Differential Equations – Lesson Notes – Chapter 4: Higher Order DEs

4.1: Higher-order linear DEs

In this section, we first identify some important properties and theorems we'll need as we learn various methods in the rest of the chapter for finding solutions to different forms of higher-order differential equations.

Initial Value Problems (IVP) vs Boundary Value Problems (BVP)

For first-order DEs, we know that an Initial Value Problem is one where in addition to the DE we also specify the value of y and y' at a single x...

For the DE: $y' + 2xy^2 = 0$

We could find that the family of solutions (general solution) is: $y = \dfrac{1}{x^2 + C}$

But if we are also given the initial condition: $y\left(\dfrac{3}{2}\right) = \dfrac{4}{5}$

We can solve for the constant: $C = -1$

...to find the particular solution: $y = \dfrac{1}{x^2 - 1}$

Also, there was a theorem that guaranteed that if certain conditions were met, then this solution would be unique over an interval.

For higher-order DEs, sometimes general solutions will involve only a single constant...

For the DE: $x^2 y'' - 2xy' + 2y = 6$

We could find that the family of solutions (general solution) is: $y = Cx^2 + x + 3$

But if we are also given the initial condition: $y(0) = 3 \quad and \quad y'(0) = 1$

...it can be shown that for this initial condition, all values of the constant *C* make the DE true, so there are many solutions (there is no unique particular solution).

This is still called an Initial Value Problem (IVP) because we are given conditions for all the derivatives in the problem but for only one value of x.

It turns out that there will only be a unique solution over an interval where the coefficients of all the derivative terms and the right side function are non-zero.

In this example: $x^2, \ -2x, \ 2, \ 6$ these coefficient do go to zero at x = 0,

so there is no unique solution over any interval which contains x = 0.

But for higher-order DEs, most often the general solution will contain multiple constants...

For the DE: $y'' + 16y = 0$

We could find that the family of solutions (general solution) is: $y = C_1 \cos(4x) + C_2 \sin(4x)$

To find a particular solution, often we are given the values of y and derivatives of y at multiple values of x...

$$y(0) = 0, \quad y\left(\frac{\pi}{8}\right) = 0$$

...and then the problem is called a **Boundary Value Problem (BVP)**, instead of an Initial Value Problem.

The reason this distinction is important is that Boundary Value Problems may have many, one, or no solutions.

(We won't be doing a lot with this in this course, but I wanted to mention it for completeness).

Homogeneous vs. Non-homogeneous DEs

One thing we will refer to frequently is whether or not a differential equation is homogeneous or non-homogeneous. This is very straightforward: A linear DE is homogeneous if the right-hand-side function is zero:

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

Non-homogeneous DE: $(x^2)y'' + (8x)y' - 5y = 2x^2$

corresponding homogeneous DE: $(x^2)y'' + (8x)y' - 5y = 0$

Superposition Principle

It can be shown that for a homogeneous linear differential equation, if you can find multiple solutions to the DE, then the linear combination of these solutions is also a solution:

If $y_1, y_2, \ldots, y_k$ are solutions of a linear homogeneous differential equation, then $y = c_1 y_1 + c_2 y_2 + \ldots + c_k y_k$ is also a solution.

Checking for Linear Dependence/Independence - the Wronskian

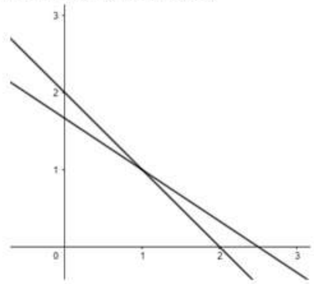
If we have a set of functions that are each solutions to a differential equation (and which could be linearly combined to also form a solution), it can be shown that the functions in this set will be **linearly independent** of one another.

Linearly independent means that no function can be formed by multiplying any other function by a constant.

As an analogy, this is similar to the idea of a system of equations.
Consider this system of two linear equations:

$$2x+3y=5$$
$$x+\ \ y=2$$

This system has a unique solution - there is an intersection, because the lines are not parallel.



