{B}_2\sin\beta t\right)e^{\alpha t}$$

$$\vec{X}_2=\left(\vec{B}_2\cos\beta t+\vec{B}_1\sin\beta t\right)e^{\alpha t}$$

An example...

Solve the system:

$$\frac{dx}{dt} = 6x - y$$

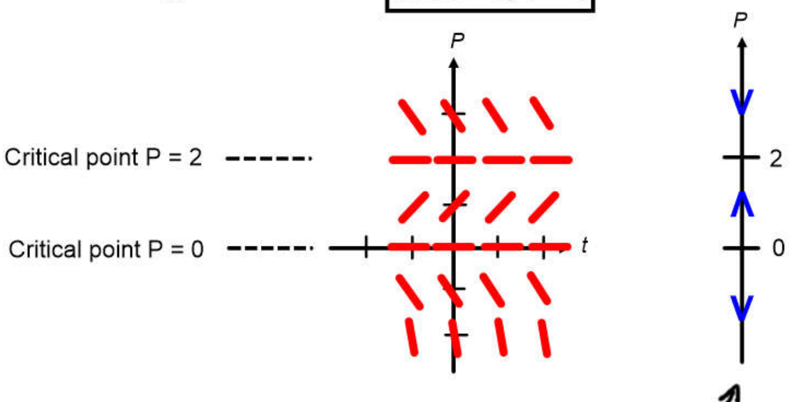
$$\frac{dy}{dt} = 5x + 4y$$

8.2 day 4: Phase Portraits

Remember 1D Phase Portraits?

For autonomous first-order DEs the RHS is only a function of the dependent variable, so the slope field has constant slope at a given y for all values of x. The zeros of the RHS function are critical points and we can make a 1D Phase Portrait:

$$\frac{dP}{dt} = P(2-P) \qquad \begin{array}{l} P(2-P)=0 \\ \boxed{when\ P=0,\ \ P=2} \end{array}$$



Such a diagram is called a **1D Phase Portrait (or Phase Line)**

For 2D systems of DEs we can define a 2D Phase Portrait

Consider this system:

$$\frac{dx}{dt} = 2x+3y \qquad \vec{X}' = \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix} \vec{X}$$

$$\frac{dy}{dt} = 2x+y$$

with initial condition:

$$\vec{X}(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

We can solve for eigenvalues:

$$\begin{vmatrix} 2-\lambda & 3 \\ 2 & 1-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)(1-\lambda)-6=0$$

$$\lambda^2-3\lambda-4=0$$

$$(\lambda+1)(\lambda-4)=0$$

$$\lambda=-1 \quad \lambda=4$$

Two distinct real roots, find eigenvector for each...

$\lambda=-1$

$$\begin{bmatrix} 2-(-1) & 3 & 0 \\ 2 & 1-(-1) & 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 3 & 0 \\ 2 & 2 & 0 \end{bmatrix} rref \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$k_1+k_2=0$$

$$k_1=-k_2$$

$$(-k_2,\ \ k_2)\ choose\ k_2=1$$

$$\begin{bmatrix} -1 \\ 1 \end{bmatrix} for\ \lambda=-1$$

$\lambda=4$

$$\begin{bmatrix} 2-(4) & 3 & 0 \\ 2 & 1-(4) & 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 3 & 0 \\ 2 & -3 & 0 \end{bmatrix} rref \begin{bmatrix} 1 & -3/2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$k_1-\frac{3}{2}k_2=0$$

$$k_1=\frac{3}{2}k_2$$

$$\left(\frac{3}{2}k_2,\ \ k_2\right)\ choose\ k_2=2$$

$$\begin{bmatrix} 3 \\ 2 \end{bmatrix} for\ \lambda=4$$

General solution:

$$\begin{bmatrix} -1 \\ 1 \end{bmatrix} for\ \lambda=-1 \qquad \begin{bmatrix} 3 \\ 2 \end{bmatrix} for\ \lambda=4$$

$$\vec{X} = C_1\begin{bmatrix} -1 \\ 1 \end{bmatrix}e^{-t} + C_2\begin{bmatrix} 3 \\ 2 \end{bmatrix}e^{4t}$$

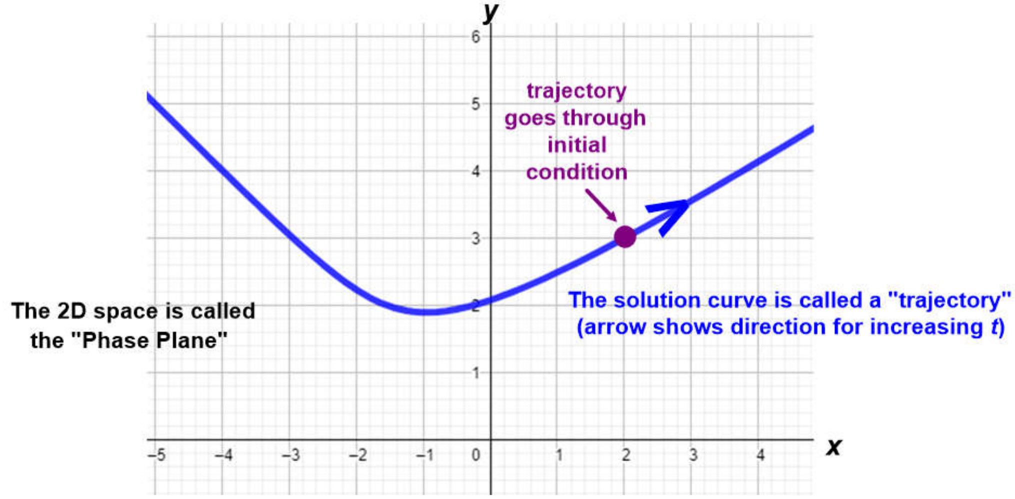
Using the initial condition, we find the constants:

$$\begin{cases} 2=-C_1+3C_2 \\ 3=C_1+2C_2 \end{cases} \qquad \begin{bmatrix} -1 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix} rref \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \quad C_1=1 \quad C_2=1$$

Specific solution... ...which can be written like this:

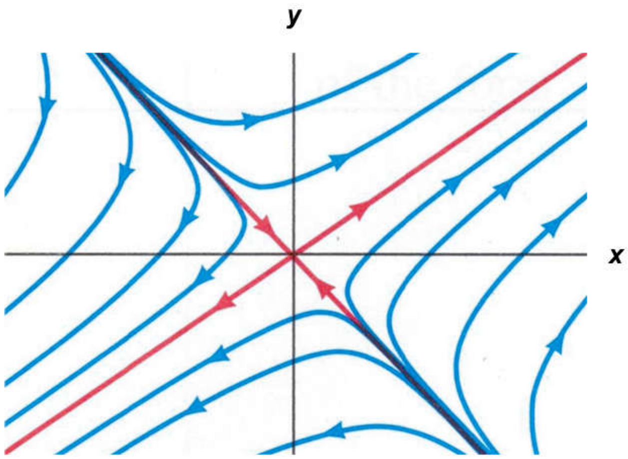
$$\vec{X} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-t} + \begin{bmatrix} 3 \\ 2 \end{bmatrix} e^{4t} \quad \dashrightarrow \quad \begin{aligned} x(t) &= -e^{-t} + 3e^{4t} \\ y(t) &= e^{-t} + 2e^{4t} \end{aligned}$$

Because the system has two dependent variables, x and y, we can use the x-y plane to represent the solution as a curve with parameter t...



$$\frac{dx}{dt} = 2x + 3y$$
$$\frac{dy}{dt} = 2x + y$$

If we solve for the constants for many different initial conditions and include multiple trajectories, the graph is called a **2D phase portrait:**

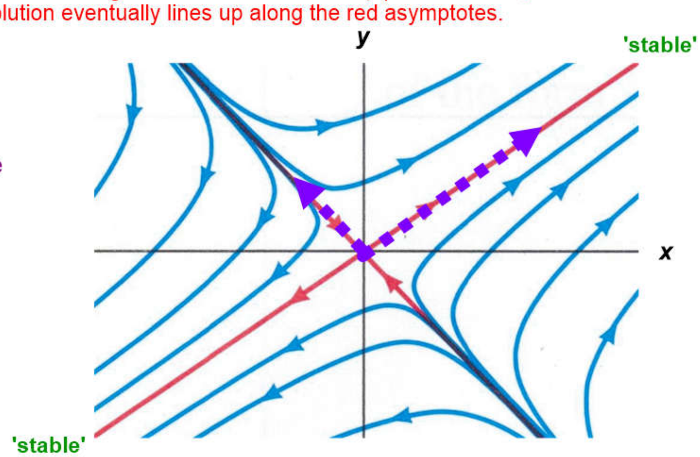


$$\frac{dx}{dt} = 2x + 3y$$
$$\frac{dy}{dt} = 2x + y$$

Notice that regardless of where in the x-y plane we start, the solution eventually lines up along the red asymptotes.

Interestingly, these asymptotes are in the direction of the eigenvectors:

$$\begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$



By carefully noting the direction along the trajectories as t increases, we can see that the system will tend towards specific paths which are 'stable'.

A 2nd example...

Consider this system:

$$\frac{dx}{dt} = 3x - 13y$$
$$\frac{dy}{dt} = 5x + y$$

$$\vec{X}' = \begin{bmatrix} 3 & -13 \\ 5 & 1 \end{bmatrix}\vec{X}$$

with initial condition:

$$\vec{X}(0) = \begin{bmatrix} 3 \\ -10 \end{bmatrix}$$

We can solve for eigenvalues:

$$\begin{vmatrix} 3-\lambda & -13 \\ 5 & 1-\lambda \end{vmatrix} = 0$$
$$(3-\lambda)(1-\lambda)+65=0$$
$$\lambda^2 - 4\lambda + 68 = 0$$
$$\lambda = \frac{4 \pm \sqrt{16-4(68)}}{2}$$
$$\lambda = 2 \pm 8i$$

Complex conjugate roots, find eigenvector...

$$\lambda = 2+8i$$
$$\begin{bmatrix} 3-(2+8i) & -13 & 0 \\ 5 & 1-(2+8i) & 0 \end{bmatrix}$$
$$\begin{bmatrix} 1-8i & -13 & 0 \\ 5 & -1-8i & 0 \end{bmatrix}$$
$$5k_1 + (-1-8i)k_2 = 0$$
$$k_1 = \frac{1}{5}(1+8i)k_2$$
$$\left(\frac{1}{5}(1+8i)k_2,\ k_2\right) \ \textit{choose}\ k_2 = 5$$
$$\vec{K} = \begin{bmatrix} 1+8i \\ 5 \end{bmatrix} \ \textit{for}\ \lambda = 2+8i$$

Now build the general solution:

$$\vec{B}_1 = \mathrm{Re}\left\{\begin{bmatrix} 1+8i \\ 5 \end{bmatrix}\right\} = \begin{bmatrix} 1 \\ 5 \end{bmatrix} \qquad \vec{B}_2 = \mathrm{Im}\left\{\begin{bmatrix} 1+8i \\ 5 \end{bmatrix}\right\} = \begin{bmatrix} 8 \\ 0 \end{bmatrix}$$

$$\vec{X} = C_1\left(\begin{bmatrix} 1 \\ 5 \end{bmatrix}\cos 8t - \begin{bmatrix} 8 \\ 0 \end{bmatrix}\sin 8t\right)e^{2t}$$
$$+ C_2\left(\begin{bmatrix} 8 \\ 0 \end{bmatrix}\cos 8t + \begin{bmatrix} 1 \\ 5 \end{bmatrix}\sin 8t\right)e^{2t}$$

$$\vec{X} = C_1\left(\begin{bmatrix} \cos 8t - 8\sin 8t \\ 5\cos 8t \end{bmatrix}\right)e^{2t} + C_2\left(\begin{bmatrix} 8\cos 8t + \sin 8t \\ 5\sin 8t \end{bmatrix}\right)e^{2t}$$

Use the initial condition to solve for the constants...

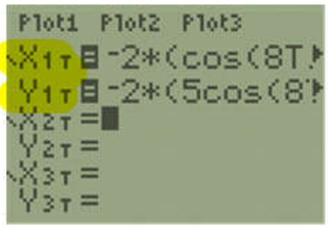
$$\begin{cases} 3 = C_1 + 8C_2 \\ -10 = 5C_1 \end{cases} \quad \begin{bmatrix} 1 & 8 & 3 \\ 5 & 0 & -10 \end{bmatrix} \textit{rref} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & \frac{5}{8} \end{bmatrix} \quad C_1 = -2 \quad C_2 = \frac{5}{8}$$

...so the specific solution is:

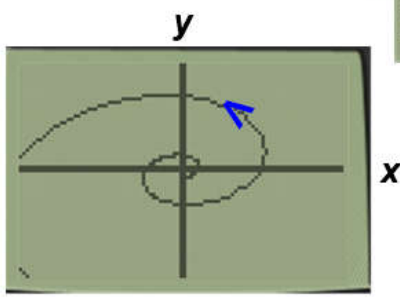
$$\vec{X} = -2\left(\begin{bmatrix} \cos 8t - 8\sin 8t \\ 5\cos 8t \end{bmatrix}\right)e^{2t} + \frac{5}{8}\left(\begin{bmatrix} 8\cos 8t + \sin 8t \\ 5\sin 8t \end{bmatrix}\right)e^{2t}$$

$$\vec{X} = -2\left(\begin{bmatrix} \cos 8t - 8\sin 8t \\ 5\cos 8t \end{bmatrix}\right)e^{2t} + \frac{5}{8}\left(\begin{bmatrix} 8\cos 8t + \sin 8t \\ 5\sin 8t \end{bmatrix}\right)e^{2t}$$

To graph, let's use the 'parametric' mode in our calculator:

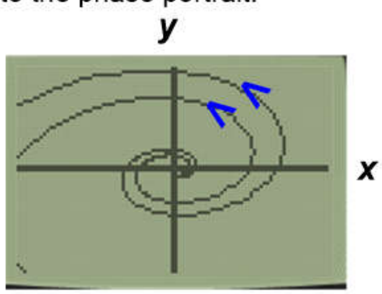


(important!)



