Differential Equations – Lesson Notes – Chapter 7: Laplace Transforms

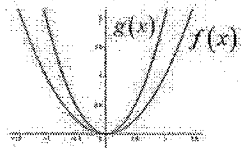
7.1: Laplace Transform

(A quick diversion from Differential Equations to learn about a related topic which in the next section we will be able to use to solve Differential Equations.)

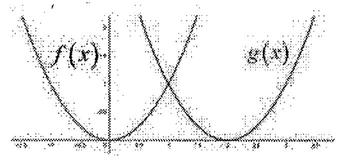
Transforms

A transform is a mathematical operation which operates on a function to transform it to a different function. The transforms that we've encountered so far transform a single-variable function into a different single-variable function, with both input and output having the same variable:

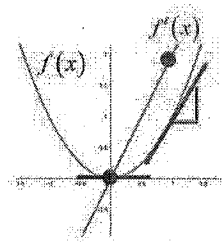
Vertical Stretch: $g(x) = Af(x)$



Horizontal Shift: $g(x) = f(x-2)$



Derivative: $f'(x) = \lim\limits_{\Delta x \to 0} \dfrac{f(x+\Delta x) - f(x)}{\Delta x}$



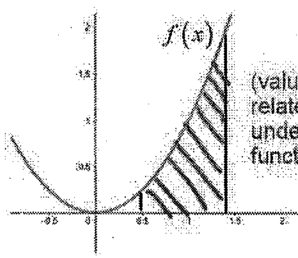
(value of derivative equals slope of tangent line to input function at that x)

Integral Transforms

An integral transform is just a transform which involves computing an integral. Our standard integral is an integral transform and the result is based on the undoing the derivative, so it is called the antiderivative (and the definite integral has the interpretation area under the input function curve):

Integral: $F(x) = \int f(x)\,dt$

Definite Integral: $F(b) - F(a) = \int_a^b f(x)\,dx$



(value of integral is related to the area under the input function curve)

Linear Transforms

A transformation is linear if it is true that when the input to the transformation can be written as separate terms combined by adding (and subtracting), the output is the result of adding (or subtracting) the transformed terms. Derivative and integral transforms are linear transforms:

$$\frac{d}{dx}\left[f(x) \pm g(x)\right] = \frac{d}{dx}\left[f(x)\right] \pm \frac{d}{dx}\left[g(x)\right]$$

$$\int \left(f(x) \pm g(x)\right) dx = \int f(x)\,dx \pm \int g(x)\,dx$$

Multivariable Function Transforms

We can now define a transform for a multivariable function and specifically the Laplace Transform. In the transformations we've seen so far, the functions have been functions of one variable. But if we multiply the function to be transformed by a multivariable function and then take an integral, we can define an integral transform as follows...

$$\int_0^\infty K(s,t)\,f(t)\,dt$$

We are starting with an input function $f(t)$ which is a function of only the variable *t*, but by multiplying by the *K* function, which is called the **kernel** of the transform, the resulting output of the integration will be a function of the newly introduced variable *s*.

The notation convention for the resulting output function is to use the same function letter, but capitalized:

$$\int_0^\infty K(s,t)\,f(t)\,dt = F(s)$$

The Laplace Transform

The **Laplace Transform** is an integral transform with the kernel function $K(s,t) = e^{-st}$

$$\boxed{\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\,dt}$$

This transform isn't defined unless the integral converges, so it only works with some functions.

The Laplace Transform of $f(t)=1$

Let's start with the simplest example: $f(t)=1$

Using the definition: $\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\,dt$

$$\begin{aligned}
&= \int_0^\infty e^{-st}(1)\,dt \\
&= \lim_{b\to\infty}\int_0^b e^{-st}\,dt \\
&= \lim_{b\to\infty}\frac{1}{-s}\left[e^{-st}\right]_0^b \\
&= \lim_{b\to\infty}\frac{1}{-s}\left[e^{-sb}-1\right] \\
&= \frac{1}{-s}(-1) \\
&= \frac{1}{s}
\end{aligned}$$

The Laplace Transform of $f(t)=\sin 2t$

As a more complex example, using the definition to compute the Laplace transform of $\sin 2t$: We would need to start with integration by parts, which we have to do twice and this is the form that reproduces the original, so we gather terms on the left and divide:

$$\int_0^\infty e^{-st}\sin 2t\,dt$$

$$by\ parts: u=\sin 2t \quad dv=e^{-st}dt$$

$$du=2\cos 2t dt \quad v=-\frac{1}{s}e^{-st}$$

$$uv-\int vdu=-\frac{1}{s}e^{-st}\sin 2t-\int\left(-\frac{1}{s}e^{-st}\right)2\cos 2t dt$$

$$-\frac{1}{s}e^{-st}\sin 2t+\frac{2}{s}\left[\int e^{-st}\cos 2t\,dt\right]$$

$$by\ parts\ again: \ u=\cos 2t \quad dv=e^{-st}dt$$

$$du=-2\sin 2t dt \quad v=-\frac{1}{s}e^{-st}$$

$$uv-\int vdu=-\frac{1}{s}e^{-st}\cos 2t-\int\left(-\frac{1}{s}e^{-st}\right)-2\sin 2t dt$$

$$\left[-\frac{1}{s}e^{-st}\cos 2t-\frac{2}{s}\int e^{-st}\sin 2t\,dt\right]$$

$$-\frac{1}{s}e^{-st}\sin 2t+\frac{2}{s}\left[-\frac{1}{s}e^{-st}\cos 2t-\frac{2}{s}\int e^{-st}\sin 2t\,dt\right]$$

$$\underline{\int e^{-st}\sin 2t\,dt}=-\frac{1}{s}e^{-st}\sin 2t-\frac{2}{s^2}e^{-st}\cos 2t-\frac{4}{s^2}\underline{\int e^{-st}\sin 2t\,dt}$$

$$\left(1+\frac{4}{s^2}\right)\int e^{-st}\sin 2t\,dt=-\frac{1}{s}e^{-st}\sin 2t-\frac{2}{s^2}e^{-st}\cos 2t$$

$$\int e^{-st}\sin 2t\,dt=\frac{\left(-\frac{1}{s}\right)}{\left(1+\frac{4}{s^2}\right)}e^{-st}\sin 2t+\frac{\left(-\frac{2}{s^2}\right)}{\left(1+\frac{4}{s^2}\right)}e^{-st}\cos 2t$$

$$\int e^{-st}\sin 2t\,dt=\frac{(-s)}{(s^2+4)}e^{-st}\sin 2t+\frac{(-2)}{(s^2+4)}e^{-st}\cos 2t$$

The Laplace Transform of $f(t)=\sin 2t$

...then we still need to evaluate this by plugging in the limits of integration, but because the top is infinity, we use a constant and take the limit for the infinity terms:

$$\int_0^\infty e^{-st}\sin 2t\,dt=\lim_{b\to\infty}\left[\frac{(-s)}{(s^2+4)}e^{-st}\sin 2t+\frac{(-2)}{(s^2+4)}e^{-st}\cos 2t\right]$$
$$-\left[\frac{(-s)}{(s^2+4)}e^{-s0}\sin 0+\frac{(-2)}{(s^2+4)}e^{-s0}\cos 0\right]$$

The last two terms we can evaluate with zero plugged in:

$$-\left[\frac{(-s)}{(s^2+4)}e^{-s0}\sin 0+\frac{(-2)}{(s^2+4)}e^{-s0}\cos 0\right]$$
$$-\left[\frac{(-s)}{(s^2+4)}(1)(0)+\frac{(-2)}{(s^2+4)}(1)(1)\right]$$
$$\frac{2}{s^2+4}$$

...for the first two terms we can move the exponent to the bottom of the fraction:

$$\lim_{b\to\infty}\left[\frac{(-s)}{(s^2+4)}e^{-st}\sin 2t+\frac{(-2)}{(s^2+4)}e^{-st}\cos 2t\right]$$
$$\lim_{b\to\infty}\left(\frac{-s}{s^2+4}\right)\frac{\sin 2t}{e^{st}}+\lim_{b\to\infty}\left(\frac{-s}{s^2+4}\right)\frac{\cos 2t}{e^{st}}$$

If s > 0, then as b gets infinitely large, the numerators of these fractions cycle between -1 and 1 but the denominators get infinitely large, so the fractions go to zero:

$$\lim_{b\to\infty}\left(\frac{-s}{s^2+4}\right)\frac{\sin 2t}{e^{st}}+\lim_{b\to\infty}\left(\frac{-s}{s^2+4}\right)\frac{\cos 2t}{e^{st}}$$
$$\left(\frac{-s}{s^2+4}\right)\frac{(-1\,to\,1)}{\infty}+\left(\frac{-s}{s^2+4}\right)\frac{(-1\,to\,1)}{\infty}$$
$$\left(\frac{-s}{s^2+4}\right)0+\lim_{b\to\infty}\left(\frac{-s}{s^2+4}\right)0$$

...which means that the transform for $f(t)=\sin 2t$ is:

$$\mathcal{L}\{\sin 2t\}=\int_0^\infty e^{-st}\sin 2t\,dt=0+0+0+\frac{2}{s^2+4}$$

$$\mathcal{L}\{\sin 2t\}=\frac{2}{s^2+4}$$ Whew!

The good news is that Laplace transforms for many commonly encountered functions are already worked out for us in the Laplace Transform table, which we will use...