Other ways to express this idea: you can't find a constant to multiply the 2nd equation by that would turn it into the first equation:

$$2(x+y)=2(4)$$
$$2x+2y=4 \quad \textit{is not the same function as } 2x+3y=5$$

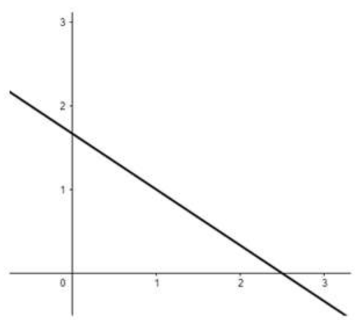
Also, if you solved this system using Cramer's rule, all the determinants are non-zero:

$$x=\frac{\begin{vmatrix}5 & 3\\ 2 & 1\end{vmatrix}}{\begin{vmatrix}2 & 3\\ 1 & 1\end{vmatrix}}=\frac{5-6}{2-3}=\frac{-1}{-1}=1 \qquad y=\frac{\begin{vmatrix}2 & 5\\ 1 & 2\end{vmatrix}}{\begin{vmatrix}2 & 3\\ 1 & 1\end{vmatrix}}=\frac{4-5}{2-3}=\frac{-1}{-1}=1$$

Now consider this system:

$$2x+3y=5$$
$$6x+9y=15$$

This system does not have a unique solution because the lines are coincident.



Other ways to express this idea: you can find a constant to multiply the 1st equation by that would turn it into the first equation:

$$3(2x+3y)=3(5)$$
$$6x+9y=15 \quad \textit{is the same function as } 2x+3y=5$$

Also, if you solved this system using Cramer's rule, all the determinants are non-zero:

$$x=\frac{\begin{vmatrix}5 & 3\\ 15 & 9\end{vmatrix}}{\begin{vmatrix}2 & 3\\ 6 & 9\end{vmatrix}}=\frac{45-45}{18-18}=\frac{0}{0} \qquad y=\frac{\begin{vmatrix}2 & 5\\ 6 & 15\end{vmatrix}}{\begin{vmatrix}2 & 3\\ 6 & 9\end{vmatrix}}=\frac{30-30}{18-18}=\frac{0}{0}$$

We don't have unique solutions when the determinant associated with the coefficients of the system have a zero determinant (and, interestingly, these coefficients happen to be partial derivatives of their respective terms).

Checking for Linear Dependence/Independence - the Wronskian

For a set of equations (or solutions to a differential equation) to be linearly independent, there has to exist no set of constants that would allow one function to be transformed into another other function in the set. When considering more than 2 functions, this is equivalent to stating the following:

> A set of functions $f_1(x),\, f_2(x)\,....f_n(x)$ is **linearly dependent** on an interval I
> if there exists constants $c_1,\, c_2.......c_n$ not all zero, such that:
>
> $$c_1 f_1(x) + c_2 f_2(x) + ...c_n f_n(x) = 0$$
>
> for every *x* in the interval. If the set of function is not linearly dependent upon the interval, then the set of functions is **linearly independent**.

The good news is that we don't have to adopt a 'trial-and-error' approach trying to find such constants, because, similar to the determinant's behavior with Cramer's Rule for linear systems, there is a determinant we can use to quickly determine if a set of functions is linearly independent. This determinant is called the **Wronskian**.

> Suppose each of the functions $f_1(x),\, f_2(x)\,....f_n(x)$ possesses at least *n* - 1 derivatives.
>
> The **Wronskian** of the functions is defined to be:
>
> $$W\left(f_1(x),\, f_2(x)\,....f_n(x)\right) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \cdots & \cdots & \cdots & \cdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}$$
>
> If the **Wronskian** of a set of functions is non-zero, the set of functions is **linearly independent.**

