Differential Equations – Lesson Notes – Chapter 3: Modeling with 1st order DEs

3.1: First-order linear models

Modeling = finding a differential equation to model a scenario

The term 'modeling' is used to mean finding a differential equation which applies to a given scenario, applying given conditions to solve for any constants and then using the differential equation and its solution to answer questions about the scenario.

Differential Equations are used in modeling many, many different scenarios. In this course, we'll consider some of the more common and interesting applications.

Growth/Decay proportion to amount

There are many situations where the rate of change of a quantity (with respect to time) is proportional to the amount of the quantity.

In problems like these, the appropriate differential equation to model this would be:

$$\frac{dy}{dt} = ky$$

...where *k* is a constant of proportionality.

Some specific examples...

Unrestricted Population Growth: $\frac{dP}{dt} = kP$

Continuously Compounded Interest: $\frac{dA}{dt} = kA$

Radioactivity Decay: $\frac{dQ}{dt} = kQ$

Let's take one of these, and solve for the form of the solution to the differential equation:

Continuously Compounded Interest: $\frac{dA}{dt} = kA$

This particular form can be solved using Separation of Variables...

$$\frac{dA}{dt} = kA$$

$$\frac{1}{A}dA = kdt$$

$$\int \frac{1}{A}dA = \int kdt$$

$$\ln|A| = kt + C_1$$

...and solving for A...

$$\ln|A| = kt + C_1$$

$$A = e^{kt+C_1}$$

$$A = e^{kt}e^{C_1}$$

$$A = Ce^{kt}$$

We usually have an initial condition: $A(0) = A_0$

...and can use it to solve for *C*...

$$A_0 = Ce^{k(0)},\ C = A_0$$

...which gives us the particular solution for the scenario: $\boxed{A = A_0e^{kt}}$

(for continuously compounded interest, the *k* has a specific meaning: the annual interest rate)

Growth/Decay proportion to amount

We could then be asked specific follow-on questions. Let's look at a specific example, but to show how a whole host of scenarios are modeled this way, let's do a bacterial growth problem:

Ex) A culture initially has 200 bacteria. At t = 1 hr, the population of bacteria has increased to 300 bacteria. If the rate of growth is proportional to the number of bacteria present, determine the time needed for the bacteria population to quadruple.

State an appropriate form DE: $\frac{dP}{dt} = kP$

Solve the DE: by separation of variables, solution is: $P = P_0 e^{kt}$

Use conditions to establish constants: $P(0) = 200$ $\quad P = 200e^{kt}$

$P(1) = 300$ $\quad 300 = 200e^{k(1)}$

$$\frac{300}{200} = e^{k(1)}$$

$$e^k = \frac{3}{2}$$

$$k = \ln\left(\frac{3}{2}\right) \approx .405465$$

$$\boxed{P = 200e^{.405465t}}$$

Now answer the question (time to quadruple): $4(200) = 200e^{.405465t}$

$$4 = e^{.405465t}$$

$$.405465t = \ln(4)$$

$$t = \frac{\ln(4)}{.405465} = \boxed{3.419 \; hours}$$

Radioactive Carbon Dating

An interesting application of this is using radioactive carbon dating to establish the age of previously living materials. In 1950, a chemist named Willard Libby found a way to use the ratio of the amount of radioactive carbon-14 to ordinary carbon in a living substance to establish the time when it died.

The action of cosmic radiation on nitrogen in the atmosphere turns some of the regular carbon into radioactive carbon-14, and the ratio of carbon-14 to regular carbon is constant and is absorbed by all living things, so the same ratio appears in the living tissues. But when an organism dies, the absorption of the carbon-14 by either breathing or eating stops. The regular carbon remains in the tissues, but the radioactive carbon-14 decays over time according to a radioactive decay model, and it is known that the 'half-life' of carbon-14 is 5,600 years (the time when only half of the initial carbon-14 remains).

Here is a specific example:

Ex) A fossilized bone is found to contain one-thousandth of the C-14 level found in living matter. Estimate the age of the fossil.

