1.1: Introduction, terminology

What is a differential equation?

A differential equation is an equation (contains an equals sign which states the two sides are equal) but where at least one term contains a derivative.

$$\frac{dy}{dx}+5y=e^x \qquad \frac{d^2y}{dx^2}-3\frac{dy}{dx}+5y=6$$

A **solution** to a differential equation is itself an equation - an equation which completely satisfies the original differential equation.

To verify if an equation is a solution to given differential equation, you first take the required derivatives, and then plug everything in:

Ex) Verify that $y=100e^{.05t}+50$
is a solution to the differential equation $\frac{dy}{dt}=5e^{.05t}$

Symbols for derivatives

There are many different symbols for writing a derivative, so we need to be aware of all notations:

Leibnitz notation: $\frac{dy}{dt}$ $\frac{d^2y}{dt^2}$ $\frac{\partial^2 z}{\partial x^2}$

Prime notation: y' y'' y''' $y^{(4)}$

Subscript notation (for partial derivatives): f_y f_{xy}

Newton's dot notation: $\dot{y}$ $\ddot{y}$ $\dddot{y}$

Terminology

This course is mainly about learning different methods of finding the solutions to given differential equations (although we will also investigate applications of differential equations in the real world).

Various techniques exist for finding the solutions to differential equations, and these techniques vary depending upon the specific form of the differential equation.

So we 'classify' differential equations into different groups and throughout the course will be learning different techniques that apply to different classifications of DEs in order to obtain solutions.

Terminology - classification by 'type'

If all the derivatives are ordinary (not partial) derivatives, the equation is called an **Ordinary Differential Equation (ODE)**:

An ODE can contain more than one dependent variable...

$$\frac{dy}{dx}+5y=e^x \qquad \frac{d^2y}{dx^2}-3\frac{dy}{dx}+5y=6 \qquad \frac{dx}{dt}+2\frac{dy}{dt}=5x-3y$$

...but only one independent variable

If there is more than one independent variable and any of the derivatives are partial derivatives, the equation is called a **Partial Differential Equation (PDE)**:

$$\frac{\partial u}{\partial y}=-\frac{\partial u}{\partial y} \qquad 3\frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial y^2}=4$$

(Partial Differential Equations can be extremely difficult to solve and are beyond the scope of this course - we will focus only on Ordinary Differential Equations).

Terminology - classification by 'order'

The **order** of the differential equation is the order of the highest derivative that appears in the equation

$$\frac{dy}{dx}+2y+\frac{d^2y}{dx^2}=x^3-4$$ This is a 2nd-order DE

$$y''+2y-y'+4y'''=3$$ This is a 3rd-order DE

$$\frac{d^2y}{dx^2}+5\left(\frac{dy}{dx}\right)^3-4y=e^x$$ This is a 2nd-order DE

(this is a 1st-order term being cubed)

Terminology - dependent vs. independent variables

The last way we classify DEs is by linearity, but to do so we must first identify the independent and dependent variables. For an ODE, there is only one independent variable (although there can be more than one dependent variable):

$$(1-y)y'+2y=e^x$$
Dependent variable: y (because of the y prime)
Independent variable: x

$$\frac{d^2y}{dx^2}+\sin y=e^y$$
Dependent variable: y
Independent variable: x (in denominator of derivative)

$$\frac{d^3x}{dy^3}+x^3=2y$$
Dependent variable: x
Independent variable: y (in denominator of derivative)

$$3e^y+(y+y^3)x'=3$$
Dependent variable: x (because of x prime)
Independent variable: y

Terminology - classification by 'linearity'

For an *n-th* order ODE where y is the dependent and x the independent variable, the DE is **linear** if it can be written in this form...

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + ... + a_2(x)\frac{d^2 y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

This means that to be linear...

- The dependent variable *y* and all its derivative *y', y''...* are all of first degree (the power of each term is 1).
- The coefficients of all the terms containing the dependent variable and its derivatives (*y,y',y''...*) depend at most on the independent variable - meaning, they can be constants or functions of x, but cannot be functions of y.
- The term which contains no dependent variable, y, or derivatives can be any function of x.