Let's see how the Wronskian works using the two simple linear systems we considered:

$$2x+3y=5$$
$$x+\ \ y=2$$
$$f_1: y=-\frac{2}{3}x+\frac{5}{3}$$
$$f_2: y=2-x$$
$$W=\begin{vmatrix} \left(-\frac{2}{3}x+\frac{5}{3}\right) & (2-x) \\ \left(-\frac{2}{3}\right) & (-1) \end{vmatrix}$$
$$=\left(-\frac{2}{3}x+\frac{5}{3}\right)(-1)-(2-x)\left(-\frac{2}{3}\right)$$
$$=\frac{2}{3}x-\frac{5}{3}+\frac{4}{3}-\frac{2}{3}x$$
$$=-\frac{1}{3}$$

$$2x+3y=5$$
$$6x+9y=15$$
$$f_1: y=-\frac{2}{3}x+\frac{5}{3}$$
$$f_2: y=-\frac{6}{9}x+\frac{15}{9}$$
$$W=\begin{vmatrix} \left(-\frac{2}{3}x+\frac{5}{3}\right) & \left(-\frac{6}{9}x+\frac{15}{9}\right) \\ \left(-\frac{2}{3}\right) & \left(-\frac{6}{9}\right) \end{vmatrix}$$
$$=\left(-\frac{2}{3}x+\frac{5}{3}\right)\left(-\frac{6}{9}\right)-\left(-\frac{6}{9}x+\frac{15}{9}\right)\left(-\frac{2}{3}\right)$$
$$=\frac{12}{27}x-\frac{30}{27}-\frac{12}{27x}+\frac{30}{27}$$
$$=0$$

When the Wronskian is non-zero, it means the set of functions is linearly independent.
On the right, the Wronskian is zero, meaning this set of functions is linearly dependent.

The Wronskian works to determine independence for any number and any form of functions:

Ex) Determine whether the set of functions is linearly independent on the interval $(-\infty, \infty)$

$$f_1(x)=0,\ f_2(x)=x,\ f_1(x)=e^x$$

Ex) Determine whether the set of functions is linearly independent on the interval $(0, \infty)$

$$f_1(x) = e^{\frac{1}{2}x}, \quad f_2(x) = xe^{\frac{1}{2}x}$$

Solution to linear homogenous differential equations

Any set $y_1, y_2, \ldots y_n$ of *n* linearly independent solutions of a linear homogeneous equation on an interval I is called a **fundamental set of solutions** for the differential equation and such a fundamental set of solutions does exist over the interval.

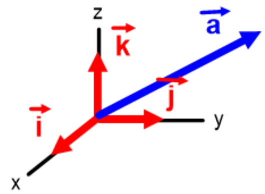
Let $y_1, y_2, \ldots y_n$ be a fundamental set of solutions of a homogeneous linear *nth*-order differential equation on an interval I. Then the **general solution** of the differential equation on the interval is:

$$y = c_1 y_1(x) + c_2 y_2(x) + \ldots + c_n y_n(x)$$

where $c_i, i = 1, 2, \ldots\ldots, n$ are arbitrary constants.

Our textbook doesn't include a proof of this, but does describe an interesting analogy...

The underlying reason why this works is similar to the way a vector in 3D space can always be expressed as a linear combination of a set of other vectors, as long as those vectors are independent (orthogonal):



$\vec{a} = c_1 \vec{i} + c_2 \vec{j} + c_3 \vec{k}$

So...solutions to homogeneous equations will involve adding together multiple functions, each multiplied by a constant. (Setting these constants differently will produce different particular solutions for particular initial values or boundary conditions).

Solution to linear non-homogenous differential equations

The non-homogeneous, linear differential equation...

$$a_n(x)\frac{d^n y}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+...+a_1(x)\frac{dy}{dx}+a_0(x)y=g(x)$$

has a corresponding homogeneous differential equation with right side = 0...

$$a_n(x)\frac{d^n y}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+...+a_1(x)\frac{dy}{dx}+a_0(x)y=0$$

If the general solution of the homogeneous DE is

$$y=c_1y_1(x)+c_2y_2(x)+...+c_ny_n(x)$$

where $c_i,\ i=1,2,....,n$ are arbitrary constants.

then we call the solution of the homogeneous DE the **complementary function**, y_c

Let y_p be any function which satisfies the non-homogeneous differential equation, then y_p is called a **particular solution**. Over the interval, the **general solution** of the non-homogenous differential equation is: $y=y_c+y_p$

This means that when we solve a non-homogeneous DE, first we will solve the associated homogeneous DE and then find any particular solution of the non-homogeneous DE. The general solution will be linear combination (by superposition) of these two solutions.