$f(t)$	$\mathscr{L}\{f(t)\} = F(s)$
1. 1	$\frac{1}{s}$
2. t	$\frac{1}{s^2}$
3. t^n	$\frac{n!}{s^{n+1}}$, n a positive integer
4. $t^{-1/2}$	$\sqrt{\frac{\pi}{s}}$
5. $t^{1/2}$	$\frac{\sqrt{\pi}}{2s^{3/2}}$
6. t^α	$\frac{\Gamma(\alpha+1)}{s^{\alpha+1}}$, $\alpha > -1$
7. $\sin kt$	$\frac{k}{s^2+k^2}$
8. $\cos kt$	$\frac{s}{s^2+k^2}$
9. $\sin^2 kt$	$\frac{2k^2}{s(s^2+4k^2)}$
10. $\cos^2 kt$	$\frac{s^2+2k^2}{s(s^2+4k^2)}$
11. e^{at}	$\frac{1}{s-a}$
12. $\sinh kt$	$\frac{k}{s^2-k^2}$
13. $\cosh kt$	$\frac{s}{s^2-k^2}$
14. $\sinh^2 kt$	$\frac{2k^2}{s(s^2-4k^2)}$
15. $\cosh^2 kt$	$\frac{s^2-2k^2}{s(s^2-4k^2)}$
16. te^{at}	$\frac{1}{(s-a)^2}$
17. $t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$, n a positive integer
18. $e^{at}\sin kt$	$\frac{k}{(s-a)^2+k^2}$
19. $e^{at}\cos kt$	$\frac{s-a}{(s-a)^2+k^2}$

Γ = 'gamma function'

$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\,dt$

but for n a positive integer,

$\Gamma(n+1) = n!$

(the reason t^α has a separate entry from t^n is that α is allowed to be a complex number)

The Laplace Transform is a linear transform

...which means that you can move constants outside of the transform and take the Laplace transforms of each term separately and add/subtract the results:

$$\mathcal{L}\{3-8\sin 2t\}$$
$$\mathcal{L}\{3\}-\mathcal{L}\{8\sin 2t\}$$
$$3\mathcal{L}\{1\}-8\mathcal{L}\{\sin 2t\}$$
$$3\frac{1}{s}-8\frac{2}{s^4+4}$$

An example

Usually, we'll find a Laplace transform by using the table. Let's do this one together:

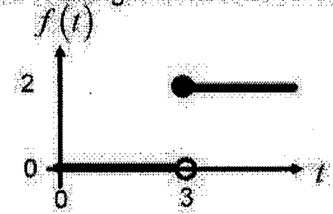
Find the Laplace transform of $f(t)=t^2-e^{-9t}+5$

$$F(s)=\mathcal{L}\{t^2\}-\mathcal{L}\{e^{-9t}\}+5\mathcal{L}\{1\}$$
$$=\frac{2!}{s^{2+1}}-\frac{1}{s-(-9)}+5\frac{1}{s}$$
$$=\boxed{\frac{2}{s^3}-\frac{1}{s+9}+\frac{5}{s}}$$

Laplace transforms of piece-wise defined functions

If we are given a piece-wise defined function which has different functions for different parts of the t domain, then we have to go back to using the definition and computing the integral by hand. Here is an example:

$$f(t)=\begin{cases}0, & 0\le t<3\\ 2, & t\ge 3\end{cases}$$



$$\begin{aligned}
F(s)&=\int_0^\infty e^{-st}f(t)\,dt\\
&=\int_0^3 e^{-st}(0)\,dt+\int_3^\infty e^{-st}(2)\,dt\\
&=\quad 0 \quad +2\int_3^\infty e^{-st}\,dt\\
&=\quad 0 \quad +2\left[-\frac{1}{s}e^{-st}\right]_{t=3}^{t\to\infty}\\
&=2\left[-\frac{1}{s}e^{-st}\right]^{t\to\infty}-2\left[-\frac{1}{s}e^{-st}\right]_{t=3}\\
&=-\frac{2}{s}\lim_{t\to\infty}e^{-st}+\frac{2}{s}e^{-3s}\\
&=0+\boxed{\frac{2}{s}e^{-3s}}\quad (s>0)
\end{aligned}$$

Why are we learning about the Laplace Transform?

Because in the next section we are going to define the Inverse Laplace Transform and learn how to take derivatives of Laplace transforms, and we will find that when you take the derivative of a Laplace transform you get the same transform but multiplied by another s.

This means that if we have a differential equation, we will be able to take the Laplace transform of it which will convert it into an algebraic equation. We will then be able to use algebra to solve the differential equation, and in the next section we will define the Inverse Laplace Transform which will allow us to transform the solution back into t.

Another reason has to do with modeling...when we imagine things like a spring/mass system that is driven by a driving function, we've been using things like sinusoidal driving functions which have continuous derivatives. But in real situation we often have abruptly changing driving functions, like an 'impulse' which is zero until $t = 0$, then suddenly it applied. In this case, the driving function would not have a continuous derivative, but we can use the Laplace transform for piecewise functions to model the driving function and to solve the differential equation.

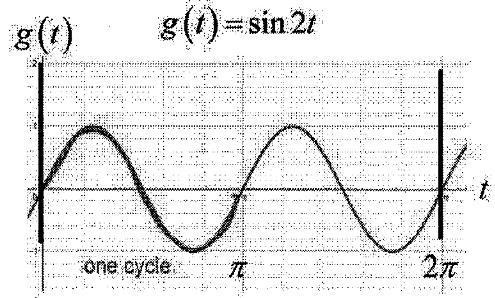
Additional info about the Laplace Transform (not officially part of this course)

The Laplace Transform has many other applications besides enabling the solution of certain differential equations. It is a generalized version of something called a Fourier Transform, and these transforms have many other applications.

We will talk more about this later, but here we can talk about one interesting application of the Laplace Transform.

We found that the Laplace transform of $g(t)=\sin 2t$ $\quad G(s)=\mathcal{L}\{\sin 2t\}=\dfrac{2}{s^2+4}$

If we were to plot the original function versus time, *t*:



A sine function goes through one period when its argument goes from 0 to 2π:

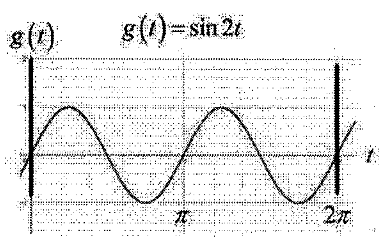
$$0<2t<2\pi$$
$$0<t<\pi$$

...which means its **Period** is π and there are two complete cycles in 2π

...which means its **Circular frequency,** ω**,** is 2.

Circular frequency is defined to be $\omega=2\pi f$ where f is the **Frequency** (cycles per second)

$g(t)=\sin 2t$ $\qquad G(s)=\mathcal{L}\{\sin 2t\}=\dfrac{2}{s^2+4}$



F(s) is in the form of a fraction...are there any values of *s* which would cause this function to be undefined?

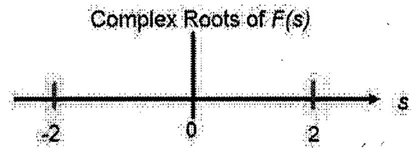
$$s^2+4=0$$
$$s^2=-4$$
$$s=\pm\sqrt{-4}$$
$$s=\pm 2i$$

Yes, but they are complex numbers, *2i* and *-2i*.

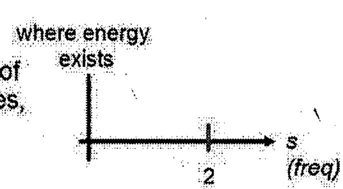
The numerical values here, 2 and -2 are called the 'roots' of the Laplace transform. Notice that the **roots are +/- the frequency of the sinusoidal time function.**

If we were to plot the values of the roots of the Laplace transform on a horizontal axis...

Complex Roots of *F(s)*



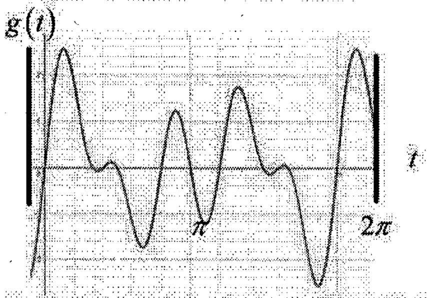
...and the looked at only the positive values, it gives a picture of where this **frequency** lies on the list of all possible frequencies, or at what frequencies this signal contains energy.



Additional info about the Laplace Transform (not officially part of this course)

Now if you had a time signal function which was made up of multiple different frequencies added together...

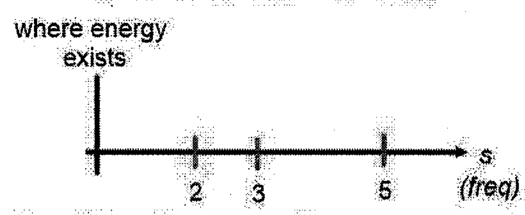
$g(t) = \sin 2t + \sin 3t + \sin 5t$



The time signal is complicated (and could represent, for example, a snippet of musical sound).

...and took the Laplace Transform:

$$G(s) = \{\sin 2t + \sin 3t + \sin 5t\}$$
$$= \frac{2}{s^2+4} + \frac{3}{s^2+9} + \frac{5}{s^2+25}$$



...the Laplace Transform roots allow us to separate out the individual frequencies that made up the time signal.

TIME DOMAIN $g(t)$ $\longrightarrow$ **Laplace Transform** $\longrightarrow$ **FREQUENCY DOMAIN** $G(s)$

Besides allowing us to solve DEs, the Laplace Transform allows us to take any time varying signal and transform from the time domain to the frequency domain to see the frequency content of the signal.

There are software algorithms which allow computationally efficient ways to perform the Fourier Transform (a cousin of the Laplace Transform) called FFTs (Fast-Fourier Transforms) to, for example, display the bass-treble frequency response of music playing, or to transform singing to the frequency domain, correct the pitch, then transform back to the time domain (auto-tune).

7.2 day 1: Inverse Laplace Transform

Inverse Laplace Transform

The Laplace Transform...

$$\boxed{\mathcal{L}\{f(t)\}=F(s)=\int_0^{\infty} e^{-st} f(t)\,dt}$$

...converts a function of the variable *t* to a function of the variable *s*:

$$\mathcal{L}\{y(t)\}=Y(s)$$

...and we usually use a table to find the *Y(s)* function for a given *y(t)* function...

The Inverse Laplace Transform reverses this procedure. It converts a function of the variable *s* to a function of the variable *t*:

$$\boxed{\mathcal{L}^{-1}\{Y(s)\}=y(t)}$$

...so we just use the table in reverse.

$$\mathcal{L}^{-1}\left\{\frac{k}{s^2+k^2}\right\}=\sin kt$$

...and we usually use a table to find the *y(t)* function for a given *Y(s)* function...