State an appropriate form DE: $\frac{dQ}{dt} = kQ$

Solve the DE: by separation of variables, solution is: $Q = Q_0 e^{kt}$

Use conditions to establish constants:

For carbon-14, half-life=5600 yrs, so $Q(5600) = \frac{1}{2}Q_0$

$$\frac{1}{2}Q_0 = Q_0 e^{k(5600)}$$

$$e^{k(5600)} = \frac{1}{2}$$

$$5600k = \ln\left(\frac{1}{2}\right)$$

(don't round until the very end if possible) $k = \frac{\ln\left(\frac{1}{2}\right)}{5600} = -1.23776283 \bullet 10^{-4}$

$$\boxed{Q = Q_0 e^{\left(-1.23776283 \bullet 10^{-4}\right)t}}$$

Now answer the question: age when quantity is one-thousandth $Q\left(t_{age}\right) = \frac{1}{1000}Q_0$

$$\frac{1}{1000}Q_0 = Q_0 e^{\left(-1.23776283 \bullet 10^{-4}\right)t}$$

$$e^{\left(-1.23776283 \bullet 10^{-4}\right)t} = \frac{1}{1000}$$

$$\left(-1.23776283 \bullet 10^{-4}\right)t = \ln\left(\frac{1}{1000}\right)$$

$$t = \frac{\ln\left(\frac{1}{1000}\right)}{-1.23776283 \bullet 10^{-4}} = \boxed{55808\ years}$$

Newton's Law of Cooling/Warming

A slightly different form DE is found in scenarios where objects starting at one temperature are immersed in a medium at a different temperature. Assuming the medium is large (the object isn't big enough to heat or cool the medium) than the rate of change (over time) of the temperature of the object is proportional to the difference between its temperature and the medium:

$$\frac{dT}{dt}=k\left(T-T_m\right) \quad \textit{where } T_m \textit{ is the temperature of the medium}$$

Solving these DEs is usually done by putting the DE into standard form for a linear first-order DE, then using an integrating factor:

$$\frac{dT}{dt}=k\left(T-T_m\right)$$

$$\frac{dT}{dt}-kT=-kT_m$$

$$I.F.=e^{\int P(x)dx}=e^{\int -kdt}=e^{-kt}$$

$$\frac{d}{dt}\left[e^{-kt}T\right]=-kT_me^{-kt}$$

$$e^{-kt}T=\int -kT_me^{-kt}dt=\frac{-kT_m}{-k}e^{-kt}+C$$

$$\boxed{T=T_m+Ce^{kt}}$$

Let's look at a more complicated example, where this model is used twice:

Ex) At $t = 0$ a sealed test tube containing a chemical is immersed in a liquid bath. The initial temperature of the chemical in the test tube is 20 °C. The bath has a constant temperature of 90 °C. After 2 minutes in the bath, the test tube temperature has risen to 30 °C. After remaining in the bath for 10 minutes, the test tube is taken out of the bath and kept in a room whose ambient temperature is constant at 40 °C. After 5 minutes in the room, the test tube temperature has decreased to 50 °C.How long, measured from the start of the entire process, will it take for the test tube to reach 45 °C?

There are two parts to this process, the liquid bath and the room. For each, the DE form and solution form are: $\frac{dT}{dt}=k\left(T-T_m\right) \qquad T=T_m+Ce^{kt}$

Liquid bath: $T = 90 + Ce^{kt}$ $T(0) = 20$ (the initial temperature of the test tube)

solving for *C*... $20 = 90 + Ce^{k(0)}$ so... $T = 90 - 70e^{kt}$

$C = -70$

after 2 minutes, T = 30: $30 = 90 - 70e^{k(2)}$ so... $T = 90 - 70e^{(-.077075)t}$

$$70e^{k(2)} = 60$$

$$e^{k(2)} = \frac{60}{70}$$

$$k = \frac{\ln\left(\frac{6}{7}\right)}{2} \approx -.077075$$

after 10 minutes: $T = 90 - 70e^{(-.077075)(10)} = 57.6\,°C$

Room air: $T = 40 + Ce^{kt}$ (to keep things simple, redefine *t* = *0* at start of this part)