If we change the initial condition: $\vec{X}(0) = \begin{bmatrix} 20 \\ 30 \end{bmatrix}$ The constants change to $C_1 = -2 \quad C_2 = \frac{11}{4}$

...and we add a second trajectory to the phase portrait:



Some books and online info label these situations in various ways 'asymptotically stable' and identify structures as 'nodes', 'centers', 'improper nodes', etc. but our book doesn't and it doesn't seem like the identifiers are consistent between different info sources, so we can just interpret the phase portrait directly from its trajectory shapes.

You try this one now...

Consider this system:

$$\frac{dx}{dt} = 2x + 8y$$

$$\frac{dy}{dt} = -x - 2y$$

with initial condition:

$$\vec{X}(0) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

8.3 day 1: Non-homogeneous Systems (Undetermined Coefficients)

Non-homogeneous systems of DEs

When we have differential equation systems like this...

$$\frac{dx}{dt} = 6x + y + 6t$$
$$\frac{dy}{dt} = 4x + 3y - 10t + 4$$

..with additional terms on the RHS of the independent variable, we can write in matrix form as...

$$\vec{X}' = \begin{bmatrix} 6 & 1 \\ 4 & 3 \end{bmatrix} \vec{X} + \begin{bmatrix} 6 \\ -10 \end{bmatrix} t + \begin{bmatrix} 0 \\ 4 \end{bmatrix}$$

...and these are known as **non-homogeneous systems of DEs**.

With single differential equations we had two methods to handle non-zero RHS and there are similar methods for each with systems:

Method of Undetermined Coefficients (table method)

Method of Variation of Parameters (kind of similar to the previous Wronskian method)

Method of Undetermined Coefficients for systems

Similar to with single equations, in the method of undetermined coefficients we postulate a solution by using a table to match RHS term forms, then take the appropriate derivative(s) and plug into the DE system, then solve for the coefficients *A,B,C,* etc.

As with single DEs, this method is faster, but doesn't always work, and there are issues with absorption along with other new issues. The bottom line is, you try to handle absorption (by multiplying by additional independent variables) and if you still can't solve for the constants, then you abandon this method and try tomorrow's more powerful (but more difficult) method.

An example...

$$\begin{cases} \dfrac{dx}{dt} = 6x + y + 6t \\ \dfrac{dy}{dt} = 4x + 3y - 10t + 4 \end{cases} \qquad \vec{X}' = \begin{bmatrix} 6 & 1 \\ 4 & 3 \end{bmatrix} \vec{X} + \begin{bmatrix} 6 \\ -10 \end{bmatrix} t + \begin{bmatrix} 0 \\ 4 \end{bmatrix}$$

First, we solve the homogeneous system to obtain the complementary solution:

$$\begin{cases} \dfrac{dx}{dt} = 6x + y \\ \dfrac{dy}{dt} = 4x + 3y \end{cases} \qquad \vec{X}' = \begin{bmatrix} 6 & 1 \\ 4 & 3 \end{bmatrix} \vec{X}$$

$$\begin{vmatrix} 6-\lambda & 1 \\ 4 & 3-\lambda \end{vmatrix} = 0$$
$$(6-\lambda)(3-\lambda) - 4 = 0$$
$$\lambda^2 - 9\lambda + 14 = 0$$
$$(\lambda-2)(\lambda-7) = 0$$
$$\lambda = 2 \quad \lambda = 7$$

$\underline{\lambda = 2}$

$$\begin{bmatrix} 4 & 1 & 0 \\ 4 & 1 & 0 \end{bmatrix} rref \begin{bmatrix} 1 & 1/4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
$$k_1 = -\frac{1}{4}k_2$$
$$\begin{bmatrix} 1 \\ -4 \end{bmatrix}$$

$\underline{\lambda = 7}$

$$\begin{bmatrix} -1 & 1 & 0 \\ 4 & -4 & 0 \end{bmatrix} rref \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
$$k_1 = k_2$$
$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

An example...

$$\begin{cases} \dfrac{dx}{dt} = 6x + y + 6t \\ \dfrac{dy}{dt} = 4x + 3y - 10t + 4 \end{cases} \qquad \vec{X}' = \begin{bmatrix} 6 & 1 \\ 4 & 3 \end{bmatrix} \vec{X} + \begin{bmatrix} 6 \\ -10 \end{bmatrix} t + \begin{bmatrix} 0 \\ 4 \end{bmatrix}$$

$$\vec{X}_C = C_1 \begin{bmatrix} 1 \\ -4 \end{bmatrix} e^{2t} + C_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{7t}$$

Now for the RHS we use the table to get a solution form. But because this is in matrix form for a system, whatever we guess <u>must be the same for all the rows</u>. So we have to build a form that includes something for all terms in all of the equations.

We then use the same table as we did back in 4.4. Here we have $6t$ and $-10t+4$ both first-degree polynomials, so we'll use this form for both rows (with unique constants):

$$\vec{X}_P = \begin{bmatrix} At + B \\ Ct + D \end{bmatrix}$$

We need to check that none of these terms match terms in the complementary function. If they did, we would multiply by extra *t* s until they didn't match, but here there is no absorption.