Examples of Linear DEs...

$$2\frac{dy}{dx} - 3y = 0$$

This common form is a linear first-order DE

$$2\frac{dy}{dx} - 3y = e^x$$

Still linear, even with the non-linear exponential because this is not a term containing the dependent variable.

$$2y'' - 3y' + y = 5$$

This common form is a linear second-order DE

$$\frac{d^3 y}{dx^3} + x\frac{dy}{dx} - 5y = e^x$$

Okay to have a coefficient that is a function instead of a constant as long as its a function of the independent variable only.

$$3e^y + (y + y^3)x' = 3$$

$$(y + y^3)x' = (3 - 3e^y)$$

This is still linear because the independent variable here is y.

Examples of Nonlinear DEs...

$$(1 - y)y' + 2y = e^x$$

derivative coefficient is a function of the dependent variable

$$\frac{d^2 y}{dx^2} + \sin y = e^y$$

non-linear terms in the dependent variable

$$\frac{d^3 x}{dy^3} + x^3 = 2y$$

non-linear term in the dependent variable (the 2y is okay)

$$3e^y + (x + x^3)x' = 3$$

here, the dependent variable is x, so the $x+x^3$ is a function in the dependent variable - not allowed for linear

Solutions, solution intervals, and solution curves

Throughout the course, we'll learn how to obtain the solutions for a given DE, but here we may just be asked to verify that a solution provided is indeed a solution. Just take the derivatives of the solution and plug into the given DE to verify. Here are a few things to know...

If the solution is $y = 0$ (just zero) than the solution is often called the **trivial solution** (because it typically doesn't give us any meaningful information about the problem scenario).

If the solution or the original DE has a restricted domain (only some values of the input(s) can be used) then the solution is defined over only an **interval of definition (existance, validity)** which would be the list of domain values that are allowed in the original DE and the solution(s).

Because the solution to a DE is an equation, it can be graphed and the graph of the solution is called a **solution curve**.

1.2: Initial Value Problems

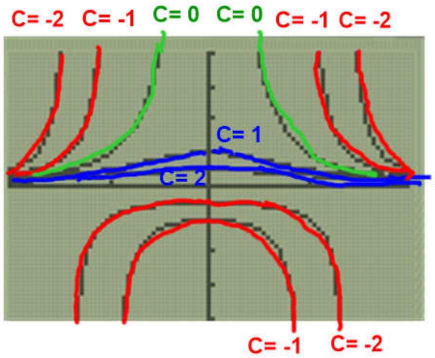
Often, a DE has a family of solutions

Frequently, the solution to a differential equation consists of a 'family' of functions where the general structure is the same but there are one or more constants which can be set to any value.

For example, it can be shown that for this differential equation: $y' + 2xy^2 = 0$

...any function of the form $y = \frac{1}{x^2 + C}$ is a solution to the DE, regardless of the value you choose for C. Such a solution is referred to as a 'family' of solutions or sometimes as a 'general solution'.

If you were to graph the solution curves for this family of solutions for *C* values of -2, -1, 0, 1, and 2 you would get:



Any of these functions satisfy the original DE.

Initial conditions

But often the DE represents some real-world situation where in addition to the DE itself, you also know something about the 'state' of the system under particular circumstances. Frequently, one of the variables is time and so this might be, for example, knowing the position of something at a given time.

This extra knowledge serves as a 'constraint' to the problem, and this is known as specifying **initial conditions**. If we know the initial conditions, we can use it to select the particular solution function curve which makes these conditions true.

