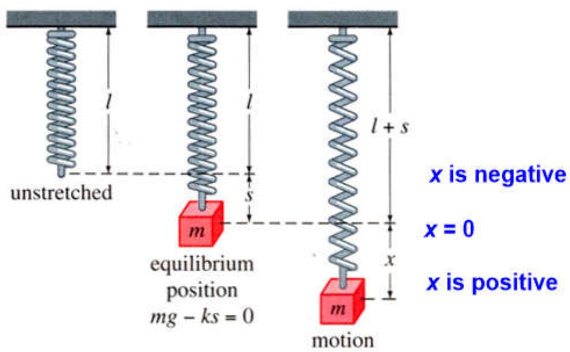
5.1: Higher-order linear models

Having 2nd or higher order derivatives opens up even more possibilities

If we use 2nd or higher order differential equations the possibilities for modeling are even greater than for first-order DEs. We only have time to focus on a couple of scenarios, but these are both really important in physics and engineering: vibrations and electrical circuits. It turns out that these are similar but easier to understand physical vibrations, so we'll start with the simplest physical vibration: a mass on a spring.

Springs: Hooke's Law

If you had a mass attached to a spring in a gravity field, the mass will stretch the spring. For all our problems of this type, we will define 'downward' as positive distance, *x*, also known as **displacement**. The spring starts with some unstretched length, and when the mass is added, it stretches to it **equilibrium position**. Then if we use an external force to either push the mass above or below its equilibrium position, we define this addition displacement as *x*:



What we are usually interested in is the position of the mass over time, so we are using a differential equation with *x* as the dependent variable and *t* as the independent variable to find a solution function for *x(t)*.

It turns out that if any force stretches a spring, the amount of stretching is well modeled using Hooke's Law, which says the force is directly proportional to the displacement:

$$F = ks$$

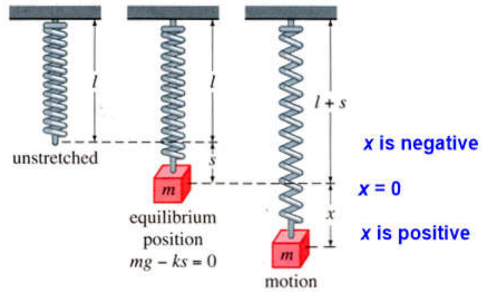
At equilibrium, the only force acting on the mass is gravity: $F = mg$

The mass will stretch the spring until the spring force exactly balances the gravitational force.

...so at equilibrium: $mg = ks$

Springs: Allow other forces with Newton's Law

If we further disturb the system, for example, by pushing the mass above or below equilibrium and releasing it, the mass will be moving, and Newton's Law says that the acceleration of this motion is described by: $F = ma$



So now there are two forces acting on the mass: gravity and this additional force which stretches it further than equilibrium displacement to a distance of *s* + *x*.

$$sum\ of\ Forces = ma = -k(s+x) + mg$$

The gravity term is positive because gravity is pulling in the direction of positive *x* but the spring force (for positive *x*) is pulling in the negative *x* direction.

We also know that acceleration is the 2nd derivative of position, so...

$$ma = -k(s+x) + mg$$

$$m\frac{d^2x}{dt^2} = -k(s+x) + mg$$

$$m\frac{d^2x}{dt^2} = -ks - kx + mg$$

$$m\frac{d^2x}{dt^2} + kx = -ks + mg$$

And because *s* is the equilibrium displacement value: $-ks + mg = 0$

$$\boxed{m\frac{d^2x}{dt^2} + kx = 0}$$

DE for free, undamped motion of a mass on a spring

Before solving this DE, it is helpful to divide by the leading coefficient and define a constant, omega:

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$

$$\frac{d^2x}{dt^2} + \omega^2 x = 0 \quad where \quad \boxed{\omega^2 = \frac{k}{m}}$$

We can now solve this DE using an auxiliary equation:

$$m^2 + \omega^2 = 0$$

$$m^2 = -\omega^2$$

$$m = \pm\omega i$$

With a complex-conjugate pair of roots (with $\alpha = 0$), the solution is:

$$\boxed{x(t) = C_1\cos(\omega t) + C_2\sin(\omega t)}$$

Alternate form of solution - as a single sinusoid

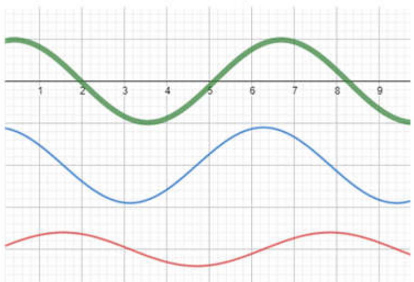
Interestingly, this solution with one cosine and one sine term with the same argument can also be written using a single sinusoid.