Now the ending temperature from the bath becomes the initial temperature for the air:

$$T(0) = 57.6$$

solving for *C*... $57.6 = 40 + Ce^{k(0)}$ so... $T = 40 + 17.6e^{kt}$

$C = 17.6$

after 5 minutes, T = 50: $50 = 40 + 17.6e^{k(5)}$ so... $T = 40 + 17.6e^{(-.11306)t}$

$$17.6e^{k(5)} = 10$$

$$e^{k(5)} = \frac{10}{17.6}$$

$$k = \frac{\ln\left(\frac{10}{17.6}\right)}{5} \approx -.11306$$

How long until temp = 45? $45 = 40 + 17.6e^{(-.11306)t}$

$$17.6e^{(-.11306)t} = 5$$

$$e^{(-.11306)t} = \frac{5}{17.6}$$

$$t = \frac{\ln\left(\frac{5}{17.6}\right)}{-.11306} = 11.1\,minutes$$ in the air

plus the 10 minutes in the bath = $\boxed{\text{21.1 minutes}}$

Series Electrical Circuits

In electrical engineering, circuits are arrangements of circuit elements connected in series with the circuit starting at the positive terminal of a voltage source and ending at the negative terminal. The response to applying a voltage to a circuit is that 'current' flows through all the elements in the circuit.

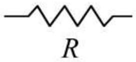
There are 3 general types of electrical elements: resistors, inductors, and capacitors (old name for capacitor 'condenser'):

Resistors: Are usually made of a partially-conductive medium like carbon with other materials mixed in to control conductivity. Resistors simply resist current generation from applied voltage according to Ohm's Law:

$$E = IR$$

$$\text{where } E = \text{voltage}(\text{volts}),\ I = \text{current}(\text{amperes}), R = \text{resistance}(\text{Ohms})$$

Symbol:



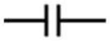
Inductors: Are usually made of a coil of wire designed to store energy in a magnetic field. Inductors have the property that they resist rapid changes in current, so current changes lag changes in voltage (there is a delay in response).

Symbol:

$L = inductance(Henrys)$

Capacitors: Are usually made of plates of metal with a non-conductive 'dielectric' material between the plates (often the plates are flexible and curled up into a form which can be inserted into a 'can'. Capacitors are designed to store energy in an electric field and have the property that they resist rapid changes in voltage across the capacitor plates, so voltage changes lag changes in current.

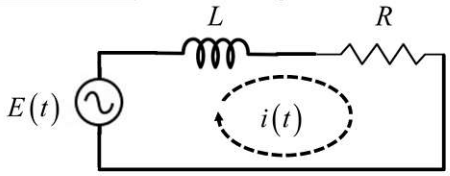
Symbol:



$C = capacitance(Farads)$

Here are the differential equations which model the two simplest forms of electrical circuits:

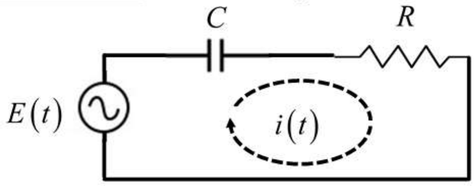
An **LR series circuit** contains a single inductor and single resistor:



DE for an LR series circuit: $L\dfrac{di}{dt} + Ri = E(t)$

(This equation comes from the fact that the voltage drop across the inductor is proportional to how fast the current is changing, and the sum of all the voltage drops must add up to the voltage source value).

An **RC series circuit** contains a single inductor and single resistor:



An equation for an LR series circuit: $Ri + \dfrac{1}{C}q = E(t)$

..where *q* is the amount of electrical charge...as the amount of charge builds up across the plates of the capacitor, the voltage across it increases and is therefore proportional to the charge.

But because current is the amount of electric charge that flows past a point in a given time, it is the derivative of charge: $i = \dfrac{dq}{dt}$

Therefore, the DE for an LR series circuit: $R\dfrac{dq}{dt} + \dfrac{1}{C}q = E(t)$

Using a Phase Portrait to establish long-term behavior, Transient Terms

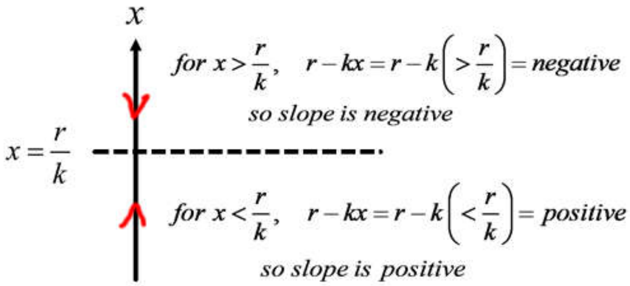
Here is one more first-order, linear DE example: A mathematical model for the rate at which a drug disseminates into the bloodstream is given by:

$$\frac{dx}{dt} = r - kx \qquad \text{where } r \text{ and } k \text{ are positive constants.}$$

The function *x(t)* describes the concentration of the drug in the bloodstream at time *t*.