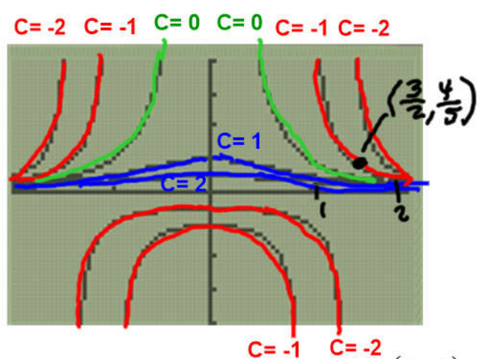
Let's say that in addition to the DE:

DE: $y' + 2xy^2 = 0$

Family of solutions: $y = \frac{1}{x^2 + C}$

We also know the initial condition:

$$y\left(\frac{3}{2}\right) = \frac{4}{5}$$



This is telling us that when x is 3/2, y must be 4/5, meaning that the solution which fully matches this scenario must pass through the point $\left(\frac{3}{2}, \frac{4}{5}\right)$

...and we can see from the graphs of the solution curves that the one with C = -1 passes through this point, so the solution with C = -1 filled in fits the scenario exactly.

$$y = \frac{1}{x^2 + C} = \frac{1}{x^2 + (-1)}$$

$$y = \frac{1}{x^2 - 1}$$ ← The resulting single solution function is often called the **particular solution**.

We can find the particular solution to an initial value problem without graphing

A problem specified in this way is known as a **initial value problem**, and we can find the particular constant algebraically, without needing to graph:

Given:

DE: $y' + 2xy^2 = 0$

Family of solutions: $y = \dfrac{1}{x^2 + C}$

and the initial condition: $y\left(\dfrac{3}{2}\right) = \dfrac{4}{5}$

We just treat the initial condition as a point, plug it in, and solve for the constant:

Higher-order DEs have multiple constants, which require a system to find

When we have higher-order DEs, we often encounter solutions with multiple constants. This means we will need a separate initial condition for each of the constants. Usually, one of the initial conditions will specify the value of the function at a given x, but the others will specify values of the derivatives at given x values. This is easiest to see through an example.

Given:

DE: $y'' - y = 0$

Family of solutions: $y = C_1 e^x + C_2 e^{-x}$

and the initial conditions: $y(1) = 0$ $\quad y'(1) = e$

Existence and Uniqueness of a solution

Beginning in Ch2, we will start learning techniques for finding the solutions families for different forms of DEs. Here, we are assuming we have a solution, and if further initial conditions are specified, determining the specific constants that give the particular solution.

But it is also important to consider: will a solution even exist for a given differential equation? If so and we find that solution, is that the only 'unique' solution? Or could there be other solutions which would also satisfy the DE?

In this introductory course, we will just use a single theorem for now which helps to determine if a unique solution exists for a given first-order DE:

> Let *R* be a rectangular region in the xy-plane defined by $a \le x \le b,\ c \le y \le d$ that contains the point (x_0, y_0) in its interior. If $f(x,y)$ and $\frac{\partial f}{\partial y}$ are continuous on *R*, then there exists some interval $I_0 : (x_0 - h, x_0 + h), h > 0$ contained in $[a,b]$ and a unique function $y(x)$, defined on I_0, that is a solution of the initial value problem.

What this is saying is if you have a first-order DE that can be solved for its derivative on the left side equals a function on the right side, you call this function on the right side f(x,y) and then take the partial derivative of that function with respect to y. Then consider both f(x,y) and the partial derivative. Any interval with both f(x,y) and the partial derivative both exist will contain a single, unique, solution.

Again, this is easiest to understand by considering a specific example...

Given the DE: $(1+y^3)y' = x^2$

Determine where in the xy-plane this DE is guaranteed to have unique solutions.