Now we take the derivative and plug into the DE system to find the constants:

$$\vec{X}'_P = \begin{bmatrix} A \\ C \end{bmatrix}$$

$$\vec{X}' = \begin{bmatrix} 6 & 1 \\ 4 & 3 \end{bmatrix} \vec{X} + \begin{bmatrix} 6 \\ -10 \end{bmatrix} t + \begin{bmatrix} 0 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} A \\ C \end{bmatrix} = \begin{bmatrix} 6 & 1 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} At + B \\ Ct + D \end{bmatrix} + \begin{bmatrix} 6 \\ -10 \end{bmatrix} t + \begin{bmatrix} 0 \\ 4 \end{bmatrix}$$

Next, matrix multiplication in first term on RHS, then write out each equation in the system:

$$\begin{bmatrix} A \\ C \end{bmatrix} = \begin{bmatrix} 6 & 1 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} At + B \\ Ct + D \end{bmatrix} + \begin{bmatrix} 6 \\ -10 \end{bmatrix} t + \begin{bmatrix} 0 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} A \\ C \end{bmatrix} = \begin{bmatrix} 6(At+B) + 1(Ct+D) \\ 4(At+B) + 3(Ct+D) \end{bmatrix} + \begin{bmatrix} 6 \\ -10 \end{bmatrix} t + \begin{bmatrix} 0 \\ 4 \end{bmatrix}$$

$$\begin{cases} A = 6At + 6B + Ct + D + 6t \\ C = 4At + 4B + 3Ct + 3D - 10t + 4 \end{cases}$$

By matching term coefficients, we obtain a system and solve for the constants...

$$\begin{cases} (6A + C + 6)t + (6B + D) = (0)t + (A) \\ (4A + 3C - 10)t + (4B + 3D + 4) = (0)t + (C) \end{cases}$$

$$\begin{cases} 6A + C + 6 = 0 \\ 6B + D = A \\ 4A + 3C - 10 = 0 \\ 4B + 3D + 4 = C \end{cases}$$

$$\begin{cases} 6A + 0B + 1C + 0D = -6 \\ -1A + 6B + 0C + 1D = 0 \\ 4A + 0B + 3C + 0D = 10 \\ 0A + 4B - 1C + 3D = -4 \end{cases}$$

$$\left[\begin{array}{cccc|c} 6 & 0 & 1 & 0 & -6 \\ -1 & 6 & 0 & 1 & 0 \\ 4 & 0 & 3 & 0 & 10 \\ 0 & 4 & -1 & 3 & -4 \end{array}\right]$$

rref

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 & -4/7 \\ 0 & 0 & 1 & 0 & 6 \\ 0 & 0 & 0 & 1 & 10/7 \end{array}\right]$$

$$A = -2, \quad B = -4/7, \quad C = 6, \quad D = 10/7$$

$$\vec{X}_P = \begin{bmatrix} At + B \\ Ct + D \end{bmatrix} = \begin{bmatrix} -2t - 4/7 \\ 6t + 10/7 \end{bmatrix}$$

$$\boxed{\vec{X} = C_1 \begin{bmatrix} 1 \\ -4 \end{bmatrix} e^{2t} + C_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{7t} + \begin{bmatrix} -2 \\ 6 \end{bmatrix} t + \begin{bmatrix} -4/7 \\ 10/7 \end{bmatrix}}$$

8.3 day 2: Non-homogeneous Systems (Variation of Parameters)

Variation of Parameters Method

In the previous section we learned the Method of Undetermined Coefficients (table method) for systems, which is nice if it works because it is faster but only works in some cases.

The Method of **Variation of Parameters** is more general and powerful, but involves more steps.

If we assume a homogeneous system of the form $\vec{X}' = \vec{A}\vec{X}$

has a solution of the form: $\vec{X} = C_1\vec{X}_1 + C_2\vec{X}_2 + \ldots + C_n\vec{X}_n$

then $\vec{X}_1, \vec{X}_2, \ldots \vec{X}_n$ is a fundamental solution set of the system, and we could write the general solution in this form:

$$\vec{X} = C_1\vec{X}_1 + C_2\vec{X}_2 + \ldots + C_n\vec{X}_n$$

$$\vec{X} = C_1\begin{bmatrix} x_{11} \\ x_{21} \\ . \\ x_{n1} \end{bmatrix} + C_2\begin{bmatrix} x_{12} \\ x_{22} \\ . \\ x_{n2} \end{bmatrix} + \ldots. + C_n\begin{bmatrix} x_{1n} \\ x_{2n} \\ . \\ x_{nn} \end{bmatrix}$$

$$\vec{X} = \begin{bmatrix} C_1x_{11} + C_2x_{12} + \ldots + C_nx_{1n} \\ C_2x_{21} + C_2x_{22} + \ldots + C_nx_{2n} \\ . \\ C_nx_{n1} + C_2x_{n2} + \ldots + C_nx_{nn} \end{bmatrix}$$

$$\vec{X} = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ & & & \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{bmatrix}\begin{bmatrix} C_1 \\ C_2 \\ . \\ C_n \end{bmatrix}$$

$$\vec{X} = \vec{\Phi}(t)\vec{C}$$

↑

Fundamental matrix of the system

Variation of Parameters Method

Because of the way the fundamental matrix is defined, its determinant is the same as the Wronskian of the fundamental solution set:

$$\det\left(\vec{\Phi}(t)\right) = W\left(\vec{X}_1, \vec{X}_2, \ldots \vec{X}_n\right)$$

...and because this is a linearly independent set of solutions to the system, the Wronskian and determinant would be non-zero, which means that...

...the inverse of the fundamental matrix must exist: $\vec{\Phi}^{-1}(t)$ exists

...and because every column of $\vec{\Phi}(t)$ is a solution vector of the system: $\vec{\Phi}'(t) = \vec{A}\vec{\Phi}(t)$

Variation of Parameters Method

If we now have a non-homogeneous system of DEs, the solution $\vec{X} = \vec{\Phi}(t)\vec{C}$ for the corresponding homogeneous system is the complementary solution of the non-homogeneous system. To get the particular solution, we will do something similar to what we did in 4.6: we will propose a vector $\vec{U}$ and assume that the particular solution is:

$$\vec{X}_P = \vec{\Phi}(t)\vec{U}$$

Then, we will take the derivative of this (using the product rule)... $\vec{X}_P' = \vec{\Phi}\vec{U}' + \vec{\Phi}'\vec{U}$

...and substitute into the non-homogeneous system (using $\vec{F}$ for the extra terms):

$$\vec{X}' = \vec{A}\vec{X} + \vec{F}$$

$$\vec{\Phi}\vec{U}' + \underline{\vec{\Phi}'}\vec{U} = \vec{A}\vec{\Phi}\vec{U} + \vec{F}$$

but we can replace $\vec{\Phi}'(t) = \vec{A}\vec{\Phi}(t)$

$$\vec{\Phi}\vec{U}' + \vec{A}\vec{\Phi}\vec{U} = \vec{A}\vec{\Phi}\vec{U} + \vec{F}$$

$$\vec{\Phi}\vec{U}' = \vec{F}$$

and because $\vec{\Phi}^{-1}$ exists...