Sometimes, we need to adjust the constant to find a match in the table

We sometimes need to make slight adjustments to a given *Y(s)* function to make it match the form in the table:

$\frac{n!}{s^5} \quad \frac{4!}{s^{4+1}} \quad (4\cdot3\cdot2=24)$

Ex) $\mathcal{L}^{-1}\left\{\frac{1}{s^2+9}\right\}=$

$\frac{1}{3}\mathcal{L}^{-1}\left\{\frac{3}{s^2+9}\right\}$

$\boxed{\frac{1}{3}\sin 3t}$

Ex) $\mathcal{L}^{-1}\left\{\frac{3}{s^5}\right\}=$

$\frac{3}{24}\mathcal{L}^{-1}\left\{\frac{24}{s^5}\right\}=\frac{1}{8}\mathcal{L}^{-1}\left\{\frac{4!}{s^{4+1}}\right\}$

$\boxed{\frac{1}{8}t^4}$

The Inverse Laplace Transform is a linear transform

Like the Laplace Transform, the Inverse Laplace Transform is a linear transform:

$$\mathcal{L}^{-1}\{Y_1(s)\pm Y_2(s)\}=\mathcal{L}^{-1}\{Y_1(s)\}\pm\mathcal{L}^{-1}\{Y_2(s)\}$$

...so you can transform term by term separately:

Ex) $\mathcal{L}^{-1}\left\{\frac{3}{s}-\frac{1}{5s-2}\right\}=3\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}-\frac{1}{5}\mathcal{L}^{-1}\left\{\frac{1}{s-\frac{2}{5}}\right\}$

$=\boxed{3-\frac{1}{5}e^{2/5t}}$

We often need to use partial fraction expansion

To split up numerators with multiple factors in order to match to the table, use partial fraction expansion:

$$\frac{1}{(s-2)(s-1)} = \frac{A}{s-2} + \frac{B}{s-1} = \frac{A(s-1)}{(s-2)(s-1)} + \frac{B(s-2)}{(s-2)(s-1)}$$

$$1 = As - A + Bs - 2B \qquad \begin{cases} A + B = 0 & B = -A \\ -A - 2B = 1 & \leftarrow -A - 2(-A) = 1 \end{cases}$$

$$1 = (A+B)s + (-A-2B) \qquad \therefore A = 1,\ B = -1$$

You are welcome to use online Partial Fraction Expansion calculators, such as this one:

so

https://www.symbolab.com/solver/partial-fractions-calculator

$$\frac{1}{(s-2)(s-1)} = (1)\frac{1}{s-2} + (-1)\frac{1}{s-1}$$

(There is a link to this on www.mrfelling.com in the 'math applets & resources' section)

One more example

Ex) $\mathcal{L}^{-1}\left\{\frac{2}{s} - \frac{3}{s-4} + \frac{s+5}{s^2+25}\right\} =$

$$2\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - 3\mathcal{L}^{-1}\left\{\frac{1}{s-4}\right\} + \mathcal{L}^{-1}\left\{\frac{s}{s^2+25}\right\} + \mathcal{L}^{-1}\left\{\frac{5}{s^2+25}\right\}$$

$$\boxed{2 - 3e^{4t} + \cos 5t + \sin 5t}$$

7.2 day 2: Inverse Laplace Transform of Derivatives, using Laplace Transforms to solve Differential Equations

Laplace Transforms of derivatives

Differential Equations involve derivatives, so in order to explore how we can use Laplace Transforms to help solve differential equations, we first must know how Laplace Transforms of derivatives work.

The Laplace Transform of the derivative of a function can be found using the definition and integrating by parts:

$$\mathcal{L}\{f'(t)\}=\int_0^{\infty} e^{-st}f'(t)\,dt$$

$$u=e^{-st} \quad dv=f'(t)\,dt$$

$$du=-se^{-st}dt \quad v=f(t)$$

$$uv-\int v\,du$$

$$e^{-st}f(t)-\int f(t)\left(-se^{-st}\,dt\right)$$

$$\left[e^{-st}f(t)\right]_0^{\infty}+s\int_0^{\infty} e^{-st}f(t)\,dt$$

$$\left[e^{-st}f(t)\right]_0^{\infty}+s\,\mathcal{L}\{f(t)\}$$

$$\left(e^{-s(\infty)}f(\infty)\right)-\left(e^{-s(0)}f(0)\right)+s\,\mathcal{L}\{f(t)\}$$

$$\boxed{\{f'(t)\}=s\,\mathcal{L}\{f(t)\}-f(0)}$$

So the Laplace Transform of a derivative of a function is *s* times the Laplace Transform of the original function, minus the initial condition of the function.

Similarly, for the Laplace Transform of a 2nd derivative of a function:

$$\mathcal{L}\{f''(t)\}=\int_0^\infty e^{-st}f''(t)\,dt$$

$$u=e^{-st} \qquad dv=f''(t)\,dt$$

$$du=-se^{-st}dt \qquad v=f'(t)$$

$$uv-\int v\,du$$

$$e^{-st}f'(t)-\int f'(t)\left(-se^{-st}\,dt\right)$$

$$\left[e^{-st}f'(t)\right]_0^\infty+s\int_0^\infty e^{-st}f'(t)\,dt$$

$$\left[e^{-st}f'(t)\right]_0^\infty+s\quad\{f(t)\}$$

$$\left(e^{-s(\infty)}f'(\infty)\right)-\left(e^{-s(0)}f'(0)\right)+s\,\mathcal{L}\{f'(t)\}$$

$$\mathcal{L}\{f''(t)\}=s\,\underline{\underline{\mathcal{L}\{f'(t)\}}}-f'(0)$$

and substituting the previous result...

$$\mathcal{L}\{f''(t)\}=s\left[s\,\mathcal{L}\{f(t)\}-f(0)\right]-f'(0)$$

$$\boxed{\mathcal{L}\{f''(t)\}=s^2\,\mathcal{L}\{f(t)\}-sf(0)-f'(0)}$$

Summary of Laplace Transforms of derivatives

This pattern can be shown to continue, and in general:

$$\boxed{\mathcal{L}\left\{f^{(n)}(t)\right\}=s^nF(s)-s^{n-1}f(0)-s^{n-2}f'(0)-\ldots-f^{(n-1)}(0)}$$

Which can be summarized as following (using the capital letter for the Laplace Transform):

$$\mathcal{L}\{0\}=0$$

$$\mathcal{L}\{y(t)\}=Y(s)$$

$$\mathcal{L}\{y'(t)\}=sY(s)-y(0)$$

$$\mathcal{L}\{y''(t)\}=s^2Y(s)-sy(0)-y'(0)$$

$$\mathcal{L}\{y'''(t)\}=s^3Y(s)-s^2y(0)-sy'(0)-y''(0)$$

Using Laplace Transforms to solve Differential Equations

The reason Laplace Transforms are so powerful in solving differential equations is that when derivatives are transformed, the result is an algebraic equation, which means we can manipulate things to solve the resulted transformed equation for *Y(s)*, then apply an inverse transform to return to the *t* domain.

Solving Differential Equations with Laplace Transforms:

1) Take the Laplace Transform of both sides of the DE.

2) Solve algebraically for *Y(s)*.

3) Find the solution *y(t)* by taking the Inverse Laplace Transform of both sides of the result.

Examples of solving differential equations using Laplace Transforms

Ex) $2y'+y=0 \quad y(0)=-3$

$$2[sY(s)-y(0)]+Y(s)=0$$

$$2[sY(s)-(-3)]+Y(s)=0$$

$$(2s+1)Y(s)+6=0$$

$$Y(s)=-\frac{6}{2s+1}=-6\frac{1}{2(s+1/2)}=-6\left(\frac{1}{2}\right)\frac{1}{s+1/2}=-3\frac{1}{s+1/2}$$

$$Y(s)=-3\frac{1}{s+1/2}$$

$$\mathcal{L}^{-1}\{Y(s)\}=-3\mathcal{L}^{-1}\left\{\frac{1}{s+1/2}\right\}$$

$$\boxed{y(t)=-3e^{-1/2t}}$$

(we could have used earlier techniques here...)

$$2y'+y=0 \quad y(0)=-3$$

$$2m+1=0$$

$$m=-1/2$$

$$y=c_1e^{-1/2t}$$

$$-3=c_1e^{-1/2(0)}=c_1$$

$$\boxed{y=-3e^{-1/2t}}$$

Examples of solving differential equations using Laplace Transforms

Ex) $y''' + 2y'' - y' - 2y = \sin 3t, \quad y(0)=0,\ y'(0)=0,\ y''(0)=1$

$$\left[s^3Y(s) - s^2y(0) - s\,y'(0) - y''(0)\right] + 2\left[s^2Y(s) - s\,y(0) - y'(0)\right]$$
$$-\left[sY(s) - y(0)\right] - 2Y(s) = \frac{3}{s^2+9}$$

$$s^3Y(s) - s^2(0) - s(0) - 1 + 2s^2Y(s) - 2s(0) - 2(0) - sY(s) + (0) - 2Y(s) = \frac{3}{s^2+9}$$
$$s^3Y(s) - 1 + 2s^2Y(s) - sY(s) - 2Y(s) = \frac{3}{s^2+9}$$
$$(s^3+2s^2-s-2)Y(s) - 1 = \frac{3}{s^2+9}$$

don't forget this

$$(s^3+2s^2-s-2)Y(s) = \frac{3}{s^2+9} + 1 = \frac{3}{s^2+9} + \frac{s^2+9}{s^2+9} = \frac{s^2+12}{s^2+9}$$

$$Y(s) = \frac{s^2+12}{(s^2+9)(s^3+2s^2-s-2)}$$

by online partial fractions expansion calculator:

$$Y(s) = \frac{3}{130}\frac{s}{s^2+9} - \left(\frac{6}{130}\right)\frac{1}{s^2+9} + \frac{16}{39}\frac{1}{s+2} - \frac{13}{20}\frac{1}{s+1} + \frac{13}{60}\frac{1}{s-1}.$$

$$\left(=\frac{3}{65}\right)$$

$$\boxed{y(t) = \frac{3}{130}\cos 3t - \frac{3}{65}\sin 3t + \frac{16}{39}e^{-2t} - \frac{13}{20}e^{-t} + \frac{13}{60}e^{t}}$$

7.3 day 1: Laplace Transform Properties: shifting

Properties turn out to be useful to solve more complicated DEs

The Laplace Transform has a number of properties which are very useful in solving more complicated differential equations, especially ones in which the driving function on the RHS is not smooth and continuous.