First, solve the DE for its derivative... $y' = \dfrac{x^2}{1+y^3}$

The function on the right is f(x,y)... $f(x,y) = \dfrac{x^2}{1+y^3}$

Take the partial derivative required by the theorem...

$$\frac{\partial f}{\partial y} = \frac{(1+y^3)\frac{\partial}{\partial y}[x^2] - (x^2)\frac{\partial}{\partial y}[1+y^3]}{(1+y^3)^2}$$

$$= \frac{0-(x^2)3y^2}{(1+y^3)^2} = \frac{-3x^2y^2}{(1+y^3)^2}$$

Now consider the function and partial derivative, are there any x,y values which cause these to be undefined?

$$f(x,y) = \frac{x^2}{1+y^3}$$

Undefined when denominator = 0

$$1+y^3 = 0$$
$$y^3 = -1$$
$$y = -1$$

$$\frac{\partial f}{\partial y} = \frac{-3x^2y^2}{(1+y^3)^2}$$

Undefined when denominator = 0

$$(1+y^3)^2 = 0$$
$$1+y^3 = 0$$
$$y^3 = -1$$
$$y = -1$$

So for all x,y except where y = -1, there is guaranteed to be a unique solution to the DE.

The theorem doesn't specify what happens at y = -1. It could be that there is no solution for these points. It could also be that there are solutions, but the solutions aren't unique (there are multiple solutions).

2.1: Slope fields, solutions curves, phase portraits

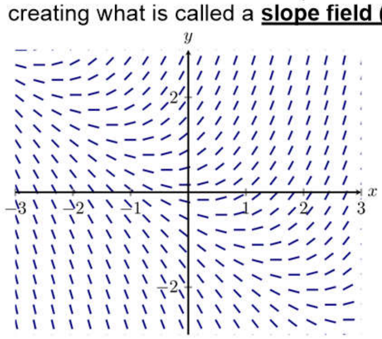
First-order DEs define the 'slope' of the solution function at each point

Throughout the course, we'll learn various ways to obtain the solutions to different forms of differential equations, but first we want to explore ways to picture what a solution to a differential equation is. In this first section, we'll consider only First-Order Differential Equations (FODEs) of the form...

$$\frac{dy}{dx} = f(x,y)$$

...which has solutions of the form: $y = \phi(x)$

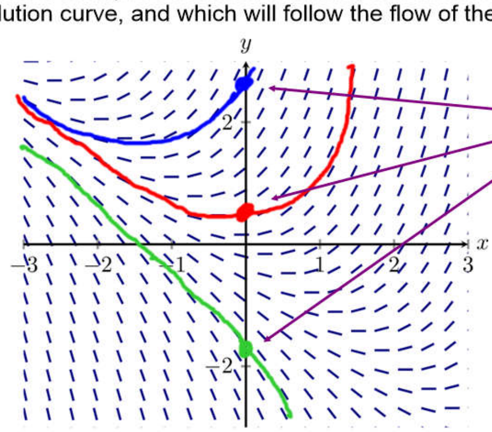
If you were to plug in various *x,y* points to the right side of the DE and evaluate, the result is the slope of the solution function at that point. This could be graphed by including slopes at many points, creating what is called a **slope field (or direction field)**:



...where the little line segments indicate the slope of the solution function curve at that *x,y* value.

The solution curves follow the 'flow' of the slope field line segments

If you then impose an initial condition, this defines a particular solution, which has a specific solution curve, and which will follow the flow of the slope field:



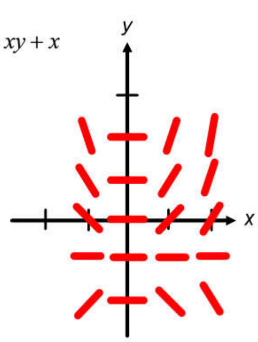
By specifying an initial condition, for example what y is when x is zero, we get different particular solution curves.