This geogebra app can visualize this: https://www.geogebra.org/calculator/c4uafzr7

But you can show with application of trig identities that:

$$\boxed{x(t) = C_1 \cos(\omega t) + C_2 \sin(\omega t) = A \sin(\omega t + \phi)}$$

$$\text{where } A = \sqrt{C_1^2 + C_2^2} \quad \text{and} \quad \tan\phi = \frac{C_1}{C_2}$$



$$C_1 \cos(\omega t) + C_2 \sin(\omega t) = A \sin(\omega t + \phi)$$

$$C_1 \cos(\omega t)$$

$$C_2 \sin(\omega t)$$

Amplitude, phase shift, period, and frequency

The constants have meanings in context:

$$x(t) = C_1 \cos(\omega t) + C_2 \sin(\omega t) = A \sin(\omega t + \phi)$$

Sine goes through one period when its argument goes from 0 to 2π :

$$0 < \omega t + \phi < 2\pi$$

$$-\phi < \omega t < 2\pi - \phi$$

$$-\frac{\phi}{\omega} < t < \frac{2\pi}{\omega} - \frac{\phi}{\omega}$$

period   phase shift

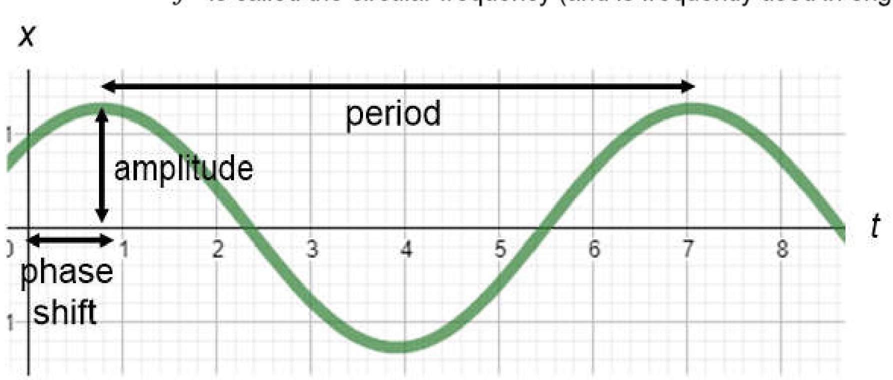
The period = $T = \dfrac{2\pi}{\omega}$ is the time between crests

The phase shift = $\dfrac{\phi}{\omega}$ is the time after zero when the first zero-crossing occurs

The amplitude = A is the height above and below of the max and min values

The frequency = $f = \dfrac{1}{T} = \dfrac{\omega}{2\pi}$ is the number of cycles per second

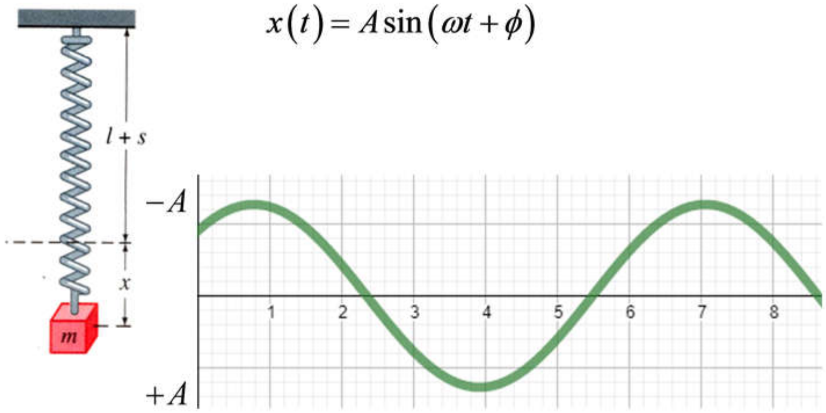
and $\omega = 2\pi f$ is called the circular frequency (and is frequently used in engineering)