$$\vec{\Phi}^{-1}\vec{\Phi}\vec{U}' = \vec{\Phi}^{-1}\vec{F}$$

$$\vec{I}\vec{U}' = \vec{\Phi}^{-1}\vec{F}$$

$$\vec{U}' = \vec{\Phi}^{-1}\vec{F}$$

to find $\vec{U}$ take the antiderivative... $\vec{U} = \int \vec{\Phi}^{-1}\vec{F}\,dt$

...and then form the particular solution: $\vec{X}_P = \vec{\Phi}\vec{U} = \vec{\Phi}\int \vec{\Phi}^{-1}\vec{F}\,dt$

...so the general solution is: $\vec{X} = \vec{X}_C + \vec{X}_P$

$$\boxed{\vec{X} = \vec{\Phi}\vec{C} + \vec{\Phi}\int \vec{\Phi}^{-1}\vec{F}\,dt}$$

Inverse of a 2x2 matrix

For us to use this, we will need to be able to take an inverse of the fundamental matrix. This is fairly straightforward for a 2x2 matrix and procedures exist for larger matrices (we'll consider 3x3 later), but here are the steps for one method of finding an inverse of a 2x2 matrix: $\vec{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ Find the determinant: $\det \vec{A} = ad - bc$

Find the inverse: $\vec{A}^{-1} = \frac{1}{\det \vec{A}}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ (swap elements on main diagonal, negate the other elements, multiple the whole thing by the reciprocal of the determinant)

An example...

We'll see how this all works by carefully doing one example together...

$$\frac{dx}{dt} = -3x + y + 3t$$
$$\frac{dy}{dt} = 2x - 4y + e^{-t}$$

$$\boxed{\vec{X} = \vec{\Phi}\vec{C} + \vec{\Phi}\int \vec{\Phi}^{-1}\vec{F}\,dt}$$

8.3 day 3: Non-homogeneous Systems with Initial Conditions

Solving a non-homogeneous system of DEs with initial conditions

So far with non-homogeneous solutions we've obtained the general solution with the C_1, C_2 constants, but haven't then been given an initial condition in order to find these final constants. Here we'll consider how to handle initial conditions.

Here is our general solutions form: $\vec{X} = \vec{\Phi}\vec{C} + \vec{\Phi}\int \vec{\Phi}^{-1}\vec{F}\,dt$

...and being more precise, everything except the constants are actually functions of t:

$$\vec{X}(t) = \vec{\Phi}(t)\vec{C} + \vec{\Phi}(t)\int \vec{\Phi}(t)^{-1}\vec{F}(t)\,dt$$

If we are given an initial condition: $\vec{X}(t_0) = \vec{X}_0$

...we know by the Fundamental Theorem of Calculus that $\int \vec{\Phi}(t)^{-1}\vec{F}(t)\,dt = \int_{t_0}^{t} \vec{\Phi}(s)^{-1}\vec{F}(s)\,ds$

...so then $\vec{X}(t) = \vec{\Phi}(t)\vec{C} + \vec{\Phi}(t)\int_{t_0}^{t} \vec{\Phi}(s)^{-1}\vec{F}(s)\,ds$

...and $\vec{X}(t_0) = \vec{X}_0 = \vec{\Phi}(t_0)\vec{C}$ so $\vec{C} = \vec{\Phi}(t_0)^{-1}\vec{X}_0$

(limits of integration chosen so that the particular solution vanishes at $t = t_0$)

...then

$$\boxed{\vec{X}(t) = \vec{\Phi}(t)\vec{\Phi}(t_0)^{-1}\vec{X}_0 + \vec{\Phi}(t)\int_{t_0}^{t} \vec{\Phi}(s)^{-1}\vec{F}(s)\,ds}$$

An example...

We'll see how this works by doing one example together...

$$\begin{cases} \dfrac{dx}{dt} = x - y + \dfrac{1}{t} \\ \dfrac{dy}{dt} = x - y + \dfrac{1}{t} \end{cases} \qquad \vec{X}(1) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$\boxed{\vec{X}(t) = \vec{\Phi}(t)\vec{\Phi}(t_0)^{-1}\vec{X}_0 + \vec{\Phi}(t)\int_{t_0}^{t}\vec{\Phi}(s)^{-1}\vec{F}(s)\,ds}$$

8.3 day 4: Higher-order single DEs as systems

You can represent a higher-order single DE as a system of first-order DEs

Interestingly, you can represent a higher-order single differential equation as a system of differential equations. This can be helpful if you have a DE which you can't solve using existing method, but could solve as a system.

A simple example to illustrate...

$$y'' - y' - 6y = 0$$

We already have methods to solve this...

$$m^2 - m - 6 = 0$$
$$(m-3)(m+2) = 0$$
$$m = 3, \quad m = -2$$
$$\boxed{y = C_1e^{3x} + C_2e^{-2x}}$$

Here is how this would be converted to a system. First, solve for the highest derivative:

$$y'' = 6y + y'$$

Add in a row above like this for form a system: $\begin{cases} y' = 0y + y' \\ y'' = 6y + y' \end{cases}$

Now define $\vec{X} = \begin{bmatrix} y \\ y' \end{bmatrix}$ and write the system in matrix form: $\vec{X}' = \begin{bmatrix} 0 & 1 \\ 6 & 1 \end{bmatrix}\vec{X}$

Now solve the system $\vec{X}' = \begin{bmatrix} 0 & 1 \\ 6 & 1 \end{bmatrix}\vec{X}$

$$\begin{vmatrix} -\lambda & 1 \\ 6 & 1-\lambda \end{vmatrix} = 0$$
$$(-\lambda)(1-\lambda) - 6 = 0$$
$$\lambda^2 - \lambda - 6 = 0$$
$$(\lambda - 3)(\lambda + 2) = 0$$
$$\lambda = 3 \quad \lambda = -2$$

$\underline{\lambda = 3}$

$$\begin{bmatrix} -3 & 1 & 0 \\ 6 & -2 & 0 \end{bmatrix} rref \begin{bmatrix} 1 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$k_1 = \frac{1}{3}k_2 \quad \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$\underline{\lambda = -2}$

$$\begin{bmatrix} 2 & 1 & 0 \\ 6 & 3 & 0 \end{bmatrix} rref \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$k_1 = -\frac{1}{2}k_2 \quad \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$\vec{X} = C_1\begin{bmatrix} 1 \\ 3 \end{bmatrix}e^{3x} + C_2\begin{bmatrix} 1 \\ -2 \end{bmatrix}e^{-2x}$$

$$\begin{bmatrix} y \\ y' \end{bmatrix} = \begin{bmatrix} C_1e^{3x} + C_2e^{-2x} \\ 3C_1e^{3x} - 2C_2e^{2x} \end{bmatrix}$$

← Here is the original DE solution...