(We're not going to do non-homogenous yet in homework, but want to lay the groundwork here)

Ex) Verify that the given functions form a fundamental set of solutions of the differential equation in the indicated interval, then form the general solution.

$$4y''-4y'+y=0 \qquad f_1(x)=e^{\frac{1}{2}x},\ f_2(x)=xe^{\frac{1}{2}x} \qquad (-\infty,\infty)$$

First, verify the first function is a solution tc

$$y=e^{\frac{1}{2}x}$$

$$y'=\frac{1}{2}e^{\frac{1}{2}x}$$

$$y''=\frac{1}{4}e^{\frac{1}{2}x}$$

$$4y''-4y'+y\overset{?}{=}0$$

$$4\left(\frac{1}{4}e^{\frac{1}{2}x}\right)-4\left(\frac{1}{2}e^{\frac{1}{2}x}\right)+e^{\frac{1}{2}}$$

$$e^{\frac{1}{2}x}-2e^{\frac{1}{2}x}+e^{\frac{1}{2}x}$$

$$0=0$$

Next, verify the second function is a solution to the DE...

$$y=xe^{\frac{1}{2}x}$$

$$y'=x\left(\frac{1}{2}e^{\frac{1}{2}x}\right)+e^{\frac{1}{2}x}(1)=\frac{1}{2}xe^{\frac{1}{2}x}+e^{\frac{1}{2}x}$$

$$y''=\frac{1}{2}\left(x\left(\frac{1}{2}e^{\frac{1}{2}x}\right)+e^{\frac{1}{2}x}(1)\right)+\frac{1}{2}e^{\frac{1}{2}x}=\frac{1}{4}xe^{\frac{1}{2}x}+\frac{1}{2}e^{\frac{1}{2}x}+\frac{1}{2}e^{\frac{1}{2}x}$$

$$=\frac{1}{4}xe^{\frac{1}{2}x}+e^{\frac{1}{2}x}$$

$$4y''-4y'+y\overset{?}{=}0$$

$$4\left(\frac{1}{4}xe^{\frac{1}{2}x}+e^{\frac{1}{2}x}\right)-4\left(\frac{1}{2}xe^{\frac{1}{2}x}+e^{\frac{1}{2}x}\right)+xe^{\frac{1}{2}x}$$

$$xe^{\frac{1}{2}x}+4e^{\frac{1}{2}x}-2xe^{\frac{1}{2}x}-4e^{\frac{1}{2}x}+xe^{\frac{1}{2}x}$$

$$0=0$$

Now that we know these functions are both solutions to the DE, a second-order DE should have 2 functions for a fundamental solution set, so verify that these two functions are linearly independent and form such a set by using the Wronskian:

$$W = \begin{vmatrix} e^{\frac{1}{2}x} & xe^{\frac{1}{2}x} \\ \frac{1}{2}e^{\frac{1}{2}x} & \frac{1}{2}xe^{\frac{1}{2}x} + e^{\frac{1}{2}x} \end{vmatrix}$$

$$= \left(e^{\frac{1}{2}x}\right)\left(\frac{1}{2}xe^{\frac{1}{2}x} + e^{\frac{1}{2}x}\right) - \left(xe^{\frac{1}{2}x}\right)\left(\frac{1}{2}e^{\frac{1}{2}x}\right)$$

$$= \frac{1}{2}xe^{\frac{1}{2}x}e^{\frac{1}{2}x} + e^{\frac{1}{2}x}e^{\frac{1}{2}x} - \frac{1}{2}xe^{\frac{1}{2}x}e^{\frac{1}{2}x}$$

$$= \frac{1}{2}xe^{x} + e^{x} - \frac{1}{2}xe^{x}$$

$$= e^{x} \neq 0$$

$W = e^{x}$

The exponential function has an asymptote at y= 0 so it is never zero, and therefore these two solution functions form a fundamental solution set for the differential equation over the interval $(-\infty, \infty)$