Physical meaning of free, undamped mass/spring system solution

Physically, this means that the mass oscillates up and down (down is positive) and if you traced out the position of the mass over time, it would be sinusoidal.

$$x(t) = A\sin(\omega t + \phi)$$

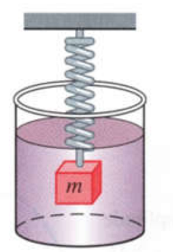


Because there are no other forces acting on the mass (there is no air resistance, and no energy lost to heat in the spring), this motion would (theoretically) occur unchanged forever.

Free, damped mass/spring motion DE

Of course, in the real world there are always other forces acting on the mass and some energy is lost to heat in the spring, so the oscillation amplitude will decrease over time.

The way this is modeled is to include **damping** - something which opposes motion of the mass similar to the way air resistance opposes motion in falling mass problems. This can be thought of as if the mass were submerged in a thick liquid:



Similar to air resistance, we can assume that this opposing force is proportional to a power of the velocity of the mass, although a very good model is obtained just assuming it is proportion to the velocity. The proportionality constant is usually labeled beta, and this provides another force in the sum of forces equation:

$$m\frac{d^2x}{dt^2} = -kx - \beta\frac{dx}{dt}$$

...which when rearranged gives the most useful form of DE for solving mass/spring problems:

$$\boxed{m\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + kx = 0}$$

Free, damped mass/spring motion solutions

As we solve, it is again useful to make some symbol substitutions:

$$m\frac{d^2x}{dt^2}+\beta\frac{dx}{dt}+kx=0$$

$$\frac{d^2x}{dt^2}+\frac{\beta}{m}\frac{dx}{dt}+\frac{k}{m}x=0$$

$$\frac{d^2x}{dt^2}+2\lambda\frac{dx}{dt}+\omega^2x=0 \qquad where \; \boxed{2\lambda=\frac{\beta}{m}} \; and \; \boxed{\omega^2=\frac{k}{m}}$$

Using an auxiliary equation:

$$m^2+2\lambda m+\omega^2=0$$

$$m=\frac{-2\lambda\pm\sqrt{4\lambda^2-4\omega^2}}{2}=-\lambda\pm\sqrt{\lambda^2-\omega^2}$$

so the solution will be of the form:

$$x(t)=C_1e^{\left(-\lambda+\sqrt{\lambda^2-\omega^2}\right)t}+C_2e^{\left(-\lambda-\sqrt{\lambda^2-\omega^2}\right)t}$$

The specific solution form depends upon the discriminant and there are 3 cases:

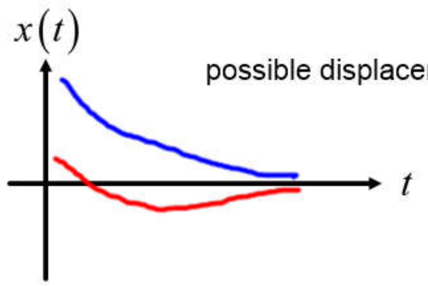
$$x(t)=C_1e^{\left(-\lambda+\sqrt{\lambda^2-\omega^2}\right)t}+C_2e^{\left(-\lambda-\sqrt{\lambda^2-\omega^2}\right)t}$$

1) If $\lambda^2-\omega^2>0$ there are two distinct real roots:

$$x(t)=C_1e^{\left(-\lambda+\sqrt{\lambda^2-\omega^2}\right)t}+C_2e^{\left(-\lambda-\sqrt{\lambda^2-\omega^2}\right)t}$$

$$\boxed{x(t)=C_1e^{-\lambda t}e^{+\sqrt{\lambda^2-\omega^2}\,t}+C_2e^{-\lambda t}e^{-\sqrt{\lambda^2-\omega^2}\,t}}$$