Sketching slope fields

To sketch a slope field, you pick a large number of *x,y* values, plug them into the DE to find the slope and add lines. Here are the slope fields for two DEs with the calculations for just a few points listed:

$$\frac{dy}{dx} = xy + x$$

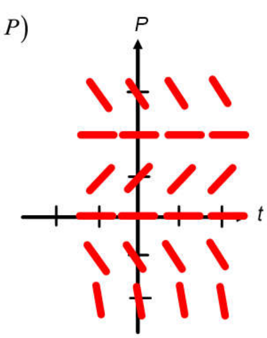
(x,y)	$\frac{dy}{dx}$
$(1,-2)$	-1
$(1,-1)$	0
$(1,0)$	1
$(1,1)$	2
$(1,2)$	3
$(2,-2)$	-2
$(2,-1)$	0



Non-autonomous DE

$$\frac{dP}{dt} = P(2-P)$$

(x,y)	$\frac{dy}{dx}$
$(1,-2)$	-1
$(1,-1)$	0
$(1,0)$	1
$(1,1)$	2
$(1,2)$	3
$(2,-2)$	-2
$(2,-1)$	0



Autonomous DE

Another classification for DEs: A Ordinary DE is **autonomous** if the independent variable does not appear explicitly. (Meaning that the function on the right side of the DE is only a function of the dependent variable and does not contain the independent variable).

The DE on the right is autonomous because the function on the right side of the DE contains only the P variable (the dependent variable) and not the independent variable, t, and notice how it does not change with t. The DE on the left is non-autonomous because the function on the right contains both variables.

Many ODEs in the real world are autonomous

Frequently, ODEs in the real world have time as their independent variable, but the derivative is a function of the quantity of the dependent variable, this does not change over time. The result is an autonomous FO-ODE:

$$\frac{dP}{dt} = kP \qquad \frac{dT}{dt} = k(T - T_m) \qquad \frac{dQ}{dt} = 6 - \frac{1}{100}Q^2$$

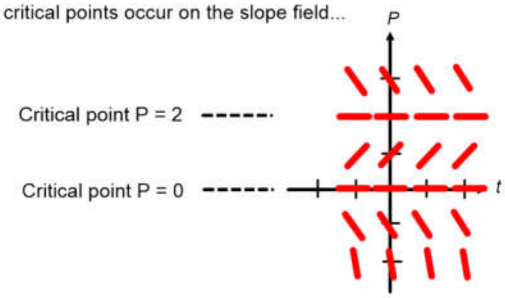
Critical Points divide the independent axis into regions

The zeros of the function on the right side of an autonomous FO-DE are called **critical points**. These are interesting because they result in what are called **equilibrium (or constant) solutions**.

Let's find the critical points for the autonomous FODE graphed above.

$$\frac{dP}{dt} = P(2-P) \qquad P(2-P) = 0 \qquad \boxed{\text{when } P = 0,\ P = 2}$$

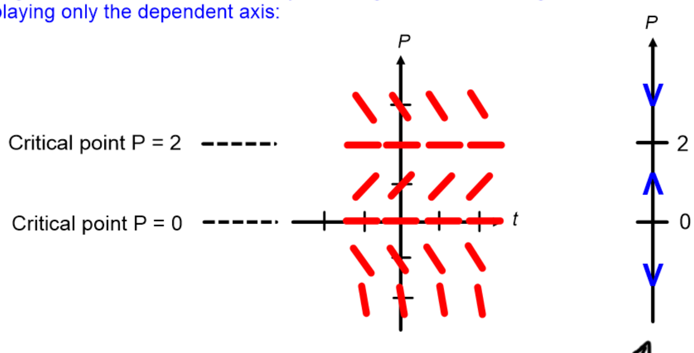
Notice where the critical points occur on the slope field...



Phase Portraits

Because the critical points are dependent variable values that cause the derivative ('slope') to be zero, these occur at locations where the slopes on the slope field are all horizontal...

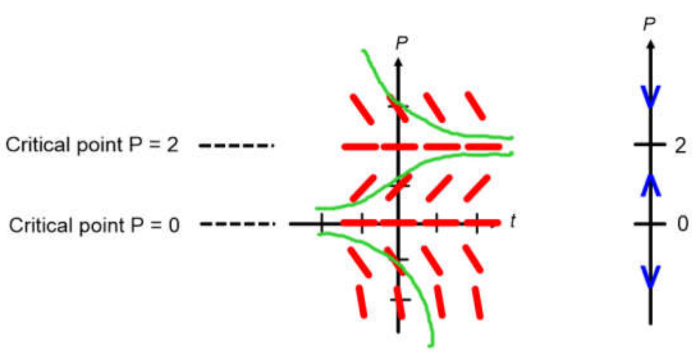
For dependent variable values between the critical points the slopes are either positive or negative, and in a 'region' between or above/below critical points, the slopes for all points will be all positive or all negative. You can indicate this by including an arrow showing direction of the slopes on a line displaying only the dependent axis:



Such a diagram is called a **1D Phase Portrait (or Phase Line)**

Equilibrium (Constant) solutions

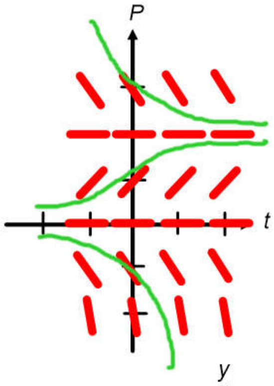
At the critical points which are defined for autonomous FO-ODEs, the solution curves are horizontal lines and solutions from other initial conditions tend to 'flow into' these lines. This is why these are referred to as **equilibrium (or constant) solutions.**

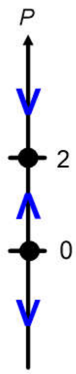


If fact, these equilibrium solutions behave like asymptotes...other solutions tend to align themselves with these constant solutions as you approach large values (positive or negative) in the independent variable.

Stable and unstable critical points, attractors and repellers

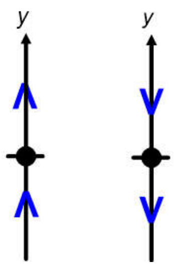
Because of this behavior of the solutions, the critical points in a phase portrait have special names depending upon the slopes above and below them...





This critical point is called **stable** and also called an **attractor**.

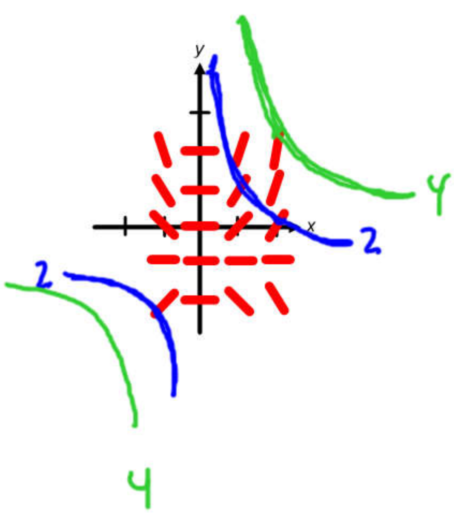
This critical point is called **unstable** and also called an **repeller**.



The only other possibility is if the slopes are the same direction on both sides of a critical point. In this case, it is called **semi-stable**.

Isoclines

Last semester when we studied multivariable functions, we sometimes chose different output values of the function and draw curves in the domain that produced those values called level curves. We have something similar for DEs called **isoclines.** We choose values for the derivative (the 'slope') and then sketch the curves in the x-y plane that would produce that slope.



$$\frac{dy}{dx} = xy + x$$

$xy + x = 2$	$xy + x = 4$
$x(y+1) = 2$	$x(y+1) = 4$
$y+1 = \frac{2}{x}$	$y+1 = \frac{4}{x}$
$y = \frac{2}{x} - 1$	$y = \frac{4}{x} - 1$

2.2: Separable Variables

If you can separate the variables, you can solve by integration

For first-order DEs, if the function on the right side can be factored into two separate functions, each of only one variable...

$$\frac{dy}{dx} = g(x)\,h(y)$$

...then it is possible to solve by integration.