Finally, we can use the superposition principle to write the solution for this differential equation:

$$y = C_1 e^{\frac{1}{2}x} + C_2 x e^{\frac{1}{2}x}$$

4.2: Reduction of Order

For a 2nd order linear DE, if we know one solutions, we use find a 2nd solution

If we have a linear 2nd-order DE and we happen to know one solution, then we can use this solution and a procedure called '**reduction of order**' to reduce the 2nd-order DE to a 1st-order DE, and then use our 1st-order techniques to solve this new DE to obtain the 2nd solution to the original DE.

Reduction of Order:

1) Given a 2nd-order DE and known solution y_1, postulate a currently unknown function, *u(x)* such that:

$$y_2(x) = u(x)y_1(x)$$

2) Take the first and second derivatives of this new y_2 function (which will require product rule multiple times).

3) Plug y_2 and its derivatives into the original 2nd-order DE. This will result in a new DE with all the x's changes to u's, and it will turn out that the non-derivative u terms will cancel out.

4) Define $w = u'$ and substitute, producing a new first-order DE. Use known techniques to solve this for w, which is u'.

5) Integrate u' to find u, then form the second solution: $y_2(x) = u(x)y_1(x)$

An example...

The indicated function is a solution of the given differential equation. Use reduction of order to find a second solution.

$$6y'' + y' - y = 0; \quad y_1 = e^{\frac{1}{3}x}$$

If the integration of the resulting function is too hard, you can use a formula instead

Because the steps are fairly procedural, our textbook develops a formula you can also use:

If you divide by the 2nd derivative coefficient to put the given DE in standard form:

$$y'' + P(x)y' + Q(x)y = 0$$

Then the 2nd solution can be found with this equation:

$$y_2(x) = y_1(x)\int \frac{e^{-\int P(x)dx}}{\left(y_1(x)\right)^2}dx$$

(I suggest only using this if you are having difficulty with the normal procedure)

4.3: Solving homogeneous linear DEs with constant coefficients

For a 2nd order linear DE, if we know one solutions, we use find a 2nd solution

For linear homogeneous equations of any order, if all coefficients are constants, we can use this section's method to solve. An example of such a DE would be:

$$2y''' - 3y'' + y' - 4y = 0$$

The method presented consists of representing the DE with a corresponding algebra equation, solution for the zeros of the algebra equation, and then using these result in some standard forms for solutions.

(The procedure is presented here without derivation. If you are interested in the background about why this works, there is a separate PDF with more information.)

Procedure for solving higher-order homogeneous linear DEs with constant coefficients

This procedures works for any order DE, but we will use a 2nd-order DE for illustration.

1) Given a DE, find its **auxiliary equation**:

DE: $ay'' + by' + cy = 0$

auxiliary equation: $am^2 + bm + c = 0$

2) Solve the auxiliary equation by factoring or quadratic formula or for higher orders by guessing a zero and using synthetic division to check and to factor.

3) This will result in some combination of *m* values which are zeros (roots) of the auxiliary equation, and will in occur in three possible ways:

a) Single (distinct) real roots, and for each you add to the solution a term of the form:

$$C_k e^{mx}$$

b) Repeated real roots (multiplicity > 1). For the first copy add a term of the form $C_k e^{mx}$ and for each 'copy' add another term by with a additional *x*.

For example, if your factored aux equation is $(m-2)^3 = 0$

...you would add the following to the solution: $y = C_1 e^{2x} + C_2 x e^{2x} + C_3 x^2 e^{2x}$

c) Complex roots (which always occur in complex-conjugate pairs).

If the *m* values are written in the form: $m = \alpha \pm i\beta$

for each pair, add to the solution:

$$C_1 e^{\alpha x} \cos \beta x + C_2 e^{\alpha x} \sin \beta x$$

Examples...