This situation is called **Overdamped** and physically this means that there is no oscillation, the position slowly and smoothly moves towards the steady-state position:



possible displacement behavior curves for overdamped systems

Free, damped mass/spring motion solutions

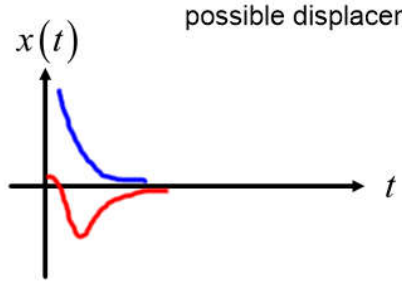
The specific solution form depends upon the discriminant and there are 3 cases:

2) If $\lambda^2 - \omega^2 = 0$ there is one, repeated real root:

$$x(t) = C_1 e^{\left(-\lambda+\sqrt{\lambda^2-\omega^2}\right)t} + C_2 e^{\left(-\lambda-\sqrt{\lambda^2-\omega^2}\right)t}$$

$$\boxed{x(t) = C_1 e^{-\lambda t} + C_2 t e^{-\lambda t}}$$

This situation is called **Critically Damped** and physically this means that there is no oscillation, but that the position is changing as rapidly as possible to get to the steady-state position:



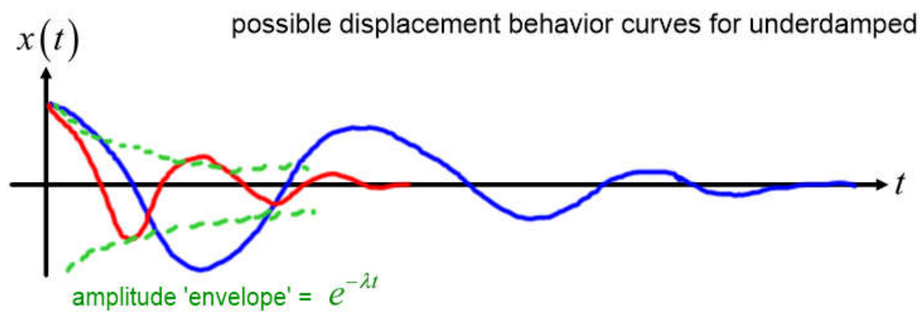
possible displacement behavior curves for critically damped systems

3) If $\lambda^2 - \omega^2 < 0$ the roots are complex-conjugates: $\alpha \pm \beta i = -\lambda \pm \sqrt{\omega^2 - \lambda^2}$

$$x(t) = C_1 e^{\left(-\lambda+\sqrt{\lambda^2-\omega^2}\right)t} + C_2 e^{\left(-\lambda-\sqrt{\lambda^2-\omega^2}\right)t} = C_1 e^{(\alpha+\beta i)t} + C_2 e^{(\alpha-\beta i)t}$$

$$\boxed{x(t) = C_1 e^{-\lambda t}\cos\left(\sqrt{\omega^2-\lambda^2}\,t\right) + C_2 e^{-\lambda t}\sin\left(\sqrt{\omega^2-\lambda^2}\,t\right)}$$

This situation is called **Underdamped** and physically this means that the damping isn't strong enough to prevent oscillation, but the exponential amplitude multiplier means the oscillations will fade out over time:



possible displacement behavior curves for underdamped systems

Solving for constants C_1 and C_2

So far we've been finding the general solutions to these mass/spring systems. To establish the constants C_1 and C_2 we need to have initial conditions.

We can specify how far the mass is moved away from equilibrium position:

'The mass is initially pulled 1 ft below equilibrium' is equivalent to $x(0)=1$

'The mass is released 2 ft above equilibrium' is equivalent to $x(0)=-2$

Since there are typically two constants, we also need to take the derivative of our solution and have an initial condition for the derivative. This corresponds to imparting an initial velocity to the mass (not just 'letting it go'):

'The mass is released with an initial velocity upward of 3 ft/sec' is equivalent to $x'(0)=-3$

Once we have the general solution, we take the derivative, plug in the initial conditions and solve for C_1 and C_2, which may involve solving a system.

Driven, damped mass/spring motion systems

So far all of our DEs have been homogeneous because once we release mass (possibly with some initial velocity) there are no additional, continuing forces acting on the mass other than the spring and gravity.