(a) Since the DE is autonomous, use the phase portrait concept to find the limiting value of $x(t)$ as $t \to \infty$

A phase portrait is a picture of the behavior of y and the dividing values are those which make the right-hand-side = zero: $r - kx = 0, \quad x = \frac{r}{k}$

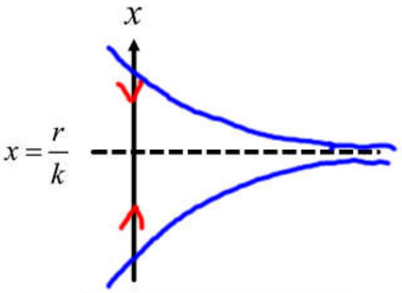


$for\ x > \frac{r}{k}, \quad r - kx = r - k\left(> \frac{r}{k}\right) = negative$

so slope is negative

$for\ x < \frac{r}{k}, \quad r - kx = r - k\left(< \frac{r}{k}\right) = positive$

so slope is positive



$$\boxed{\text{as } t \to \infty \quad x(t) \to \frac{r}{k}}$$

Using a Phase Portrait to establish long-term behavior, Transient Terms

(b) Solve the DE subject to the condition that the drug concentration is 0 at *t* = 0. Use a calculator to verify your sketch from part (a).

$$\frac{dx}{dt} = r - kx, \quad \frac{dx}{dt} + kx = r \qquad e^{kt}x = \int re^{kt}dt = \frac{r}{k}e^{kt} + C$$

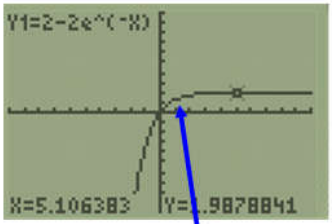
$$IF = e^{\int kdt} = e^{kt} \qquad x = \frac{r}{k} + Ce^{-kt}$$

applying the condition... $0 = \frac{r}{k} + Ce^{-k(0)}, \quad C = -\frac{r}{k}, \quad \boxed{x(t) = \frac{r}{k} - \frac{r}{k}e^{-kt}}$

'steady-state term'

'transient term'

calculator (I used r=2, k=1):



$\frac{r}{k} = 2$ **'steady-state' value or 'terminal' value**

'transient behavior' (changing)

The system starts in a changing state, but over time the effects of any terms with e to a negative exponent fade out, leaving a steady-state value.

Other first-order linear DE models

There are many other scenarios that can be modeled with first-order, linear differential equations: how materials flow between containers where the materials are mixing, velocity and air resistance for falling masses in physics, and many others. These scenarios have so many variations that there isn't really a standard form for the differential equations, so you need to build the differential equation to match the specific scenario and then use one of the methods you know to solve the differential equation to answer the question. But the pattern is the same:

1) Build a DE to match the scenario.

2) Solve the DE using a known method which results in a general solution with one or more constants.

3) Use given conditions/information in the problem to solve for the constants to get the particular solution.

4) Use the solution to answer the follow-on question(s) in the problem.

3.2: First-order non-linear models

First-order non-linear DE models

If we open our differential equations model up to having non-linear coefficient terms or other variations that mean the differential equation no longer qualifies as a 'linear' DE, then we can model a wider variety of scenarios. The downside is that most of the methods we know so far for solving DEs require linear DEs. The main method we know that works with non-linear DEs is Separation of Variables. Later in the course, we will learn additional methods that may also work with non-linear DEs, and we can always resort to numerical methods/computers to find approximate solutions as well.

Unfortunately, when the DE are non-linear, often the solutions are much more involved, because the integrals needed are complicated enough to require using an integral table or methods such as partial fraction expansion.

But non-linear DEs can model a variety of interesting scenarios. We'll consider just two complicated examples here.