Find the general solution for $y'' - 36y = 0$

Find the general solution for $y'' - 10y' + 25y = 0$

Find the general solution for $2y'' - 3y' + 4y = 0$

Find the general solution for $y'' + 9y = 0$

Find the general solution for $y''' + 3y'' - 4y' - 12y = 0$

4.4: Undetermined Coefficients (solving non-homogeneous DEs)

Solving a DE with non-zero right side (nonhomogeneous)

In order to solve nonhomogeneous DEs, we will use the superposition principle that if we can first solve the corresponding homogeneous DE and then find *any* particular solution which includes the right hand side, then the complete general solution will be:

$$y = y_c + y_p$$

where y_c is the solution to the corresponding homogeneous DE
and y_p is any particular solution of the nonhomogenous DE

The **method of undetermined coefficients** is one method we can use to find this solution which works with the original DE is linear and has only constant coefficients, and for which the right-hand-side function is a linear combination of only polynomials, exponential, and sin or cos function terms.

So we could use this method for DEs like these...

$$2y''' + y'' - 2y' + 3y = 4x^2e^{2x} - 2\cos(3x)$$

$$2y''' + y'' - 2y' + 3y = 20x^2 + x - 4e^{2x} + 4\cos(3x)$$

...but not these...

$$2y''' + y'' - 2y' + 3y = \ln x$$

$$2y''' + y'' - 2y' + 3y = \frac{1}{x} + 2\tan x$$

Solving a DE with non-zero right side (nonhomogeneous)

The way we find the particular solution, y_p, is essentially to guess its form (with its own associated constants) and then take derivatives, plug into the original DE and establish the constants. If we are able to establish constants which work, then the y_p was valid. We use the following table of forms to make the initial guess for the form of y_p.

$g(x)$	Form of y_p
1. 1 (any constant)	A
2. $5x + 7$	$Ax + B$
3. $3x^2 - 2$	$Ax^2 + Bx + C$
4. $x^3 - x + 1$	$Ax^3 + Bx^2 + Cx + E$
5. $\sin 4x$	$A\cos 4x + B\sin 4x$
6. $\cos 4x$	$A\cos 4x + B\sin 4x$
7. e^{5x}	Ae^{5x}
8. $(9x - 2)e^{5x}$	$(Ax + B)e^{5x}$
9. x^2e^{5x}	$(Ax^2 + Bx + C)e^{5x}$
10. $e^{3x}\sin 4x$	$Ae^{3x}\cos 4x + Be^{3x}\sin 4x$
11. $5x^2\sin 4x$	$(Ax^2 + Bx + C)\cos 4x + (Ex^2 + Fx + G)\sin 4x$
12. $xe^{3x}\cos 4x$	$(Ax + B)e^{3x}\cos 4x + (Cx + E)e^{3x}\sin 4x$

Method of Undetermined Coefficients Procedure

1) Find the solution y_c of the corresponding homogeneous DE.

2) Select a form from the table for y_p which most closely matches the right-hand-side.

2a) If the form matches any term in the y_c solution, then multiply the selected y_p by x^n where n is the smallest integer which allows the term to no longer match a term in y_c

3) Take all derivatives of the y_p solution required to plug into the original DE.

4) Plug y_p and its derivatives into the original DE and solve for the A,B, etc. constants.

5) Solve the remaining equation as a system for the constants A, B, etc. and plug in to get y_p.

6) Form the general solution by combining: $y = y_c + y_p$

(this will still contain constants C_1, C_2, etc.)

Initial Value Problem?

7) Take derivatives of the complete general solution, and plug in the initial conditions to solve for C_1, C_2, etc. to find the solution for the initial conditions.

Examples...

$$y'' - y' + \frac{1}{4}y = 3x + 1$$

Examples...

$$y'' - y' + \frac{1}{4}y = 3 + e^{\frac{1}{2}x}$$

$$y'' - y' + \frac{1}{4}y = 3 + e^{\frac{1}{2}x} \qquad y(0) = 14, \quad y'(0) = 4$$

Examples...

$$y''' - 6y'' = 3 - \cos x$$

4.6: Variation of Parameters

Solving linear non-homogeneous DEs with non-constant coefficients

If we have a non-homogeneous DE like this...

$$x^2y'' + xy' + y = x^3$$

...we are unable to use the method of undetermined coefficients, because that procedure requires that all coefficients of the derivatives be constants.