But what if something is continuing to push on the math, potentially in some time varying manner? This is known as a **driven system** or a **forced system**, and the way this is modeled is to include a **driving function** on the RHS of the DE:

$$\boxed{m\frac{d^2x}{dt^2}+\beta\frac{dx}{dt}+kx=f(t)}$$

...so the DE is now non-homogenous, which means that we need to use methods such as Undetermined Coefficients (guess particular solution for y_p using the table) or Separation of Variables to solve.

Before we solve, we typically divide by the leading coefficient to get:

$$\frac{d^2x}{dt^2}+\frac{\beta}{m}\frac{dx}{dt}+\frac{k}{m}x=\frac{f(t)}{m}$$

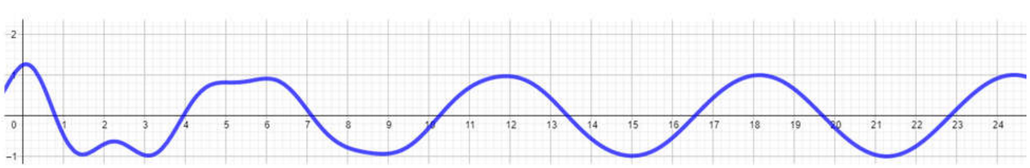
$$\boxed{\frac{d^2x}{dt^2}+2\lambda\frac{dx}{dt}+\omega^2x=F(t)}$$

Steady-state and Transient terms

When you solve driven mass/spring system problems, the solutions are often in forms similar to this:

$$x(t)=\frac{1}{2}e^{-\frac{1}{4}t}\cos(3t)+\frac{2}{5}e^{-\frac{1}{4}t}\sin(3t)+\frac{3}{4}\cos(t)-\frac{2}{3}\sin(t)$$

transient terms (solution) steady-state terms (solution)



At first, all 4 terms affect the solution..

...but after awhile, the envelope on the transient terms causes their effect to fade out.

At this point, the mass will move according to the driving function.

Natural Frequency

So what does the behavior of the transient terms represent?

$$x(t)=\frac{1}{2}e^{-\frac{1}{4}t}\cos(3t)+\frac{2}{5}e^{-\frac{1}{4}t}\sin(3t)+\frac{3}{4}\cos(t)-\frac{2}{3}\sin(t)$$

transient terms (solution) — steady-state terms (solution)

In this example, the driving function has a circular frequency $\omega=1$
but even in the absence of a driving function, system spring system, if disturbed, will oscillate with a circular frequency $\omega=3$

The frequency of the transient terms is called the system's **natural frequency.**

Resonance

What happens if you try to drive a mass/spring system with no damping at its natural frequency?

Ex) A mass of 2 kg is attached to a spring with k = 32 N/m and it comes to rest at equilibrium. Starting at $t = 0$ a driving force $f(t)=16\cos 4t$ is applied to the system. Find the equation of motion in the absence of damping.

$$mx''+\beta x'+kx=16\cos 4t$$
$$2x''+0x'+32x=16\cos 4t$$
$$x''+16x=8\cos 4t$$

Solve the homogeneous DE:

$$x''+16x=0$$
$$m^2+16=0$$
$$m=\pm 4i \quad (\alpha=0)$$
$$x(t)=C_1\cos 4t+C_2\sin 4t$$

then solve the non-homogeneous DE:

for RHS $8\cos 4t$ table suggests

y_P *here* $x_P=A\cos 4t+B\sin 4t$

but when you take derivatives and plug in, everything cancels so you can't solve for constants.

Next, try multiplying terms by t:

y_P *here* $x_P=At\cos 4t+Bt\sin 4t$

when you take derivatives and plug into DE, A=0, B=1

so $x_P=t\sin 4t$ *(steps omitted)*

and the general solution is: $x(t)=C_1\cos 4t+C_2\sin 4t+t\sin 4t$

If we then impose some initial conditions, such as: $x(0)=1,\ x'(0)=2$

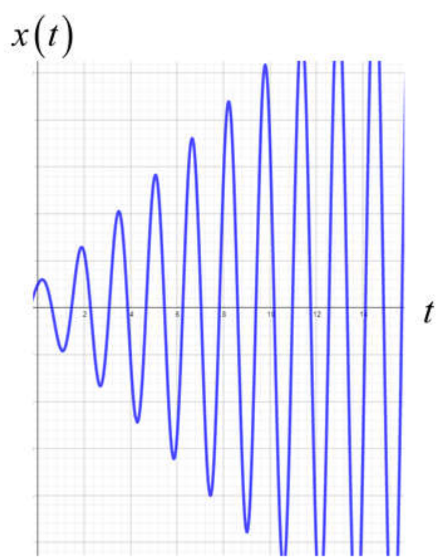
taking derivative: $x'(t)=-4C_1\sin 4t+4C_2\cos 4t+4t\cos 4t+\sin 4t$

and using initial conditions, you can solve for C_1 and C_2: *(steps omitted)* $C_1=1, C_2=\frac{1}{2}$