We'll see more later about exactly how this helps, but first we must learn and practice using some of the properties of the Laplace Transform.

Laplace Transform / Inverse Laplace Transform with Shifting

The following properties can be shown to be true...

$$\boxed{\begin{gathered}\mathcal{L}\{e^{at}f(t)\} = F(s-a) \\ \mathcal{L}^{-1}\{F(s-a)\} = e^{at}f(t)\end{gathered}}$$

Shifting in the s domain is equivalent to multiplying by an exponential in the t domain.

Examples

Ex) $\mathcal{L}\{e^{-2t}\cos(4t)\}$

$$\boxed{\frac{(s+2)}{(s+2)^2+16}}$$

Ex) $\mathcal{L}^{-1}\left\{\frac{2}{(s-3)}\right\} = 2\mathcal{L}^{-1}\left\{\frac{1}{(s-3)}\right\}$

$$\boxed{2e^{3t}}$$

Ex) $\mathcal{L}^{-1}\left\{\frac{1}{(s-6)^3}\right\}$

$$\frac{n!}{s^{n+1}} = t^n$$

$$\frac{2!}{s^3} = t^2$$

$$\frac{1}{2}\frac{2}{s^3} = \frac{1}{2}t^2$$

$$\boxed{\frac{1}{2}e^{6t}t^2}$$

Examples

Ex) $\mathcal{L}^{-1}\left\{\dfrac{2s+5}{(s-3)^2}\right\}$

$$\frac{2s+5}{(s-3)^2} = \frac{2(s+\frac{5}{2})}{(s-3)^2} = \frac{2(s-3+3+\frac{5}{2})}{(s-3)^2}$$

$$= \frac{2(s-3+\frac{11}{2})}{(s-3)^2} = 2\frac{(s-3)}{(s-3)^2} + 11\frac{1}{(s-3)^2}$$

$$= 2\frac{1}{(s-3)} + 11\frac{1}{(s-3)^2}$$

$$2\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} + 11\mathcal{L}^{-1}\left\{\frac{1}{(s-3)^2}\right\}$$

$$\boxed{2e^{3t} + 11e^{3t}t}$$

Ex) $\mathcal{L}^{-1}\left\{\dfrac{\frac{1}{2}s+\frac{5}{3}}{s^2+4s+6}\right\}$

s^2+4s+6 not factorable so complete the square:

$$(s^2+4s+\underline{4})+6-\underline{4}$$

$$(s+2)^2+2$$

then

$$\frac{\frac{1}{2}s+\frac{5}{3}}{s^2+4s+6} = \frac{\frac{1}{2}s+\frac{5}{3}}{(s+2)^2+2} = \frac{\frac{1}{2}(s+\frac{10}{3})}{(s+2)^2+2}$$

$$= \frac{\frac{1}{2}(s+2-2+\frac{10}{3})}{(s+2)^2+2} = \frac{\frac{1}{2}(s+2+\frac{4}{3})}{(s+2)^2+2}$$

this one

$$= \frac{\frac{1}{2}(s+2)+\frac{2}{3}}{(s+2)^2+2} = \frac{1}{2}\frac{(s+2)}{(s+2)^2+2} + \frac{2}{3}\frac{1}{(s+2)^2+2}\cdot\frac{\sqrt{2}}{\sqrt{2}}$$

$$= \frac{1}{2}\frac{(s+2)}{(s+2)^2+2} + \frac{2}{3\sqrt{2}}\frac{\sqrt{2}}{(s+2)^2+2}$$

now $\mathcal{L}^{-1}$:

$$\boxed{y(t) = \frac{1}{2}e^{-2t}\cos\sqrt{2}t + \frac{2}{3\sqrt{2}}e^{-2t}\sin\sqrt{2}t}$$

Ex) $\mathcal{L}^{-1}\left\{\frac{2s+5}{s^2+6s+34}\right\}$

$s^2+6s+34$ complete the square:

$(s^2+6s+\underline{9})+34-\underline{9}$

$(s+3)^2+25$

$$\frac{2s+5}{(s+3)^2+25} = \frac{2(s+\frac{5}{2})}{(s+3)^2+25} = \frac{2(s+3-3+\frac{5}{2})}{(s+3)^2+25} = \frac{2(s+3)-1}{(s+3)^2+25}$$

$$= 2\frac{(s+3)}{(s+3)^2+25} - \frac{1}{(s+3)^2+25}\cdot\frac{5}{5}$$

$$= 2\frac{(s+3)}{(s+3)^2+25} - \frac{1}{5}\frac{5}{(s+3)^2+25}$$

now $\mathcal{L}^{-1}$.

$$\boxed{y(t) = 2e^{-3t}\cos 5t - \frac{1}{5}e^{-3t}\sin 5t}$$

Ex) $y''-6y'+9y=t^2e^{3t}$ $\quad y(0)=2 \quad y'(0)=17$

$\mathcal{L}$ of both sides:

$$[s^2Y(s)-sy(0)-y'(0)]-6[sY(s)-y(0)]+9Y(s) = \frac{2!}{(s-3)^{2+1}} = \frac{2}{(s-3)^3}$$

$$(s^2-6s+9)Y(s)-s(2)-17-6(-2) = \frac{2}{(s-3)^3}$$

$$(s^2-6s+9)Y(s) = \frac{2}{(s-3)^3}+2s+5 \qquad s^2-6s+9=(s-3)^2$$

so $Y(s) = \frac{2}{(s-3)^5} + \frac{2s+5}{(s-3)^2}$ $\quad$ partial fraction expansion $\quad \frac{2s+5}{(s-3)^2} = 2\frac{1}{s-3}+11\frac{1}{(s-3)^2}$

$$Y(s) = 2\frac{1}{(s-3)^5}+2\frac{1}{s-3}+11\frac{1}{(s-3)^2}$$

$(t^4) \qquad (1) \qquad (t)$

$$\frac{2}{4!}\frac{4!}{(s-3)^{4+1}}+2\frac{1}{s-3}+11\frac{1}{(s-3)^2}$$

now $\mathcal{L}^{-1}$

$$\boxed{y(t) = \frac{1}{12}e^{3t}t^4+2e^{3t}+11e^{3t}t}$$

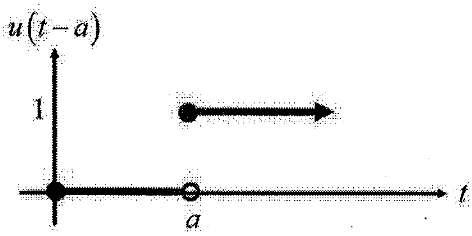
7.3 day 2: The Unit Step Function

The Unit Step Function

The Laplace Transform is particularly useful for solving DEs when the RHS driving function is piecewise defined, with abrupt changes in behavior.

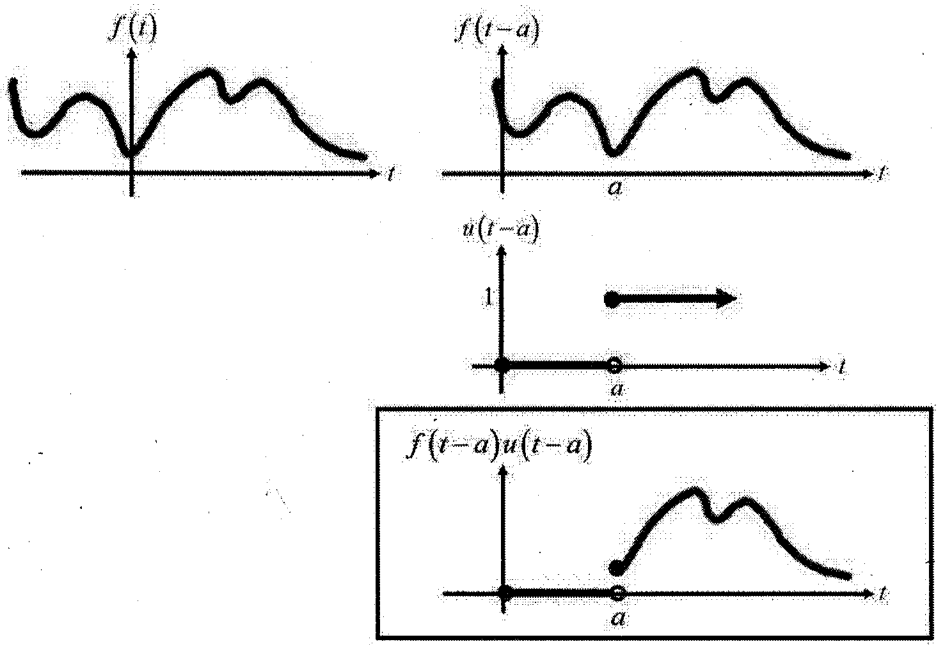
We can model functions in time which change behavior abruptly by using the **Unit Step Function**, which is defined as follows:

$$\boxed{u(t-a)=\begin{cases}0, & 0\le t<a\\ 1, & t\ge a\end{cases}}$$



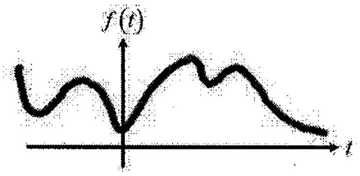
You can cause a function to start at a new time...