Separable DE

$$\frac{dy}{dx} = y^2 e^{3x+4y}$$

$$\frac{dy}{dx} = y^2 e^{3x} e^{4y}$$

$$\frac{dy}{dx} = \left(e^{3x}\right)\left(y^2 e^{4y}\right)$$

Non-Separable DE

$$\frac{dy}{dx} = y + \sin x$$

To solve a separable DE, first, we separate the variables (each side contains only one variable):

$$\frac{1}{h(y)}\,dy = g(x)\,dx$$

...then we integrate each side:

$$\int \frac{1}{h(y)}\,dy = \int g(x)\,dx$$

This produces an equation with a function of y on the left and a function of x on the right...a solution to the original differential equation.

Ex) Solve $\frac{dy}{dx} = xy$

$$\frac{dy}{dx} = xy$$

$$\frac{1}{y}\,dy = x\,dx$$

$$\int \frac{1}{y}\,dy = \int x\,dx$$

$$\ln|y| + C_1 = \frac{1}{2}x^2 + C_2$$

$$\ln|y| = \frac{1}{2}x^2 + C$$

Produces a family of solutions with different integration constants. If we have an initial condition, we can plug in x,y and solve for the C to get a particular solution.

Partial fraction expansions often involved for higher-degree polynomial functions

When the function on the right side involved higher-degree polynomials, we sometimes need partial fraction expansion. Here is an example to review how this works...

Ex) Solve $\frac{dy}{dx} = y^2 - 4$

$$\frac{1}{y^2-4}dy = dx$$

If we take the integral of both sides now:

$$\int \frac{1}{y^2-4}dy = \int dx$$

we have no easy way to integrate the left side (integration by substitution or by parts doesn't work).

The left function can be expanded with partial fractions:

$$\frac{1}{y^2-4} = \frac{1}{(y-2)(y+2)} = \frac{A}{y-2} + \frac{B}{y+2}$$

← So we often use Partial Fractions to split a complicated fraction.

$$\frac{1}{(y-2)(y+2)} = \frac{A(y+2)}{(y-2)(y+2)} + \frac{B(y-2)}{(y-2)(y+2)}$$

$$Ay + 2A + By - 2B = 1$$

$$(A+B)y + (2A-2B) = 0y + 1$$

$$so \begin{cases} A+B=0 \\ 2A-2B=1 \end{cases}$$

$$A = \frac{1}{4},\ B = -\frac{1}{4}$$

$$\frac{1}{y^2-4} = \frac{\left(\frac{1}{4}\right)}{y-2} + \frac{\left(-\frac{1}{4}\right)}{y+2}$$

Back to the DE:

$$\frac{\left(\frac{1}{4}\right)}{y-2}dy - \frac{\left(\frac{1}{4}\right)}{y+2}dy = dx$$

$$\int \frac{\left(\frac{1}{4}\right)}{y-2}dy - \int \frac{\left(\frac{1}{4}\right)}{y+2}dy = \int 1\,dx$$

$$\frac{1}{4}\ln|y-2| - \frac{1}{4}\ln|y+2| = x + C$$

$$\frac{1}{4}\ln\left|\frac{y-2}{y+2}\right| = x + C_1$$

This result is called an **implicit solution** because it isn't solved explicitly for the dependent variable.

...but this solution could be solved explicitly for y:

$$\frac{1}{4}\ln\left|\frac{y-2}{y+2}\right| = x + C_1$$

$$\ln\left|\frac{y-2}{y+2}\right| = 4x + 4C_1$$

$$e^{\ln\left|\frac{y-2}{y+2}\right|} = e^{4x+4C_1}$$

$$\frac{y-2}{y+2} = \pm e^{4x+C_2}$$

$$\frac{y-2}{y+2} = \pm e^{C_2}e^{4x}$$

$$\frac{y-2}{y+2} = \pm Ce^{4x}$$

$$y-2 = \pm Ce^{4x}(y+2)$$

$$y = Ce^{4x}y + 2Ce^{4x} + 2 \quad or \quad y = -Ce^{4x}y - 2Ce^{4x} + 2$$

$$y - Ce^{4x}y = 2Ce^{4x} + 2 \quad or \quad y + Ce^{4x}y = -2Ce^{4x} + 2$$

$$y(1 - Ce^{4x}) = 2Ce^{4x} + 2 \quad or \quad y(1 + Ce^{4x}) = -2Ce^{4x} + 2$$

$$y = 2\frac{1+Ce^{4x}}{1-Ce^{4x}} \quad or \quad y = 2\frac{1-Ce^{4x}}{1+Ce^{4x}}$$