Then the particular solution is: $x(t)=\cos 4t+\frac{1}{2}\sin 4t+t\sin 4t$

If you graph this solution:

$$x(t)=\cos 4t+\frac{1}{2}\sin 4t+t\sin 4t$$



...the system rapidly oscillates at the natural frequency and the amplitude increases without bound (until the system destroys itself physically).

This phenomenon is known as **resonance.** Any system without damping is subject to resonance at its natural frequency. To prevent this, system designers include damping that prevents resonance.

The Tacoma-Narrows Bridge

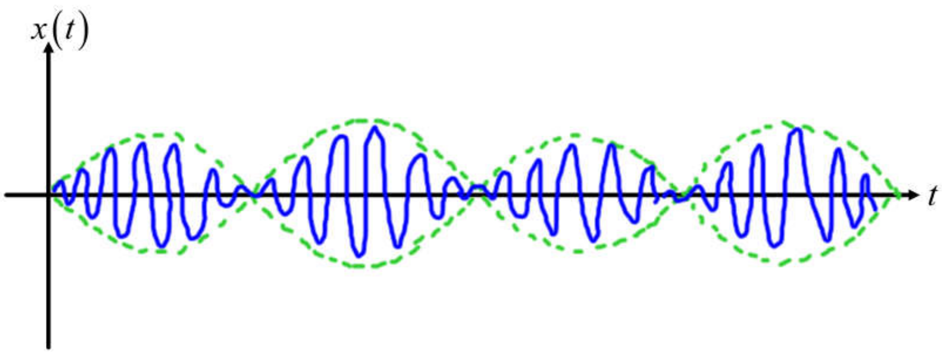
https://www.youtube.com/watch?v=j-zczJXSxnw&ab_channel=TonyC

A famous example of a bridge which was excited by high winds as a driving force that turned out to have insufficient damping at its natural frequency.

(This is a more difficult example to analyze because motion is happening in more than one direction).

Beat Phenomenon

If you drive a system with a sinusoidal driving function, not at the natural frequency but at a frequency much different than the natural frequency, under some circumstances something called the **beat phenomenon** arises. In this case, the lower frequency acts as a sinusoidal 'envelope' and you get a solution function like this:



(We're not going to work problems like this in this class, but I'm including in the notes a page from another book with more information about resonance and beat phenomenon with more specific formulas in case you want to look into it further.)

Mechanical vibrations: beats and resonance

Both mechanical vibrations and electric circuits can be accurately modeled by linear differential equations with constant coefficients. Here we will consider only the case of mechanical vibrations, but exactly the same principles apply to electric circuits.

I have posted notes on the algebra and trig underlying some of the analysis.

If u measures the displacement of a mass m from equilibrium, then

$$mu'' + \gamma u' + ku = F. \tag{1}$$

where γ and k are nonnegative constants, and $F = F(t)$ is the external force. The parameter k appears courtesy of Hooke's Law, and is called the "spring constant", even when there is no actual spring.

The damping coefficient γ might be due to friction, internal or external. The term $\gamma u'$ is the nonconservative contribution: it accounts for the energy lost to heat. Often it is the least accurate approximation to the contributing forces, but is accurate enough for many applications to give robust qualitative conclusions about many mechanical systems.

Our strategy to understand the behavior of such systems is to begin with the assumption that there is no damping ($\gamma = 0$) and no forcing ($F = 0$). We then add sinusoidal forcing, where we first encounter the phenomena of "beats" and "resonance". We then consider the effect of damping on this situation.