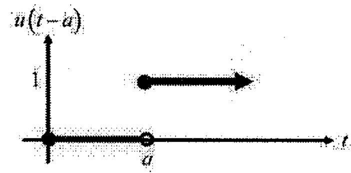
To shift a function so that it started at a new time, a, and is zero before, you multiply a time-shifted version of it by the Unit Step Function:

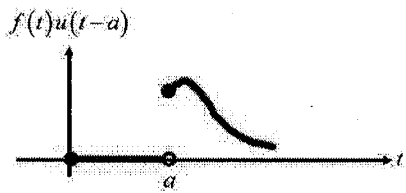


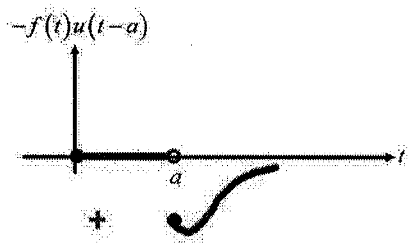
You can cause a function to stop at a specified time...

To cause a function to be applied for a specified time and then stop, you add a negative version of the function to the original, but one which begins at the specified time. The positive and negative versions of the function then sum to zero:

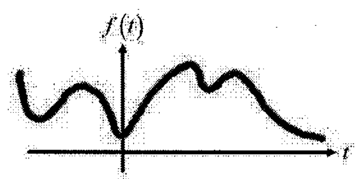




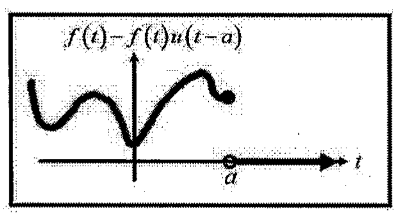




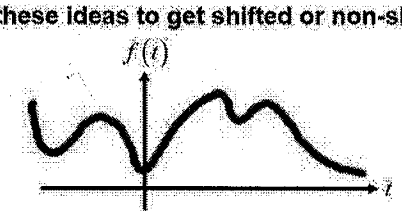
So when you add this to the original function...

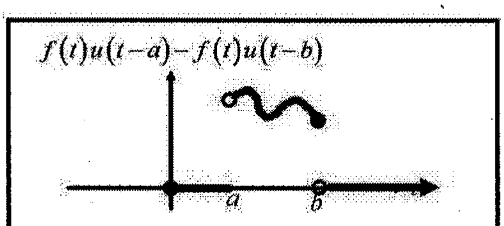


...you get...



You can combine these ideas to get shifted or non-shifted pieces of the function...





7.3 day 2: Laplace Transform Properties: shifting on *t*-axis

Laplace Transform / Inverse Laplace Transform: Shifting in *s*-axis

Earlier, we described the result of taking the Laplace transform of a function in the *t*-domain which is multiplied by an exponential function...the result is the Laplace Transform of the function, but shifted (translated) in the *s*-axis:

$$\boxed{\begin{aligned}\mathcal{L}\{e^{at}f(t)\} &= F(s-a)\\ \mathcal{L}^{-1}\{F(s-a)\} &= e^{at}f(t)\end{aligned}}$$

Now that we've explored the Unit Step Function, we can define the Laplace Transform of a function which is translated in the *t*-axis:

$$\boxed{\begin{aligned}\mathcal{L}\{f(t-a)u(t-a)\} &= e^{-as}F(s)\\ \mathcal{L}^{-1}\{e^{-as}F(s)\} &= f(t-a)u(t-a)\end{aligned}}$$

It turns out that the Laplace Transform of a *t*-domain function shifted by *a* multiplied by the Unit Step Function for *a* is the Laplace Transform of the function but multiplied by an exponential function in *s*.

It is this property which will allow us to take the Laplace Transform of RHS driving functions which are piecewise defined or shifted in time in order to solve differential equations.

Other useful Laplace Transform functions

The definitions of these Laplace Transforms can be used to show the following additional properties are also true, which are useful in solving problems...

Sometimes, we will need to take the Laplace Transform of just a Unit Step Function without any additional function multiplied, which is:

$$\boxed{\mathcal{L}\{u(t-a)\} = \frac{e^{-as}}{s}}$$

Sometimes, we will need to take the Laplace Transform of something where the function is not time-shifted but the Unit Step Function is time-shifted. In these cases, the following is useful:

$$\boxed{\mathcal{L}\{g(t)u(t-a)\} = e^{-as}\,\mathcal{L}\{g(t+a)\}}$$

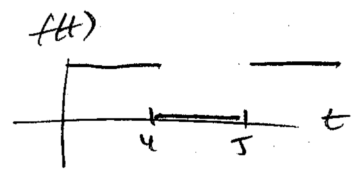
It is easiest to understand these properties by looking at examples showing how they are used...

Laplace Transform examples

Ex) Find $\mathcal{L}\{e^{2-t}u(t-2)\} = \mathcal{L}\{e^{-(t-2)}u(t-2)\}$ $a=2$, $f(t)=e^{-t} \to \frac{1}{s+1}$

$$= \boxed{e^{-2s}\frac{1}{s+1}}$$

Ex) Find $\mathcal{L}\{f(t)\}$ where $f(t)=\begin{cases}1 & 0\le t<4\\ 0 & 4\le t<5\\ 1 & t\ge 5\end{cases}$



$$f(t) = u(t) - u(t-4) + u(t-5)$$

$$\mathcal{L}\{f(t)\} = \frac{e^{0s}}{s} - \frac{e^{-4s}}{s} + \frac{e^{-5s}}{s}$$

$$= \boxed{\frac{1}{s} - \frac{e^{-4s}}{s} + \frac{e^{-5s}}{s}}$$

Ex) Find $\mathcal{L}\{\sin t\, u(t-2\pi)\}$ $a=2\pi$, $g(t)=\sin t$

$$= e^{-2\pi s}\mathcal{L}\{\sin(t+2\pi)\}$$ but $\sin(t+2\pi)=\sin(t)$

$$= e^{-2\pi s}\mathcal{L}\{\sin t\}$$

$$= \boxed{e^{-2\pi s}\frac{1}{s^2+1}}$$

Inverse Laplace Transform example

Ex) Find $\mathcal{L}^{-1}\left\{\dfrac{se^{-\frac{\pi}{2}s}}{s^2+4}\right\} = \mathcal{L}^{-1}\left\{e^{-\frac{\pi}{2}s}\dfrac{s}{s^2+4}\right\}$

$a=\frac{\pi}{2}$

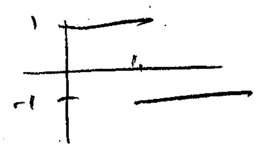
$\frac{s}{s^2+4}\to\cos 2t$

$= \boxed{\cos 2(t-\tfrac{\pi}{2})\,u(t-\tfrac{\pi}{2})}$

Solving DEs with Laplace Transforms

The real beauty of all this is it gives us the ability to solve DEs with RHS functions with abrupt changes in behavior. We will take the Laplace Transform of both sides (using our new rules to handle the RHS), then solve algebraically for Y(s), and take the Inverse Laplace Transform of both sides to find the solution:

Ex) Solve $y'+y=f(t)$ where $f(t)=\begin{cases}1 & 0\le t<1\\ -1 & t\ge 1\end{cases}$ $\quad y(0)=0$



$f(t)=u(t)-2u(t-1)$

$y'+y=u(t)-2u(t-1)$

$\mathcal{L}$: $sY(s)-\underset{(0)}{y(0)}+Y(s)=\frac{1}{s}-2\frac{e^{-s}}{s}$

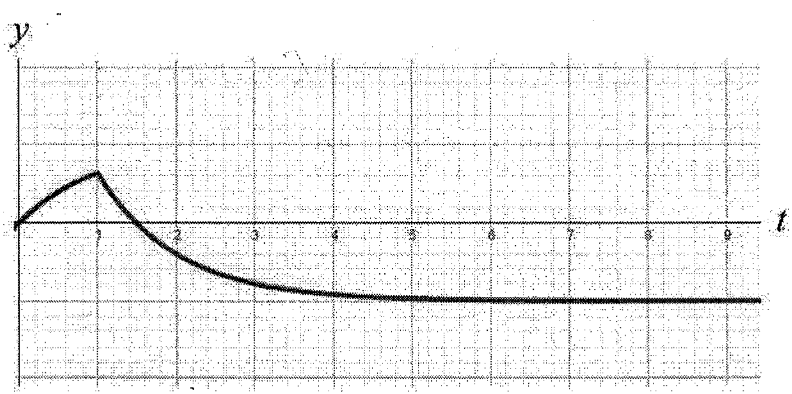
PFE:
$\frac{1}{s(s+1)}=\frac{1}{s}-\frac{1}{s+1}$

$(s+1)Y(s)=\frac{1}{s}-2\frac{e^{-s}}{s}$

$Y(s)=\frac{1}{s(s+1)}-2e^{-s}\frac{1}{s(s+1)}$

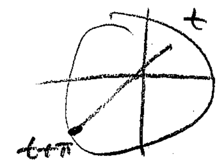
$Y(s)=\frac{1}{s}-\frac{1}{s+1}-2e^{-s}\frac{1}{s}+2e^{-s}\frac{1}{s+1}$

$\mathcal{L}^{-1}$: $\boxed{y(t)=1-e^{-t}-2u(t-1)+2e^{-(t-1)}u(t-1)}$



Another solving DE example

Ex) Solve $y'' + 4y = \cos t\, u(t-\pi) \quad y(0)=0 \quad y'(0)=1$



$\cos(t+\pi) = -\cos(t)$

$\mathcal{L}$: $s^2Y(s) - s\underset{0}{y(0)} - \underset{1}{y'(0)} + 4Y(s) = e^{-\pi s}\,\mathcal{L}\{\cos(t+\pi)\}$