← ...and you get the derivative too!

We won't work on solving, but let's practice putting DEs into system form...

Write as a system of first-order differential equations:

1) $y'' + 4y' + 5y = 35e^{-4t} \qquad y(0) = -3, \quad y'(0) = 1$

2) $y''' + 2y'' + 8y = 8t^2 + 2t - 5 \qquad y(0) = -5, \quad y'(0) = 3, \quad y''(0) = -4$

Why do this? Because you can solve higher-order for which you have no method

This is more work, so why do it? Only if you have a higher-order DE for which you have no other method. This also works with non-homogeneous DEs but the system then involves finding an inverse of 3x3 or higher fundamental matrices. To illustrate, I'll show the steps for a 3rd-order system which is just barely solvable using methods we already know.

We'll need to know how to find an inverse for a 3x3 matrix at one point. Here is an overview of how that is done...(we'll see specifically in the example):

In general, an inverse of matrix *A* is found by first finding the determinant of *A,* then finding the signed cofactors of every element in the matrix. A signed cofactor is the determinant you would use for that element if you were using it in finding the determinant of the entire matrix.

$$\vec{A} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix}$$

The signed cofactor matrix, *C*, of matrix *A* would be:

$$\vec{C} = \begin{bmatrix} \begin{vmatrix} e & f \\ h & k \end{vmatrix} & -\begin{vmatrix} d & f \\ g & k \end{vmatrix} & \begin{vmatrix} d & e \\ g & h \end{vmatrix} \\ -\begin{vmatrix} b & c \\ h & k \end{vmatrix} & \begin{vmatrix} a & c \\ g & k \end{vmatrix} & -\begin{vmatrix} a & b \\ g & h \end{vmatrix} \\ \begin{vmatrix} b & c \\ e & f \end{vmatrix} & -\begin{vmatrix} a & c \\ d & f \end{vmatrix} & \begin{vmatrix} a & b \\ d & e \end{vmatrix} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

You then find the transpose of the signed cofactor matrix by writing the rows as the columns:

$$\vec{C}^{T} = \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

Then the inverse of *A* is the transpose of the signed cofactor matrix times the reciprocal of the determinant of *A*:

$$\vec{A}^{-1} = \frac{1}{\det \vec{A}} \vec{C}^{T}$$

A higher-order non-homogeneous example solved as a system

$$y''' - y'' - 4y' + 4y = 5 - e^{x} + e^{2x}$$

solve for highest derivative...

$$y''' = -4y + 4y' + 1y'' + 5 - e^{x} + e^{2x}$$

write as a system...

$$\begin{cases} y' = 0y + 1y' + 0y'' + 0 \\ y'' = 0y + 0y' + 1y'' + 0 \\ y''' = -4y + 4y' + 1y'' + 5 - e^{x} + e^{2x} \end{cases}$$

write in matrix form...

$$\vec{X}' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & 4 & 1 \end{bmatrix} \vec{X} + \begin{bmatrix} 0 \\ 0 \\ 5 - e^{x} + e^{2x} \end{bmatrix}$$

now solve the corresponding homogeneous system...

$$\vec{X}' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & 4 & 1 \end{bmatrix} \vec{X} \qquad \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ -4 & 4 & 1-\lambda \end{vmatrix} = 0$$

$$(-\lambda)\left[(-\lambda)(1-\lambda) - 4\right] - 1\left[0 - (-4)\right] + 0 = 0$$

$$(-\lambda)\left(\lambda^2 - \lambda - 4\right) - 4 = 0$$

$$-\lambda^3 + \lambda^2 + 4\lambda - 4 = 0$$

$$\lambda^3 - \lambda^2 - 4\lambda + 4 = 0$$

$$\left(\lambda^3 - \lambda^2\right) + \left(-4\lambda + 4\right) = 0$$

$$\lambda^2(\lambda - 1) - 4(\lambda - 1) = 0$$

$$(\lambda - 1)\left(\lambda^2 - 4\right) = 0$$

$$(\lambda - 1)(\lambda - 2)(\lambda + 2) = 0$$

$$\lambda = 1 \quad \lambda = 2 \quad \lambda = -2$$

$\underline{\lambda = 1}$

$$\begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -4 & 4 & 0 & 0 \end{bmatrix} rref \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{cases} k_1 = k_3 \\ k_2 = k_3 \end{cases} \quad (k_3,\ k_3,\ k_3) \quad \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$\underline{\lambda = 2}$

$$\begin{bmatrix} -2 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ -4 & 4 & -1 & 0 \end{bmatrix} rref \begin{bmatrix} 1 & 0 & -1/4 & 0 \\ 0 & 1 & -1/2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{cases} k_1 = \frac{1}{4}k_3 \\ k_2 = \frac{1}{2}k_3 \end{cases} \quad \left(\frac{1}{4}k_3,\ \frac{1}{2}k_3,\ k_3\right) \quad \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$

$\underline{\lambda = -2}$

$$\begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ -4 & 4 & 3 & 0 \end{bmatrix} rref \begin{bmatrix} 1 & 0 & -1/4 & 0 \\ 0 & 1 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{cases} k_1 = \frac{1}{4}k_3 \\ k_2 = -\frac{1}{2}k_3 \end{cases} \quad \left(\frac{1}{4}k_3,\ -\frac{1}{2}k_3,\ k_3\right) \quad \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}$$

$$\vec{X}_C = C_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{x} + C_2 \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} e^{2x} + C_3 \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} e^{-2x}$$

then use variation of parameters to find the particular solution for the non-homogeneous system...

$$\vec{X}' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & 4 & 1 \end{bmatrix} \vec{X} + \begin{bmatrix} 0 \\ 0 \\ 5 - e^x + e^{2x} \end{bmatrix} \qquad \vec{X}_C = C_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^x + C_2 \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} e^{2x} + C_3 \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} e^{-2x}$$

from the corresponding solution we find the fundamental matrix for the system...

$$\vec{\Phi} = \begin{bmatrix} e^x & e^{2x} & e^{-2x} \\ e^x & 2e^{2x} & -2e^{-2x} \\ e^x & 4e^{2x} & 4e^{-2x} \end{bmatrix}$$

now we find the inverse of this matrix, $\vec{\Phi}^{-1}$ ...

first we need the determinant...