In cases like these, we can try the method of **Variation of Parameters**.

(The procedure is described here, but if you want to see the derivation of it (it is interesting), I've published a separate PDF with the details.)

Method of Variation of Parameters

1) Find the solution of the corresponding homogeneous DE $y_C = C_1y_1 + C_2y_2$

2) Divide by the leading coefficient of the highest order derivative to put in standard form. This may give a new right-hand-side, which we'll call $f(x)$

3) Postulate a particular solution of the form: $y_P = u_1y_1 + u_2y_2$
where y_1 and y_2 are the same forms found in the homogeneous solution and u_1 and u_2 are new functions of x (not necessarily constants).

4) Use the following fractions of determinants to solve for the derivatives of the u functions:

$$u_1' = \frac{\begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}}, \quad u_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}}$$

5) Integrate the derivatives of the u functions to find the functions:

$$u_1 = \int u_1'\,dx, \qquad u_2 = \int u_2'\,dx$$

6) Form the particular solution... $y_P = u_1y_1 + u_2y_2$

and then the complete general solution...

$$y = y_C + y_P = C_1y_1 + C_2y_2 + u_1y_1 + u_2y_2$$

Method of Variation of Parameters

Pros...

- Can solve DEs with non-constant coefficients.
- You don't have to worry about the 'absorption' issue.
- Can solve everything we've already learned to solve, but...

Cons...

- Usually takes longer than earlier methods.
- The integrals at the end can be really difficult to compute.

Examples...

$$y'' + y = \tan x$$

Examples...

$$y'' - 9y = \frac{9x}{e^{3x}}$$

Examples...

$$y'' - 2y' + y = \frac{e^x}{1+x^2}$$

Examples...

$$y'' - y = \frac{1}{x}$$

4.7: Cauchy-Euler Equations

Variation of parameters doesn't always work

The Variation of Parameters method in the last section always works for linear differential equations with constant coefficients, but in general doesn't work if the coefficients are variable functions.

We will continue studying methods later in the course for handling more complicated cases, but there is one class of differential equations which appears frequently and has two methods we can use to solve now. These functions are called **Cauchy-Euler equations** and have the form:

matches

$$ax^2 \frac{d^2y}{dx^2} + bx\frac{dy}{dx} + cy = g(x)$$

The coefficients of each term is a function with a constant and an *x* raised to the power that matches that term's derivative order.

For DEs in this form, we have two different (related) procedures we can use to solve the complementary solution for the corresponding homogeneous DE. (Then we can use methods such as variation of parameters or undetermined coefficients to find the particular solution).

(The derivation of these procedure is available in a separate PDF)

Solving Cauchy-Euler equations - method 1

1) Find the Cauchy-Euler differential equation's **auxiliary equation**:

DE: $$ax^2y'' + bxy' + cy = 0$$

auxiliary equation: $$am^2 + (b-a)m + c = 0$$

2) Solve the auxiliary equation by factoring or quadratic formula.

3) This will result in some combination of *m* values which are zeros (roots) of the auxiliary equation, and will in occur in three possible ways:

a) Two distinct, real roots: $y = C_1x^{m_1} + C_2x^{m_2}$

(note: it is x^m, not e^{mx} as in the previous procedure).

b) One real root, repeated: $y = C_1x^m + C_2x^m \ln x$

c) Complex-conjugate roots: $y = C_1x^\alpha \cos(\beta \ln x) + C_2x^\alpha \sin(\beta \ln x)$

Solving Cauchy-Euler equations - method 2

The 2nd method involves substituting $x = e^t$ which produces a new DE for *y* with respect to *t* as the independent variable. The result of doing this for a Cauchy-Euler DE always produces a linear DE with constant coefficients, which can then be solved using earlier methods.

1) Perform the following substitutions into the Cauchy-Euler differential equation:

DE: $ax^2y'' + bxy' + cy = 0$

Substitute:

$$x = e^t$$

$$t = \ln x$$

$$y' = \frac{1}{x}\frac{dy}{dt}$$

$$y'' = \frac{1}{x^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)$$

2) The resulting equation will have constant coefficients.
Solve this equation using previous methods.