The Logistic Growth Model (growth restrained by environment)

Unrestrained growth can be modeled using a linear DE such as: $\frac{dP}{dt} = kP$

But in a more realistic model, things like natural resource availability restrain growth. If you assume that there is a maximum number of individuals that the environment can support, this is called the 'carrying capacity', *L*.

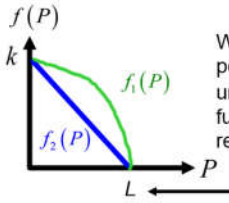
If you start with the assumption of unrestrained growth, you can rearrange the DE to obtain:

$$\frac{\left(\frac{dP}{dt}\right)}{P} = k$$

...which is saying that the ratio of how much the population changes to the population (the population change as a percentage of the population) is constant.

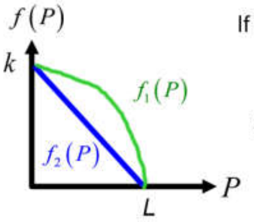
More realistic models allow this ratio to be a function of *P* instead of a constant:

$$\frac{\left(\frac{dP}{dt}\right)}{P} = f(P)$$



the 'carrying capacity' of the environment

When the population is near zero, the environment can support unrestricted population growth, so $f(P) = k$ the system starts growing with the constant for unrestrained growth. But as the population increases, $f(P)$ decreases according to some functions (two are shown here) until the growth becomes zero (shuts off) when the population reaches the carrying capacity, maximum population, supported by the environment.



If we use the simplest function, the line, it's equation would be:

$$f(P) = k - \frac{k}{L}P$$

So the DE would be:

$$\frac{\left(\frac{dP}{dt}\right)}{P} = f(P)$$

$$\frac{\left(\frac{dP}{dt}\right)}{P} = k - \frac{k}{L}P$$

$$\frac{dP}{dt} = P\left(k - \frac{k}{L}P\right) \quad \textit{factoring out k...}$$

$$\boxed{\frac{dP}{dt} = kP\left(1 - \frac{P}{L}\right)}$$

This is the **Logistic Model** for growth restrained by environment presented in AP Calculus BC.

Some textbooks leave the constant unfactored and more general like this...

$$\boxed{\frac{dP}{dt} = P(a - bP)}$$

The Logistic Growth Model (growth restrained by environment)

Let's solve using Separation of Variables and Partial Fraction Expansion to find the standard form of the solution for the Logistic Growth model...

$$\boxed{\frac{dP}{dt}=kP\left(1-\frac{P}{L}\right)}$$

$$\frac{1}{P\left(1-\frac{P}{L}\right)}dP=kdt$$

$$\int\frac{1}{P\left(1-\frac{P}{L}\right)}dP=\int kdt$$

Partial Fraction Expansion:

$$\frac{1}{P\left(1-\frac{P}{L}\right)}=\frac{A}{P}+\frac{B}{\left(1-\frac{P}{L}\right)}$$

$$\frac{1}{P\left(1-\frac{P}{L}\right)}=\frac{A\left(1-\frac{P}{L}\right)}{P\left(1-\frac{P}{L}\right)}+\frac{BP}{P\left(1-\frac{P}{L}\right)}$$

$$A-\frac{A}{L}P+BP=1$$

$$\left(B-\frac{A}{L}\right)P+(A)=(0)P+(1)$$

System: $\begin{cases}B-\dfrac{A}{L}=0\\ A=1\end{cases}$ $\quad A=1,\quad B-\dfrac{(1)}{L}=0,\quad B=\dfrac{1}{L}$

$$\int\frac{1}{P\left(1-\frac{P}{L}\right)}dP=\int kdt$$

$$\int\left[\frac{1}{P}+\frac{\left(\frac{1}{L}\right)}{1-\frac{P}{L}}\right]dP=\int kdt$$