$= e^{-\pi s}\,\mathcal{L}\{-\cos t\}$

$(s^2+4)Y(s) - 1 = -e^{-\pi s}\dfrac{s}{s^2+1}$

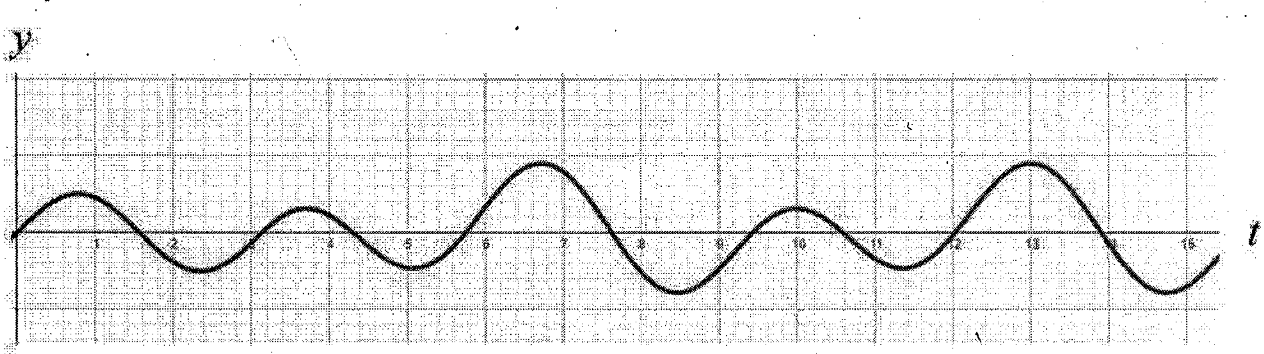
PFE: $\dfrac{s}{(s^2+4)(s^2+1)} = -\dfrac{1}{3}\dfrac{s}{s^2+4} + \dfrac{1}{3}\dfrac{s}{s^2+1}$

$Y(s) = \dfrac{1}{s^2+4} - e^{-\pi s}\dfrac{s}{(s^2+4)(s^2+1)}$

$Y(s) = \dfrac{1}{2}\dfrac{2}{s^2+4} - e^{-\pi s}\left[-\dfrac{1}{3}\dfrac{s}{s^2+4} + \dfrac{1}{3}\dfrac{s}{s^2+1}\right]$

$Y(s) = \dfrac{1}{2}\dfrac{2}{s^2+4} + \dfrac{1}{3}e^{-\pi s}\dfrac{s}{s^2+4} - \dfrac{1}{3}e^{-\pi s}\dfrac{s}{s^2+1}$

$\mathcal{L}^{-1}$: $\boxed{y(t) = \frac{1}{2}\sin 2t + \frac{1}{3}\cos 2(t-\pi)\,u(t-\pi) - \frac{1}{3}\cos(t-\pi)\,u(t-\pi)}$



7.4: Derivatives of a Transform

Summary of Laplace Transforms of derivatives

Earlier we learned that Laplace Transform of a derivative of a function in t...

$$\mathcal{L}\{0\}=0$$
$$\mathcal{L}\{y(t)\}=Y(s)$$
$$\mathcal{L}\{y'(t)\}=sY(s)-y(0)$$
$$\mathcal{L}\{y''(t)\}=s^2Y(s)-sy(0)-y'(0)$$
$$\mathcal{L}\{y'''(t)\}=s^3Y(s)-s^2y(0)-sy'(0)-y(0)$$
$$\mathcal{L}\{f^{(n)}(t)\}=s^nF(s)-s^{n-1}f(0)-s^{n-2}f'(0)-...-f^{(n-1)}(0)$$

...results in multiplying the s side function by multiples of s (along with some initial conditions).

What would happen if we took derivatives on the s side?

Derivatives of a Transform

The following property can be shown to be true:

$$\mathcal{L}\{t^n f(t)\}=(-1)^n\frac{d^n}{ds^n}[F(s)]$$

A derivative of a Laplace transform function is equivalent to the Laplace transform of the original function, but multiplied by a power of t (the power equals the degree of the derivative).

This expands functions we can find Laplace or Inverse Laplace Transforms for in the RHS function.

Ex) Find $\mathcal{L}\{te^{3t}\}$

$n=1$

$$=(-1)^1\frac{d}{ds}\left[\frac{1}{s-3}\right]$$

$$=(-1)\frac{(s-3)(0)-(1)(1)}{(s-3)^2}$$

$$=\boxed{\frac{1}{(s-3)^2}}$$

Ex) Find $\mathcal{L}\{te^{-3t}\cos 3t\}$

$n=1$

$$=(-1)^1\frac{d}{ds}\left[\mathcal{L}\{e^{-3t}\cos 3t\}\right]$$

shifted 3 is s

$$=(-1)\frac{d}{ds}\left[\frac{(s+3)}{(s+3)^2+9}\right]$$

$$=(-1)\frac{((s+3)^2+9)(1)-(s+3)(2(s+3)(1))}{[(s+3)^2+9]^2}$$

$$=(-1)\frac{(s+3)^2+9-2(s+3)^2}{[(s+3)^2+9]^2}$$

$$=\boxed{\frac{(s+3)^2-9}{[(s+3)^2+9]^2}}$$

This is also helpful when we have to take Inverse Laplace Transform not in table

Ex) Find $\mathcal{L}^{-1}\left\{\dfrac{s}{\left(s^2+16\right)^2}\right\}$ ← not in table, but...

$\int \frac{s}{(s^2+16)^2}\,ds \qquad u=s^2+16$

$du = 2s\,ds$

$\int u^{-2}\frac{1}{2}du = -\frac{1}{2}u^{-1} = -\frac{1}{2}\frac{1}{s^2+16} = -\frac{1}{8}\frac{4}{s^2+16}$

$-\frac{1}{8}\mathcal{L}^{-1}\left\{\frac{4}{s^2+16}\right\} = -\frac{1}{8}\sin 4t$

because we undid a derivative, check:

$\mathcal{L}\{t\sin 4t\} = (-1)^1\frac{d}{ds}\left[\frac{4}{s^2+16}\right] = (-1)\frac{(s^2+16)(0)-(4)(2s)}{(s^2+16)^2} = 8\frac{s}{(s^2+16)^2}$

so $\mathcal{L}^{-1}\left\{\frac{s}{(s^2+16)^2}\right\} = \frac{1}{8}\mathcal{L}^{-1}\left\{\frac{8s}{(s^2+16)^2}\right\} = \boxed{\frac{1}{8}t\sin 4t}$

What about this one?

Ex) Find $\mathcal{L}^{-1}\left\{\dfrac{1}{\left(s^2+1\right)^2}\right\}$

We wouldn't be able to easily integrate this to see if it was a derivative of a table structure.
But...

$$\begin{aligned}
\mathcal{L}\{\sin t - t\cos t\} &= \frac{1}{s^2+1} - (-1)^1\frac{d}{ds}\left[\frac{s}{s^2+1}\right] \\
&= \frac{1}{s^2+1} - (-1)^1\frac{\left(s^2+1\right)(1)-(s)(2s)}{\left(s^2+1\right)^2} \\
&= \frac{1}{s^2+1} - (-1)^1\frac{s^2+1-2s^2}{\left(s^2+1\right)^2} \\
&= \frac{1}{s^2+1} + \frac{-s^2+1}{\left(s^2+1\right)^2} \\
&= \frac{1\left(s^2+1\right)}{\left(s^2+1^2\right)} + \frac{-s^2+1}{\left(s^2+1\right)^2} \\
&= \frac{2}{\left(s^2+1\right)^2}
\end{aligned}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{\left(s^2+1\right)^2}\right\} = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{\left(s^2+1\right)^2}\right\} = \frac{1}{2}(\sin t - t\cos t)$$

It isn't intuitive that we would try this, so this is why we have an extended Laplace Table...

Extended Laplace Transform Table

$f(t)$	$\mathscr{L}\{f(t)\} = F(s)$
1. 1	$\dfrac{1}{s}$
2. t	$\dfrac{1}{s^2}$
3. t^n	$\dfrac{n!}{s^{n+1}}$, n a positive integer
4. $t^{-1/2}$	$\sqrt{\dfrac{\pi}{s}}$
5. $t^{1/2}$	$\dfrac{\sqrt{\pi}}{2s^{3/2}}$
6. t^{α}	$\dfrac{\Gamma(\alpha+1)}{s^{\alpha+1}}$, $\alpha > -1$
7. $\sin kt$	$\dfrac{k}{s^2+k^2}$
8. $\cos kt$	$\dfrac{s}{s^2+k^2}$
9. $\sin^2 kt$	$\dfrac{2k^2}{s(s^2+4k^2)}$
10. $\cos^2 kt$	$\dfrac{s^2+2k^2}{s(s^2+4k^2)}$
11. e^{at}	$\dfrac{1}{s-a}$
12. $\sinh kt$	$\dfrac{k}{s^2-k^2}$
13. $\cosh kt$	$\dfrac{s}{s^2-k^2}$
14. $\sinh^2 kt$	$\dfrac{2k^2}{s(s^2-4k^2)}$
15. $\cosh^2 kt$	$\dfrac{s^2-2k^2}{s(s^2-4k^2)}$
16. te^{at}	$\dfrac{1}{(s-a)^2}$
17. $t^n e^{at}$	$\dfrac{n!}{(s-a)^{n+1}}$, n a positive integer
18. $e^{at}\sin kt$	$\dfrac{k}{(s-a)^2+k^2}$
19. $e^{at}\cos kt$	$\dfrac{s-a}{(s-a)^2+k^2}$