$$\det \vec{\Phi}^{-1} = e^x \begin{vmatrix} 2e^{2x} & -2e^{-2x} \\ 4e^{2x} & 4e^{-2x} \end{vmatrix} - e^{2x} \begin{vmatrix} e^x & -2e^{-2x} \\ e^x & 4e^{-2x} \end{vmatrix} + e^{-2x} \begin{vmatrix} e^x & 2e^{2x} \\ e^x & 4e^{2x} \end{vmatrix}$$

$$= e^x \left|8 + 8\right| - e^{2x} \left|4e^{-x} + 2e^{-x}\right| + e^{-2x} \left|4e^{3x} - 2e^{3x}\right|$$

$$= 16e^x - 6e^x + 2e^2$$

$$= 12e^x$$

next, we'll write out the signed cofactor matrix...

$$\vec{C} = \begin{bmatrix} \begin{vmatrix} 2e^{2x} & -2e^{-2x} \\ 4e^{2x} & 4e^{-2x} \end{vmatrix} & -\begin{vmatrix} e^x & -2e^{-2x} \\ e^x & 4e^{-2x} \end{vmatrix} & \begin{vmatrix} e^x & 2e^{2x} \\ e^x & 4e^{2x} \end{vmatrix} \\ -\begin{vmatrix} e^{2x} & e^{-2x} \\ 4e^{2x} & 4e^{-2x} \end{vmatrix} & \begin{vmatrix} e^x & e^{-2x} \\ e^x & 4e^{-2x} \end{vmatrix} & -\begin{vmatrix} e^x & e^{2x} \\ e^x & 4e^{2x} \end{vmatrix} \\ \begin{vmatrix} e^{2x} & e^{-2x} \\ 2e^{2x} & -2e^{-2x} \end{vmatrix} & -\begin{vmatrix} e^x & e^{-2x} \\ e^x & -2e^{-2x} \end{vmatrix} & \begin{vmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{vmatrix} \end{bmatrix}$$

computing all the minor determinants...

$$\vec{C} = \begin{bmatrix} 16 & -6e^{-x} & 2e^{3x} \\ 0 & 3e^{-x} & -3e^{3x} \\ -4 & 3e^{-x} & e^{3x} \end{bmatrix}$$

next, we find the transpose of the signed cofactor matrix...

$$\vec{C}^T = \begin{bmatrix} 16 & 0 & -4 \\ -6e^{-x} & 3e^{-x} & 3e^{-x} \\ 2e^{3x} & -3e^{3x} & e^{3x} \end{bmatrix}$$

then, we use the determinant and transpose to find the inverse matrix:

$$\vec{\Phi}^{-1} = \frac{1}{\det \vec{A}} \vec{C}^T$$

$$= \frac{1}{12e^x} \begin{bmatrix} 16 & 0 & -4 \\ -6e^{-x} & 3e^{-x} & 3e^{-x} \\ 2e^{3x} & -3e^{3x} & e^{3x} \end{bmatrix}$$

$$\vec{\Phi}^{-1} = \begin{bmatrix} \frac{4}{3}e^{-x} & 0 & -\frac{1}{3}e^{-x} \\ -\frac{1}{2}e^{-2x} & \frac{1}{4}e^{-2x} & \frac{1}{4}e^{-2x} \\ \frac{1}{6}e^{2x} & -\frac{1}{4}e^{2x} & \frac{1}{12}e^{2x} \end{bmatrix}$$

okay, now we can find the particular solution for the non-homogeneous system...

$$\vec{X}_P = \vec{\Phi} \int \vec{\Phi}^{-1} \vec{F}\, dx$$

$$= \begin{bmatrix} e^x & e^{2x} & e^{-2x} \\ e^x & 2e^{2x} & -2e^{-2x} \\ e^x & 4e^{2x} & 4e^{-2x} \end{bmatrix} \int \begin{bmatrix} \frac{4}{3}e^{-x} & 0 & -\frac{1}{3}e^{-x} \\ -\frac{1}{2}e^{-2x} & \frac{1}{4}e^{-2x} & \frac{1}{4}e^{-2x} \\ \frac{1}{6}e^{2x} & -\frac{1}{4}e^{2x} & \frac{1}{12}e^{2x} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 5 - e^x + e^{2x} \end{bmatrix} dx$$

multiplying the matrices inside in the integral...

$$\vec{X}_P = \begin{bmatrix} e^x & e^{2x} & e^{-2x} \\ e^x & 2e^{2x} & -2e^{-2x} \\ e^x & 4e^{2x} & 4e^{-2x} \end{bmatrix} \int \begin{bmatrix} -\frac{5}{3}e^{-x} & \frac{1}{3} & -\frac{1}{3}e^{x} \\ \frac{5}{4}e^{-2x} & -\frac{1}{4}e^{-x} & \frac{1}{4} \\ \frac{5}{12}e^{2x} & -\frac{1}{12}e^{3x} & \frac{1}{12}e^{4x} \end{bmatrix} dx$$

now taking the integral (of each row separately)...

$$\vec{X}_P = \begin{bmatrix} e^x & e^{2x} & e^{-2x} \\ e^x & 2e^{2x} & -2e^{-2x} \\ e^x & 4e^{2x} & 4e^{-2x} \end{bmatrix} \begin{bmatrix} \frac{5}{3}e^{-x} + \frac{1}{3}x - \frac{1}{3}e^x \\ -\frac{5}{8}e^{-2x} + \frac{1}{4}e^{-x} + \frac{1}{4}x \\ \frac{5}{24}e^{2x} - \frac{1}{36}e^{3x} + \frac{1}{48}e^{4x} \end{bmatrix}$$

then multiplying these two matrices...

$$\vec{X}_P = \begin{bmatrix} \frac{5}{3} + \frac{1}{3}xe^x - \frac{1}{3}e^{2x} - \frac{5}{8} + \frac{1}{4}e^x + \frac{1}{4}xe^{2x} + \frac{5}{24} - \frac{1}{36}e^x + \frac{1}{48}e^{2x} \\ \frac{5}{3} + \frac{1}{3}xe^x - \frac{1}{3}e^{2x} - \frac{5}{4} + \frac{1}{2}e^x + \frac{1}{2}xe^{2x} - \frac{5}{12} + \frac{1}{18}e^x - \frac{1}{24}e^{2x} \\ \frac{5}{3} + \frac{1}{3}xe^x - \frac{1}{3}e^{2x} - \frac{5}{2} + e^x + xe^{2x} + \frac{5}{6} - \frac{1}{9}e^x + \frac{1}{12}e^{2x} \end{bmatrix}$$

$$\vec{X}_P = \begin{bmatrix} \frac{5}{4} + \frac{1}{3}xe^x + \frac{1}{4}xe^{2x} - \frac{5}{16}e^{2x} + \frac{2}{9}e^x \\ \frac{1}{3}xe^x + \frac{1}{2}xe^{2x} - \frac{3}{8}e^{2x} + \frac{5}{9}e^x \\ \frac{1}{3}xe^x + xe^{2x} - \frac{1}{4}e^{2x} + \frac{8}{9}e^x \end{bmatrix}$$

combining with the complementary to form the full general solution...