3) Re-substitute $t = \ln x$ to express the solution using *x* as the independent variable.

Examples...

$$x^2y'' - 2xy' - 4y = 0$$

Examples...

$$x^2 y'' - 3xy' + 3y = 2x^4 e^x$$

Examples...

$$x^2 y'' - xy' + y = \ln x$$

Higher-Order DE procedures for solving homogeneous DEs

2nd-order linear, have one solution and need another?

$$a(x)y'' + b(x)y' + c(x)y = 0$$

yes

Reduction of Order

$$y_2(x) = u(x)y_1(x)$$

(take derivatives and substitute w=u', solve resulting 1st-order DE, then integrate to get u).

no

homogeneous with constant coefficients?

$$ay'' + by' + cy = 0$$

yes

use auxiliary equation

$$am^2 + bm + c = 0$$

single root:	$y = C_1 e^{mx}$
repeated roots:	$y = C_1 e^{mx} + C_2 x e^{mx} + C_3 x^2 e^{mx} \ldots$
complex pairs:	$y = C_1 e^{\alpha x}\cos\beta x + C_2 e^{\alpha x}\sin\beta x$

no

2nd-order, Cauchy-Euler form?

yes

use auxiliary equation

$$am^2 + (b-a)m + c = 0$$

distinct roots:	$y = C_1 x^{m_1} + C_2 x^{m_2}$
repeated roots:	$y = C_1 x^m + C_2 x^m \ln x$
complex pairs:	$y = C_1 x^\alpha \cos(\beta \ln x) + C_2 x^\alpha \sin(\beta \ln x)$

no

We'll need methods from future chapters...

Check if solutions are Linearly Independent (form a fundamental solution set) using Wronskian:

$$W(f_1(x), f_2(x) \ldots f_n(x)) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \cdots & \cdots & \cdots & \cdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}$$

Linearly Independent if Wronskian is non-zero:

Higher-Order DE procedures for finding y_P from y_C

Linear non-homogeneous DE with constant coefficients?

$$ay'' + by' + cy = g(x)$$

(can be higher order too)

yes

Undetermined Coefficients

use table to find form...

$g(x)$	Form of y_p
1. 1 (any constant)	A
2. $5x + 7$	$Ax + B$
3. $3x^2 - 2$	$Ax^2 + Bx + C$
4. $x^3 - x + 1$	$Ax^3 + Bx^2 + Cx + E$
5. $\sin 4x$	$A\cos 4x + B\sin 4x$
6. $\cos 4x$	$A\cos 4x + B\sin 4x$
7. e^{5x}	Ae^{5x}
8. $(9x - 2)e^{5x}$	$(Ax + B)e^{5x}$
9. x^2e^{5x}	$(Ax^2 + Bx + C)e^{5x}$
10. $e^{3x}\sin 4x$	$Ae^{3x}\cos 4x + Be^{3x}\sin 4x$
11. $5x^2\sin 4x$	$(Ax^2 + Bx + C)\cos 4x + (Ex^2 + Fx + G)\sin 4x$
12. $xe^{3x}\cos 4x$	$(Ax + B)e^{3x}\cos 4x + (Cx + E)e^{3x}\sin 4x$

...take derivatives, plug into DE and solve for constants, then:

$$y = y_C + y_P$$

no

Variation of Parameters

assume terms have same form as homogeneous solution multiplied by *u* functions:

$$y_C = C_1 y_1 + C_2 y_2$$

$$y_P = u_1(x)y_1 + u_2(x)y_2$$

Divide by leading term to get $f(x) = \dfrac{g(x)}{a_n(x)}$

then find *u* function derivatives with Cramer's rule:

$$u_1' = \frac{\begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}}, \quad u_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}}$$

then integrate to find *u* functions, and solution:

$$u_1 = \int u_1'\,dx, \qquad u_2 = \int u_2'\,dx$$

$$y_P = u_1 y_1 + u_2 y_2$$

$$y = y_C + y_P$$