$$\int\frac{1}{P}dP+\int\frac{1}{L-P}dP=\int kdt$$

$$u=L-P,\ du=-dP$$

$$\int\frac{1}{P}dP-\int\frac{1}{u}du=\int kdt$$

$$\ln|P|-\ln|L-P|=kt+C$$

$$\ln|P|-\ln|L-P|=kt+C_1$$

$$\ln\left|\frac{P}{L-P}\right|=kt+C_1$$

$$e^{\ln\left|\frac{P}{L-P}\right|}=e^{kt+C_1}=e^{C_1}e^{kt}=C_2e^{kt}$$

$$\frac{P}{L-P}=C_2e^{kt}$$

$$P=C_2e^{kt}(L-P)$$

Differential Equation

$$\boxed{\frac{dP}{dt}=kP\left(1-\frac{P}{L}\right)}$$

$$P=C_2e^{kt}(L-P)$$

$$P=LC_2e^{kt}-C_2e^{kt}P$$

$$\left(1+C_2e^{kt}\right)P=LC_2e^{kt}$$

$$P=\frac{LC_2e^{kt}}{1+C_2e^{kt}}=\frac{\left(\frac{LC_2e^{kt}}{C_2e^{kt}}\right)}{\left(\frac{1+C_2e^{kt}}{C_2e^{kt}}\right)}=\frac{L}{\frac{1}{C_2e^{kt}}+1}$$

$$\boxed{P=\frac{L}{1+Ce^{-kt}}}$$

Solution

Differential Equation

Given this form... $\boxed{\dfrac{dP}{dt}=P(a-bP)}$

Rewrite to other form: $\dfrac{dP}{dt}=aP\left(1-\dfrac{b}{a}P\right)$

$$\frac{dP}{dt}=aP\left(1-\frac{P}{\left(\frac{a}{b}\right)}\right)$$

$$k=a,\quad L=\frac{a}{b}$$

$$\frac{dP}{dt}=kP\left(1-\frac{P}{L}\right)$$

A logistic growth model example

We'll solve a specific example so we can see how we can use this standard model.

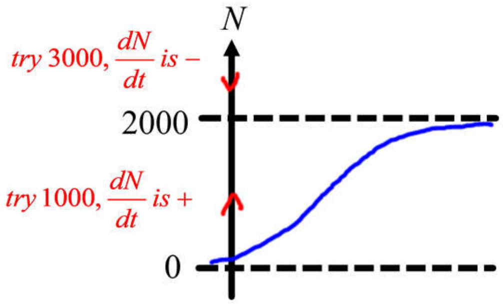
Ex) The number of supermarkets N throughout the country that are using a computerized checkout system is described by a logistic model:

$$\frac{dN}{dt}=N\left(1-0.0005N\right) \qquad with\ \ N\left(0\right)=1$$

(a) Use a phase portrait to predict how many supermarkets are expected to adopt the new procedure in the long term. Sketch your predicted solution curve.

Phase portrait has divisions at the RHS zeros:

$$N\left(1-0.0005N\right)=0 \quad at\ N=0\ and\ N=2000$$



(b) Solve the initial-value problem.

Instead of resolving from scratch, let's use the standard solution:

If we rewrite to standard form: $\frac{dP}{dt}=kP\left(1-\frac{P}{L}\right)$ ...we can use the standard solution: $P=\frac{L}{1+Ce^{-kt}}$

$$\frac{dN}{dt}=N\left(1-0.0005N\right)$$

$$\frac{dN}{dt}=N\left(1-\frac{N}{2000}\right)=kP\left(1-\frac{P}{L}\right)$$

$$if\ \ k=1,\ L=2000$$

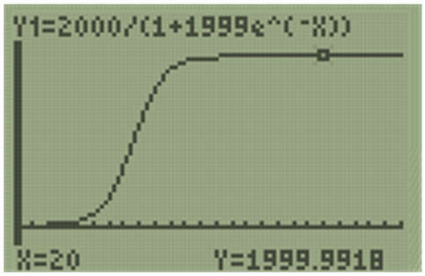
The general solution: $P=\frac{2000}{1+Ce^{-t}}$

Now, P = 1 when t = 0: $1=\frac{2000}{1+Ce^{-(0)}}=1=\frac{2000}{1+C}$

$$C+1=2000, \quad C=1999$$

The particular solution: $so\ \boxed{N\left(t\right)=\frac{2000}{1+1999e^{-t}}}$

Let's graph our solution... $N\left(t\right)=\frac{2000}{1+1999e^{-t}}$



Y1=2000/(1+1999e^(-X))

X=20 Y=1999.9918

...which matches our phase portrait.