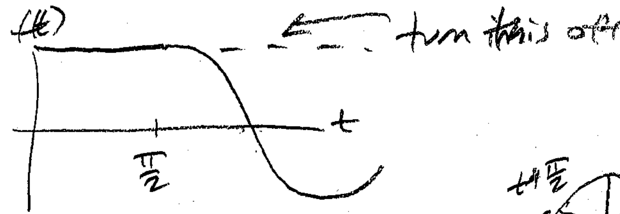
$f(t)$	$\mathcal{L}\{f(t)\} = F(s)$
20. $e^{at}\sinh kt$	$\dfrac{k}{(s-a)^2-k^2}$
21. $e^{at}\cosh kt$	$\dfrac{s-a}{(s-a)^2-k^2}$
22. $t\sin kt$	$\dfrac{2ks}{(s^2+k^2)^2}$
23. $t\cos kt$	$\dfrac{s^2-k^2}{(s^2+k^2)^2}$
24. $\sin kt + kt\cos kt$	$\dfrac{2ks^2}{(s^2+k^2)^2}$
25. $\sin kt - kt\cos kt$	$\dfrac{2k^3}{(s^2+k^2)^2}$
26. $t\sinh kt$	$\dfrac{2ks}{(s^2-k^2)^2}$
27. $t\cosh kt$	$\dfrac{s^2+k^2}{(s^2-k^2)^2}$
28. $\dfrac{e^{at}-e^{bt}}{a-b}$	$\dfrac{1}{(s-a)(s-b)}$
29. $\dfrac{ae^{at}-be^{bt}}{a-b}$	$\dfrac{s}{(s-a)(s-b)}$
30. $1-\cos kt$	$\dfrac{k^2}{s(s^2+k^2)}$
31. $kt-\sin kt$	$\dfrac{k^3}{s^2(s^2+k^2)}$
32. $\dfrac{a\sin bt - b\sin at}{ab(a^2-b^2)}$	$\dfrac{1}{(s^2+a^2)(s^2+b^2)}$
33. $\dfrac{\cos bt - \cos at}{a^2-b^2}$	$\dfrac{s}{(s^2+a^2)(s^2+b^2)}$
34. $\sin kt\sinh kt$	$\dfrac{2k^2s}{s^4+4k^4}$
35. $\sin kt\cosh kt$	$\dfrac{k(s^2+2k^2)}{s^4+4k^4}$
36. $\cos kt\sinh kt$	$\dfrac{k(s^2-2k^2)}{s^4+4k^4}$
37. $\cos kt\cosh kt$	$\dfrac{s^3}{s^4+4k^4}$
38. $J_0(kt)$	$\dfrac{1}{\sqrt{s^2+k^2}}$

$f(t)$	$\mathscr{L}\{f(t)\} = F(s)$
39. $\dfrac{e^{bt} - e^{at}}{t}$	$\ln \dfrac{s-a}{s-b}$
40. $\dfrac{2(1 - \cos kt)}{t}$	$\ln \dfrac{s^2 + k^2}{s^2}$
41. $\dfrac{2(1 - \cosh kt)}{t}$	$\ln \dfrac{s^2 - k^2}{s^2}$
42. $\dfrac{\sin at}{t}$	$\arctan\left(\dfrac{a}{s}\right)$
43. $\dfrac{\sin at \cos bt}{t}$	$\dfrac{1}{2}\arctan \dfrac{a+b}{s} + \dfrac{1}{2}\arctan \dfrac{a-b}{s}$
44. $\dfrac{1}{\sqrt{\pi t}} e^{-a^2/4t}$	$\dfrac{e^{-a\sqrt{s}}}{\sqrt{s}}$
45. $\dfrac{a}{2\sqrt{\pi t^3}} e^{-a^2/4t}$	$e^{-a\sqrt{s}}$
46. $\operatorname{erfc}\left(\dfrac{a}{2\sqrt{t}}\right)$	$\dfrac{e^{-a\sqrt{s}}}{s}$
47. $2\sqrt{\dfrac{t}{\pi}}\, e^{-a^2/4t} - a \operatorname{erfc}\left(\dfrac{a}{2\sqrt{t}}\right)$	$\dfrac{e^{-a\sqrt{s}}}{s\sqrt{s}}$
48. $e^{ab}e^{b^2t} \operatorname{erfc}\left(b\sqrt{t} + \dfrac{a}{2\sqrt{t}}\right)$	$\dfrac{e^{-a\sqrt{s}}}{\sqrt{s}(\sqrt{s} + b)}$
49. $-e^{ab}e^{b^2t} \operatorname{erfc}\left(b\sqrt{t} + \dfrac{a}{2\sqrt{t}}\right) + \operatorname{erfc}\left(\dfrac{a}{2\sqrt{t}}\right)$	$\dfrac{be^{-a\sqrt{s}}}{s(\sqrt{s} + b)}$
50. $e^{at} f(t)$	$F(s - a)$
51. $\mathscr{U}(t - a)$	$\dfrac{e^{-as}}{s}$
52. $f(t - a)\mathscr{U}(t - a)$	$e^{-as}F(s)$
53. $g(t)\mathscr{U}(t - a)$	$e^{-as}\mathscr{L}\{g(t + a)\}$
54. $f^{(n)}(t)$	$s^n F(s) - s^{(n-1)} f(0) - \cdots - f^{(n-1)}(0)$
55. $t^n f(t)$	$(-1)^n \dfrac{d^n}{ds^n} F(s)$
56. $\displaystyle\int_0^t f(\tau) g(t - \tau)\, d\tau$	$F(s)G(s)$
57. $\delta(t)$	1
58. $\delta(t - t_0)$	e^{-st_0}

A solving DE example...

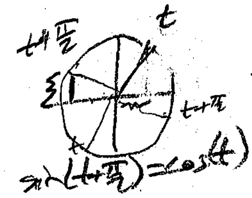
Ex) Solve $y''+y=f(t)$ $\quad y(0)=1 \quad y'(0)=0$

where $f(t)=\begin{cases}1 & 0\le t<\frac{\pi}{2}\\ \sin t & t\ge\frac{\pi}{2}\end{cases}$



$f(t)=1-(1)u(t-\frac{\pi}{2})+\sin t\, u(t-\frac{\pi}{2})$

$y''+y=1-u(t-\frac{\pi}{2})+\sin t\, u(t-\frac{\pi}{2})$



$\mathcal{L}$: $s^2Y(s)-s\underset{(1)}{y(0)}-\underset{(0)}{y'(0)}+Y(s)=\frac{1}{s}-e^{-\frac{\pi}{2}s}\frac{1}{s}+e^{-\frac{\pi}{2}s}\underset{\mathcal{L}\{\cos t\}}{\mathcal{L}\{\sin(t+\frac{\pi}{2})\}}$

$(s^2+1)Y(s)-s=\frac{1}{s}-e^{-\frac{\pi}{2}s}\frac{1}{s}+e^{-\frac{\pi}{2}s}\left(\frac{s}{s^2+1}\right)$

$Y(s)=\frac{s}{s^2+1}+\frac{1}{s(s^2+1)}-e^{-\frac{\pi}{2}s}\frac{1}{s(s^2+1)}+e^{-\frac{\pi}{2}s}\left(\frac{s}{(s^2+1)^2}\right)$

PFE

$Y(s)=\cancel{\frac{s}{s^2+1}}+\frac{1}{s}-\cancel{\frac{s}{s^2+1}}-e^{-\frac{\pi}{2}s}\frac{1}{s}+e^{-\frac{\pi}{2}s}\frac{s}{s^2+1}+e^{-\frac{\pi}{2}s}\left(\frac{s\,2}{(s^2+1)^2}\right)\frac{1}{2}$

$\mathcal{L}^{-1}$: $y(t)=1-u(t-\frac{\pi}{2})+\underset{\sin(t)}{\cos(t-\frac{\pi}{2})}u(t-\frac{\pi}{2})+\frac{1}{2}(t-\frac{\pi}{2})\underset{\cos(t)}{\sin(t-\frac{\pi}{2})}u(t-\frac{\pi}{2})$

$$\boxed{y(t)=1-u(t-\tfrac{\pi}{2})+\sin(t)u(t-\tfrac{\pi}{2})+\tfrac{1}{2}(t-\tfrac{\pi}{2})\cos(t)u(t-\tfrac{\pi}{2})}$$

7.5: The Dirac Delta Function

Is the Laplace Transform of any function = 1 ?

The simplest Laplace Transform shown in the regular table is

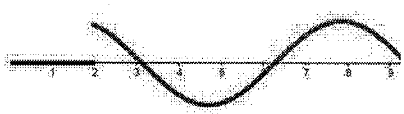
$$\mathcal{L}\{1\}=\frac{1}{s}$$

Is there any function whose Laplace Transform would be simpler than 1/s, for example 1?

The answer is yes, but it isn't a normal function.

Unit Impulse

So far we've been able to use things like the Unit Step Function to allow for driving functions for differential equations which are abruptly turned on or off:

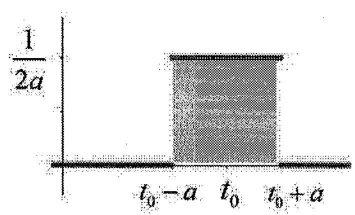


...but these function, once on, are defined (are 'on') for some period of time.

What if we wanted to model something like the vibrations of an object if it is struck by an another object? Where there is a driving function but it acts only for a brief instant in time?

We could define such a function, which is called the **Unit Impulse Function** like this:

$$\delta_a(t-t_o)=\begin{cases} 0, & 0\le t<t_0-a \\ \dfrac{1}{2a}, & t_0-a\le t<t_0+a \\ 0 & t\ge t_0+a \end{cases}$$



Where the height is defined such that the area under the curve is 1:

$$\int_0^\infty \delta_a(t-t_o)\,dt=1$$

The Dirac Delta Function

In practice, we instead work with another type of unit impulse, a 'function' that approximates the Unit Impulse Function but is defined by a limit:

$$\delta(t-t_o)=\lim_{a\to 0}\delta_a(t-t_o)$$

As a decreases, the height of the unit impulse function increase to match so that the area under the function curve remains 1. This means that in the limit, the height is approaching infinity as the width is approaching zero:

$$\delta(t-t_o)=\begin{cases}\infty, & t=t_0\\ 0, & t\neq t_0\end{cases} \qquad \int_0^\infty \delta(t-t_o)\,dt=1$$

...and this special unit impulse is called the **Dirac Delta Function**.

We can use this function to represent a driving function whose effect is felt over an infinitesimally small period of time, for example if an object is 'struck' by another object.

Laplace Transform of the Dirac Delta Function

It can be shown (separate PDF available on website) that the Laplace Transform of the Dirac Delta Function is:

$$\boxed{\mathcal{L}\{\delta(t-t_o)\}=e^{-st_0}}$$

(Like the Laplace Transform of the Unit Step Function, but without dividing by s)

And if $t_o=0$: $\boxed{\mathcal{L}\{\delta(t)\}=1}$

An example...