$$\vec{X}_P = C_1\begin{bmatrix}1\\1\\1\end{bmatrix}e^x + C_2\begin{bmatrix}1\\2\\4\end{bmatrix}e^{2x} + C_3\begin{bmatrix}1\\-2\\4\end{bmatrix}e^{-2x} + \begin{bmatrix} \frac{5}{4} + \frac{1}{3}xe^x + \frac{1}{4}xe^{2x} - \frac{5}{16}e^{2x} + \frac{2}{9}e^x \\ \frac{1}{3}xe^x + \frac{1}{2}xe^{2x} - \frac{3}{8}e^{2x} + \frac{5}{9}e^x \\ \frac{1}{3}xe^x + xe^{2x} - \frac{1}{4}e^{2x} + \frac{8}{9}e^x \end{bmatrix}$$

but the particular solution terms which match the complementary solution terms will be absorbed into new constants...

$$\vec{X}_P = C_4\begin{bmatrix}1\\1\\1\end{bmatrix}e^x + C_5\begin{bmatrix}1\\2\\4\end{bmatrix}e^{2x} + C_3\begin{bmatrix}1\\-2\\4\end{bmatrix}e^{-2x} + \begin{bmatrix} \frac{5}{4} + \frac{1}{3}xe^x + \frac{1}{4}xe^{2x} \\ \frac{1}{3}xe^x + \frac{1}{2}xe^{2x} \\ \frac{1}{3}xe^x + xe^{2x} \end{bmatrix}$$

or...

$$\vec{X}_P = C_4\begin{bmatrix}1\\1\\1\end{bmatrix}e^x + C_5\begin{bmatrix}1\\2\\4\end{bmatrix}e^{2x} + C_3\begin{bmatrix}1\\-2\\4\end{bmatrix}e^{-2x} + \begin{bmatrix}\frac{5}{4}\\0\\0\end{bmatrix} + \begin{bmatrix}\frac{1}{3}\\\frac{1}{3}\\\frac{1}{3}\end{bmatrix}xe^x + \begin{bmatrix}\frac{1}{4}\\\frac{1}{2}\\1\end{bmatrix}xe^{2x}$$

and the top row is the solution to the original DE:

$$\boxed{y = C_4e^x + C_5e^{2x} + C_3e^{-2x} + \frac{5}{4} + \frac{1}{3}xe^x + \frac{1}{4}xe^{2x}}$$

Ch8 Summary

Homogeneous systems…

Finding eigenvalues: $\left|\vec{A}-\lambda\vec{I}\right|=0$

Finding eigenvector for an eigenvalue: $\left(\vec{A}-\lambda\vec{I}\right)\vec{K}=\vec{0}$

Distinct real eigenvalues: $\vec{X}_C = C_1\begin{bmatrix}k_{11}\\k_{21}\end{bmatrix}e^{\lambda_1 t}+C_2\begin{bmatrix}k_{12}\\k_{22}\end{bmatrix}e^{\lambda_2 t}$

Repeated real eigenvalues: find 2nd eigenvalue using $\left(\vec{A}-\lambda\vec{I}\right)\vec{P}=\vec{K}$

$$\vec{X}_C = C_1\begin{bmatrix}k_1\\k_2\end{bmatrix}e^{\lambda t}+C_2\left(\begin{bmatrix}k_1\\k_2\end{bmatrix}te^{\lambda t}+\begin{bmatrix}p_1\\p_2\end{bmatrix}e^{\lambda t}\right)$$

Complex conjugate eigenvalues:

use positive version $\lambda=\alpha+\beta i$ to find eigenvector $\vec{K}=\begin{bmatrix}a+bi\\c+di\end{bmatrix}$

$$\vec{B}_1=\operatorname{Re}\vec{K}=\begin{bmatrix}a\\c\end{bmatrix}\qquad \vec{B}_2=\operatorname{Im}\vec{K}=\begin{bmatrix}b\\d\end{bmatrix}$$

$$\vec{X}_C = C_1\left(\vec{B}_1\cos\beta t-\vec{B}_2\sin\beta t\right)e^{\alpha t}+C_2\left(\vec{B}_2\cos\beta t+\vec{B}_1\sin\beta t\right)e^{\alpha t}$$

Non-Homogeneous systems…

$$\vec{X}'=\vec{A}\vec{X}+\vec{F}$$

Method of Variation of Parameters: $\vec{\Phi}$ = fundamental matrix (from $\vec{X}_C$) $\quad \vec{X}_P=\vec{\Phi}\int\vec{\Phi}^{-1}\vec{F}\,dt$

$$\vec{\Phi}^{-1}=\frac{1}{\det\vec{\Phi}}\vec{\Phi}^{T}$$

$$\vec{\Phi}^{T} = transpose = for\ 2x2\ reverse\ elements\ on\ diagonal, negate\ everything\ else$$

Solving with initial condition: $\vec{X}=\vec{\Phi}(t)\vec{\Phi}^{-1}(t_0)\vec{X}_0+\vec{\Phi}\int_{t_0}^{t}\vec{\Phi}^{-1}(s)\vec{F}(s)ds$

Method of Undetermined Coefficients:

$$\vec{X}_p = \begin{bmatrix} \textit{function from table to match forms of terms of } F \\ \textit{function from table to match forms of terms of } F \end{bmatrix}$$

(must be the same form for all rows – so selected terms must cover all terms in all rows of F)

$g(x)$	Form of y_p
1. 1 (any constant)	A
2. $5x + 7$	$Ax + B$
3. $3x^2 - 2$	$Ax^2 + Bx + C$
4. $x^3 - x + 1$	$Ax^3 + Bx^2 + Cx + E$
5. $\sin 4x$	$A\cos 4x + B\sin 4x$
6. $\cos 4x$	$A\cos 4x + B\sin 4x$
7. e^{5x}	Ae^{5x}
8. $(9x - 2)e^{5x}$	$(Ax + B)e^{5x}$
9. x^2e^{5x}	$(Ax^2 + Bx + C)e^{5x}$
10. $e^{3x}\sin 4x$	$Ae^{3x}\cos 4x + Be^{3x}\sin 4x$
11. $5x^2\sin 4x$	$(Ax^2 + Bx + C)\cos 4x + (Ex^2 + Fx + G)\sin 4x$
12. $xe^{3x}\cos 4x$	$(Ax + B)e^{3x}\cos 4x + (Cx + E)e^{3x}\sin 4x$

Then take derivative, plug into system of DEs, and solve for constants.

(Note: you need to multiply by extra *t*s if terms match any terms in X_C)