Undamped free vibrations

A free or unforced vibration is one where $F = 0$. An undamped vibration is one where $\gamma = 0$. We say that the system is *perfectly elastic*. It does not dissipate energy due to frictional or other nonconservative forces.

The general solution to the equation $u'' + ku = 0$ is

$$u = A\cos(\omega_0 t) + B\sin(\omega_0 t), \text{ where } \omega_0 = \sqrt{k/m}, \tag{2}$$

and A and B are arbitrary constants. [Check this!] We call ω_0 the *natural harmonic* of the system.

This homogeneous solution can be rewritten in the form

$$u = R\cos(\omega_0 t - \delta), \text{ where } R = \sqrt{A^2 + B^2},\ R\cos(\delta) = A. \tag{3}$$

The constant R is the amplitude of the wave. The phase shift is δ/ω_0. [Check this!]

Undamped vibrations with periodic forcing

Specificaly, we look at perhaps the simplest periodic forcing, namely $F(t) = \cos(\omega t)$. For our first look at this situation we continue with the assumption that $\gamma = 0$. For simplicity we also take as initial conditions $u(0) = u'(0) = 0$. Other initial conditions will merely effect the contribution from the homogeneous solution, and hence not affect the conclusions.

If $\omega \neq \omega_0$ then

$$u = \frac{F_0/m}{\omega_0^2 - \omega^2}\left[\cos(\omega t) - \cos(\omega_0 t)\right] \tag{4}$$

$$= \frac{F_0/m}{\frac{1}{2}|\omega_0^2 - \omega^2|}\sin(\tfrac{1}{2}|\omega_0 - \omega|\, t)\sin(\tfrac{1}{2}|\omega_0 + \omega|\, t). \tag{5}$$

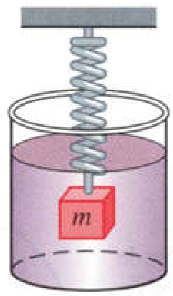
[Check this!] The first sine factor is a lower frequencey "envelope" which modulates the higher-frequency second sine factor. This is the phenomonon of *beats*. As $\omega \to \omega_0$ the solution changes to

$$u = \frac{F_0/m}{2\omega_0}\, t\, \sin(\omega_0 t). \tag{6}$$

[Check this!] Now the amplitude grows without bound. Thus is the phenomenom called *resonance*.

Mass/spring systems and energy flow

Engineers would think of a mass/spring system as a system which energy is conserved and may be changing forms from kinetic energy to potential energy. Any energy imparted to the system via an initial displacement or velocity, or continuously through a driving function cause the mass to oscillate up and down.



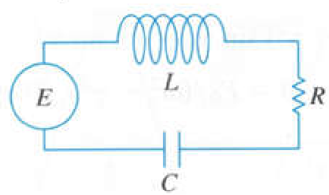
$$m\frac{d^2x}{dt^2}+\beta\frac{dx}{dt}+kx=f(t)$$

When it is 'down' it is stretching the spring, which is converting some of its kinetic energy of motion into potential energy stored in the spring. On the way back up, the spring is releasing its potential energy back into kinetic energy of the mass motion.

If there is damping, then some energy is being removed from the mass/spring - by converting some of this energy and releasing it as heat to the surroundings, thus removing energy from the system. Eventually, all the energy is released and the system is back at equilibrium.

Electrical Series Circuits have an analogy to mass/spring systems

Another system which behaves similarly are **series electrical circuits**:



In series circuits, the energy is the flow of electrical charge through the components. The inductor has the ability to store energy in the magnetic field surrounding the coil and the capacitor has the ability to store energy in the electric field between its plates.

In the mass/spring system, the mass oscillated up and down. In a circuit, the electrical charge energy flows back and forth between being stored in the inductor's magnetic field and the capacitor's electric field. The resistor serves the same function as damping: it removes energy from the system in the form of heat.

Therefore, you can model a **Series LRC Circuit** with the following differential equation based upon the mass/spring equation:

$$m\frac{d^2x}{dt^2}+\beta\frac{dx}{dt}+kx=f(t)$$

$$\boxed{L\frac{d^2q}{dt^2}+R\frac{dq}{dt}+\frac{1}{C}q=E(t)}$$

Solution techniques and phenomenon like resonance and beat phenomenon are the same for electrical circuits as for mass/spring systems.