Solve $y''+y=4\delta(t-2\pi) \quad y(0)=1 \quad y'(0)=0$

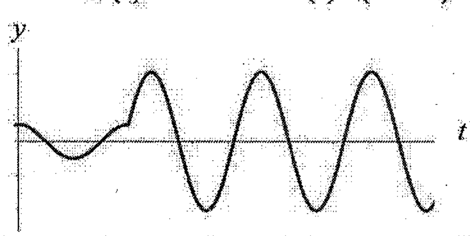
This could represent a undamped mass-spring system which has the mass initially displaced and released, but with no initial velocity, which causes some oscillation at the natural frequency, but then at a later time, the mass is 'struck' by a hammer.

Using the new definition to take the Laplace Transform of the RHS:

$$s^2Y(s)-sy(0)-y'(0)+sY(s)=4e^{-2\pi s}$$
$$s^2Y(s)-s(1)-(0)+sY(s)=4e^{-2\pi s}$$
$$(s^2+1)Y(s)-s=4e^{-2\pi s}$$
$$Y(s)=\frac{s}{s^2+1}+4\frac{1}{s^2+1}e^{-2\pi s}$$
$$y(t)=\cos t+4\sin(t-2\pi)u(t-2\pi)$$

Since $\sin(t-2\pi)=\sin(t)$

$$y(t)=\cos t+4\sin(t)u(t-2\pi)$$



The mass initially oscillates at the natural frequency due to the initial displacement, and then when the mass is struck and new oscillation is started, also at the natural frequency by the impulse strike, beginning at the point where the mass was when struck.

Summary

$$\mathcal{L}\{0\}=0$$

$$\mathcal{L}\{y(t)\}=Y(s)$$

$$\mathcal{L}\{y'(t)\}=sY(s)-y(0)$$

$$\mathcal{L}\{y''(t)\}=s^2Y(s)-sy(0)-y'(0)$$

$$\mathcal{L}\{y'''(t)\}=s^3Y(s)-s^2y(0)-sy'(0)-y(0)$$

$$\mathcal{L}\{t^n f(t)\}=(-1)^n \frac{d^n}{ds^n}[F(s)]$$

$$\boxed{\begin{gathered}\mathcal{L}\{e^{at}f(t)\}=F(s-a)\\ \mathcal{L}^{-1}\{F(s-a)\}=e^{at}f(t)\end{gathered}}$$

$$\boxed{\begin{gathered}\mathcal{L}\{f(t-a)u(t-a)\}=e^{-as}F(s)\\ \mathcal{L}^{-1}\{e^{-as}F(s)\}=f(t-a)u(t-a)\end{gathered}}$$

$$\boxed{\mathcal{L}\{u(t-a)\}=\frac{e^{-as}}{s}}$$

$$\boxed{\mathcal{L}\{g(t)u(t-a)\}=e^{-as}\mathcal{L}\{g(t+a)\}}$$

$$\boxed{\mathcal{L}\{\delta(t-t_o)\}=e^{-st_0}}$$

$f(t)$	$\mathscr{L}\{f(t)\} = F(s)$
1. 1	$\dfrac{1}{s}$
2. t	$\dfrac{1}{s^2}$
3. t^n	$\dfrac{n!}{s^{n+1}}$, n a positive integer
4. $t^{-1/2}$	$\sqrt{\dfrac{\pi}{s}}$
5. $t^{1/2}$	$\dfrac{\sqrt{\pi}}{2s^{3/2}}$
6. t^{α}	$\dfrac{\Gamma(\alpha+1)}{s^{\alpha+1}}$, $\alpha > -1$
7. $\sin kt$	$\dfrac{k}{s^2+k^2}$
8. $\cos kt$	$\dfrac{s}{s^2+k^2}$
9. $\sin^2 kt$	$\dfrac{2k^2}{s(s^2+4k^2)}$
10. $\cos^2 kt$	$\dfrac{s^2+2k^2}{s(s^2+4k^2)}$
11. e^{at}	$\dfrac{1}{s-a}$
12. $\sinh kt$	$\dfrac{k}{s^2-k^2}$
13. $\cosh kt$	$\dfrac{s}{s^2-k^2}$
14. $\sinh^2 kt$	$\dfrac{2k^2}{s(s^2-4k^2)}$
15. $\cosh^2 kt$	$\dfrac{s^2-2k^2}{s(s^2-4k^2)}$
16. te^{at}	$\dfrac{1}{(s-a)^2}$
17. $t^n e^{at}$	$\dfrac{n!}{(s-a)^{n+1}}$, n a positive integer
18. $e^{at}\sin kt$	$\dfrac{k}{(s-a)^2+k^2}$
19. $e^{at}\cos kt$	$\dfrac{s-a}{(s-a)^2+k^2}$

$f(t)$	$\mathscr{L}\{f(t)\} = F(s)$
20. $e^{at} \sinh kt$	$\dfrac{k}{(s-a)^2 - k^2}$
21. $e^{at} \cosh kt$	$\dfrac{s-a}{(s-a)^2 - k^2}$
22. $t \sin kt$	$\dfrac{2ks}{(s^2+k^2)^2}$
23. $t \cos kt$	$\dfrac{s^2-k^2}{(s^2+k^2)^2}$
24. $\sin kt + kt \cos kt$	$\dfrac{2ks^2}{(s^2+k^2)^2}$
25. $\sin kt - kt \cos kt$	$\dfrac{2k^3}{(s^2+k^2)^2}$
26. $t \sinh kt$	$\dfrac{2ks}{(s^2-k^2)^2}$
27. $t \cosh kt$	$\dfrac{s^2+k^2}{(s^2-k^2)^2}$
28. $\dfrac{e^{at}-e^{bt}}{a-b}$	$\dfrac{1}{(s-a)(s-b)}$
29. $\dfrac{ae^{at}-be^{bt}}{a-b}$	$\dfrac{s}{(s-a)(s-b)}$
30. $1 - \cos kt$	$\dfrac{k^2}{s(s^2+k^2)}$
31. $kt - \sin kt$	$\dfrac{k^3}{s^2(s^2+k^2)}$
32. $\dfrac{a \sin bt - b \sin at}{ab(a^2-b^2)}$	$\dfrac{1}{(s^2+a^2)(s^2+b^2)}$
33. $\dfrac{\cos bt - \cos at}{a^2-b^2}$	$\dfrac{s}{(s^2+a^2)(s^2+b^2)}$
34. $\sin kt \sinh kt$	$\dfrac{2k^2s}{s^4+4k^4}$
35. $\sin kt \cosh kt$	$\dfrac{k(s^2+2k^2)}{s^4+4k^4}$
36. $\cos kt \sinh kt$	$\dfrac{k(s^2-2k^2)}{s^4+4k^4}$
37. $\cos kt \cosh kt$	$\dfrac{s^3}{s^4+4k^4}$
38. $J_0(kt)$	$\dfrac{1}{\sqrt{s^2+k^2}}$

$f(t)$	$\mathcal{L}\{f(t)\} = F(s)$
39. $\dfrac{e^{bt} - e^{at}}{t}$	$\ln \dfrac{s - a}{s - b}$
40. $\dfrac{2(1 - \cos kt)}{t}$	$\ln \dfrac{s^2 + k^2}{s^2}$
41. $\dfrac{2(1 - \cosh kt)}{t}$	$\ln \dfrac{s^2 - k^2}{s^2}$
42. $\dfrac{\sin at}{t}$	$\arctan\left(\dfrac{a}{s}\right)$
43. $\dfrac{\sin at \cos bt}{t}$	$\dfrac{1}{2}\arctan\dfrac{a + b}{s} + \dfrac{1}{2}\arctan\dfrac{a - b}{s}$
44. $\dfrac{1}{\sqrt{\pi t}} e^{-a^2/4t}$	$\dfrac{e^{-a\sqrt{s}}}{\sqrt{s}}$
45. $\dfrac{a}{2\sqrt{\pi t^3}} e^{-a^2/4t}$	$e^{-a\sqrt{s}}$
46. $\operatorname{erfc}\left(\dfrac{a}{2\sqrt{t}}\right)$	$\dfrac{e^{-a\sqrt{s}}}{s}$
47. $2\sqrt{\dfrac{t}{\pi}} e^{-a^2/4t} - a \operatorname{erfc}\left(\dfrac{a}{2\sqrt{t}}\right)$	$\dfrac{e^{-a\sqrt{s}}}{s\sqrt{s}}$
48. $e^{ab}e^{b^2t} \operatorname{erfc}\left(b\sqrt{t} + \dfrac{a}{2\sqrt{t}}\right)$	$\dfrac{e^{-a\sqrt{s}}}{\sqrt{s}(\sqrt{s} + b)}$
49. $-e^{ab}e^{b^2t} \operatorname{erfc}\left(b\sqrt{t} + \dfrac{a}{2\sqrt{t}}\right) + \operatorname{erfc}\left(\dfrac{a}{2\sqrt{t}}\right)$	$\dfrac{be^{-a\sqrt{s}}}{s(\sqrt{s} + b)}$
50. $e^{at} f(t)$	$F(s - a)$
51. $\mathcal{U}(t - a)$	$\dfrac{e^{-as}}{s}$
52. $f(t - a)\mathcal{U}(t - a)$	$e^{-as}F(s)$
53. $g(t)\mathcal{U}(t - a)$	$e^{-as}\mathcal{L}\{g(t + a)\}$
54. $f^{(n)}(t)$	$s^n F(s) - s^{(n-1)} f(0) - \cdots - f^{(n-1)}(0)$
55. $t^n f(t)$	$(-1)^n \dfrac{d^n}{ds^n} F(s)$
56. $\displaystyle\int_0^t f(\tau) g(t - \tau)\, d\tau$	$F(s)G(s)$
57. $\delta(t)$	1
58. $\delta(t - t_0)$	e^{-st_0}