Falling Masses

Example: A falling mass subject to air resistance. In a vacuum, a mass in a gravitational field would fall subject to Newton's Law: $F = ma$

The mass would be specified in either kg or slugs, and for Earth, $g = 32\ ft/s^2\ \ or\ 9.8\ m/s^2$

We also know that if we specify position as a function of time: $s(t)$

then velocity is the derivative of position, and acceleration is the derivative of velocity:

$$v(t) = s'(t), \quad a(t) = v'(t) = s''(t)$$

So Newton's Law with gravity as the force can be written as a differential equation:

$$F = ma$$

$$ma = F_g$$

$$ma = mg$$

$$\boxed{m\frac{dv}{dt} = mg}$$

Actually, this is a special case. The more general equation is force = change in momentum:

$$F = \frac{d}{dt}[mv] \quad \textit{so by the Chain Rule...}$$

$$F = m\frac{d}{dt}[v] + v\frac{d}{dt}[m]$$

$$F = m\frac{dv}{dt} + v\frac{dm}{dt}$$

$$F = ma + v\frac{dm}{dt} \qquad \textit{so } F = ma \textit{ only if mass is constant}$$

But we have an atmosphere on Earth, so there is another force, air resistance, which opposes motion. In general, the faster an object is moving, the more air resistance is experienced, but we could use different models for this air resistance:

$$ma = F_g - F_{air}, \qquad m\frac{dv}{dt} = mg - F_{air}$$

We could assume that air resistance is proportional to instantaneous velocity:

$$m\frac{dv}{dt} = mg - kv$$

...which results in a first-order, linear differential equation to solve.

But we could also assume (more realistically) that air resistance is proportional to the square of instantaneous velocity:

$$m\frac{dv}{dt} = mg - kv^2$$

...which results in a first-order, non-linear differential equation to solve.

A non-linear Falling Mass example

A 16-pound cannonball is shot vertically upward from the ground with an initial velocity of 300 ft/s. Assume air resistance is proportional to the square of instantaneous velocity with *k*=0.0003. (a) Solve the DE. (b) How high would the cannonball be 4 seconds after launch?

(a) Let's define up as the positive direction, then the DE would be: $m\frac{dv}{dt} = -mg - kv^2$

(because gravity and air resistance are now both in the negative direction)

Rearranging, we can separate the variables:

$$m\frac{dv}{dt} = -mg - kv^2$$

$$-mdv = \left(mg + kv^2\right)dt$$

$$\frac{1}{mg + kv^2}dv = -\frac{1}{m}dt$$

A non-linear Falling Mass example

A 16-pound cannonball is shot vertically upward from the ground with an initial velocity of 300 ft/s. Assume air resistance is proportional to the square of instantaneous velocity with *k*=0.0003. (a) Solve the DE. (b) How high would the cannonball be 4 seconds after launch?

Solve by integrating both sides (manipulating the left side to match the arctan integral shortcut form):

$$\int \frac{1}{mg+kv^2}\,dv = -\int \frac{1}{m}\,dt$$

$$\int \frac{1}{\left(\sqrt{mg}\right)^2+\left(\sqrt{k}v\right)^2}\,dv = -\int \frac{1}{m}\,dt$$

$$a=\sqrt{mg} \quad u=\sqrt{k}v$$

$$du=\sqrt{k}dv$$

$$dv=\frac{1}{\sqrt{k}}du$$

$$\frac{1}{\sqrt{k}}\int \frac{1}{(a)^2+(u)^2}\,du = -\int \frac{1}{m}\,dt$$

$$\frac{1}{\sqrt{k}}\left[\frac{1}{a}\arctan\left(\frac{u}{a}\right)\right]+C_1 = -\frac{1}{m}t+C_2$$

$$\frac{1}{\sqrt{k}}\left[\frac{1}{\sqrt{mg}}\arctan\left(\frac{\sqrt{k}v}{\sqrt{mg}}\right)\right] = -\frac{1}{m}t+C_3$$

$$\frac{1}{\sqrt{kmg}}\arctan\left(\sqrt{\frac{k}{mg}}v\right) = -\frac{1}{m}t+C_3$$

$$\arctan\left(\sqrt{\frac{k}{mg}}v\right) = -\frac{\sqrt{kmg}}{m}t+\sqrt{kmg}C_3$$

$$\arctan\left(\sqrt{\frac{k}{mg}}v\right) = -\sqrt{\frac{kg}{m}}t+C$$

$$\arctan\left(\sqrt{\frac{k}{mg}}v\right) = -\sqrt{\frac{kg}{m}}t+C$$

$$\sqrt{\frac{k}{mg}}v = \tan\left(-\sqrt{\frac{kg}{m}}t+C\right)$$

$$\boxed{v(t)=\sqrt{\frac{mg}{k}}\tan\left(-\sqrt{\frac{kg}{m}}t+C\right)}$$

Now we have a function for velocity at any time, *t*.

To answer the 'how high' question, we need position (height) as a function of time, so we need to integrate the velocity function:

Define two constants to clean up the work... $A=\sqrt{\frac{mg}{k}}, \quad B=\sqrt{\frac{kg}{m}}$

$$v(t)=\sqrt{\frac{mg}{k}}\tan\left(-\sqrt{\frac{kg}{m}}t+C\right)$$

$$v(t)=A\tan(-Bt+C)$$

$$s(t)=\int A\tan(-Bt+C)\,dt$$

$$u=-Bt+C$$

$$du=-Bdt,\ dt=\frac{-1}{B}du$$

$$s(t)=\frac{-A}{B}\int \tan(u)\,du$$

$$s(t)=\frac{-A}{B}\int \tan(u)\,du$$

$$s(t)=\frac{-A}{B}\int \frac{\sin(u)}{\cos(u)}\,du$$

$$define\ w=u=-Bt+C$$

$$s(t)=\frac{-A}{B}\int \frac{\sin(w)}{\cos(w)}\,dw$$

$$new\ u=\cos(w)$$

$$du=-\sin(w)dw$$

$$s(t)=\frac{A}{B}\int \frac{1}{u}\,du$$

$$s(t)=\frac{A}{B}\int \frac{1}{u}\,du$$

$$s(t)=\frac{A}{B}\ln|u|+K$$

$$s(t)=\frac{A}{B}\ln|\cos(w)|+K$$

$$s(t)=\frac{A}{B}\ln|\cos(-Bt+C)|+K$$

$$s(t)=\frac{\sqrt{\frac{mg}{k}}}{\sqrt{\frac{kg}{m}}}\ln\left|\cos\left(-\sqrt{\frac{kg}{m}}t+C\right)\right|+K$$

Function for position (height) at any time, *t*.

$$\boxed{s(t)=\sqrt{\frac{m^2}{k^2}}\ln\left|\cos\left(-\sqrt{\frac{kg}{m}}t+C\right)\right|+K}$$

Now to answer (b), we first must establish all constants...

$$v(t)=\sqrt{\frac{mg}{k}}\tan\left(-\sqrt{\frac{kg}{m}}t+C\right)$$

$$mg=16,\ so\ m=\frac{16}{32}=0.5\ slugs\ and\ k=0.0003$$

$$v(t)=\sqrt{\frac{(0.5)32}{0.0003}}\tan\left(-\sqrt{\frac{(0.0003)32}{0.5}}t+C\right)$$

$$v(t)=230.9401\tan(-0.138564t+C)$$

$$initial\ velocity=300\ ft/s$$

$$300=230.9401\tan(-0.138564(0)+C)$$

$$\tan(C)=\frac{300}{230.9401}$$

$$C=\arctan\left(\frac{300}{230.9401}\right)=0.91474$$

$$s(t)=\sqrt{\frac{m^2}{k^2}}\ln\left|\cos\left(-\sqrt{\frac{kg}{m}}t+C\right)\right|+K$$

$$s(t)=\sqrt{\frac{0.5^2}{0.0003^2}}\ln\left|\cos\left(-\sqrt{\frac{(0.0003)32}{0.5}}t+0.91474\right)\right|+K$$

$$s(t)=1666.6667\ln|\cos(-0.138564t+0.91474)|+K$$

$$initial\ position=0\ ft$$

$$0=1666.6667\ln|\cos(-0.138564(0)+0.91474)|+K$$

$$K=-1666.6667\ln|\cos(0.91474)|=823.837$$

$$s(t)=1666.6667\ln|\cos(-0.138564t+0.91474)|+823.837$$

Then use the final position function:

$$s(4)=1666.6667\ln|\cos(-0.138564(4)+0.91474)|+823.837$$

$$\boxed{s(4)=713.12\ ft}$$