...and since *C* could be positive or negative, these can both be represented by:

$$y = 2\frac{1+Ce^{4x}}{1-Ce^{4x}}$$

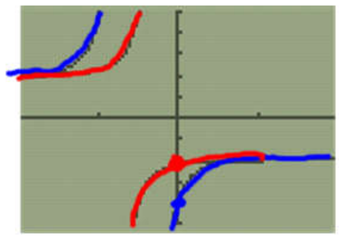
Solution curves for this example

Here was our family of solutions to the DE $\frac{dy}{dx} = y^2 - 4$

$$y = 2\frac{1 + Ce^{4x}}{1 - Ce^{4x}}$$

For different initial conditions, we would have different constants, C and different solution curves:

If $y(0) = -4$
$C = 3$
and $y = 2\frac{1 + 3e^{4x}}{1 - 3e^{4x}}$



If $y(0) = -2.2$
$C = 21$
and $y = 2\frac{1 + 21e^{4x}}{1 - 21e^{4x}}$

Be aware, you can 'lose a solution' with this method

Because when we separate the variables we are typically dividing both sides of the DE by the *y* variable function, we could have values of this function which would go to zero, thus undefined in our solution, but may still represent a solution.

In our last example, the DE was $\frac{dy}{dx} = y^2 - 4$

and we started by separating variables which caused us to divide by the right hand side:

$$\frac{1}{y^2 - 4}dy = dx$$

The left side is undefined at y = 2 and y = -2, so our solution would not capture these solutions.

You should manually check any values which make the functions undefined after dividing. In this case, this means checking two solutions:

$$\begin{array}{ll} y = 2 & y = -2 \\ \frac{dy}{dx} = 0 & \frac{dy}{dx} = 0 \\ (0) = (2)^2 - 4 & (0) = (-2)^2 - 4 \\ 0 = 0 & 0 = 0 \end{array}$$

Which means a complete family of solutions is: $y = 2\frac{1 + Ce^{4x}}{1 - Ce^{4x}}$ and $y = 2,\ y = -2$

(and actually y = 2 is included in our first solutions for the case C=0)

Try this one...

Ex) Solve $\frac{dy}{dx} = ky$

Once you have the implicit solution, try to find the explicit solution (and let's assume that y is always positive for this)

Try this initial value problem...

Ex) Solve $\frac{dy}{dt} + 2y = 1 \qquad y(0) = \frac{5}{2}$

One more...(a homework problem)

#27) Solve and find the explicit solution for

$$\sqrt{1-y^2}\,dx - \sqrt{1-x^2}\,dy = 0 \qquad y(0) = \frac{\sqrt{3}}{2}$$

2.3: Solving First-Order Linear Differential Equations

First-order, linear differential equation forms and terminology

We have one solution method so far to find solutions: we can find a solution if the variables can be separated by integrating. But this is frequently not the case, and for non-separable DEs, the solutions methods depend upon the form of the equation.

A first-order differential equations is defined as **linear** if it can be written in the form:

$$\boxed{a_1(x)\frac{dy}{dx}+a_0(x)y=g(x)}$$

Further, if we divide everything by $a_1(x)$ we obtain what is called the **standard form**:

$$\frac{dy}{dx}+\frac{a_0}{a_1(x)}(x)y=\frac{g(x)}{a_1(x)}$$

$$\boxed{\frac{dy}{dx}+P(x)y=f(x)}$$

Also, if the function on the RHS is zero, we call this a **homogenous** DE:

homogenous	**nonhomogenous**
$\frac{dy}{dx}+P(x)y=0$	$\frac{dy}{dx}+P(x)y=f(x)$ $f(x)\neq 0$

Property: Solution is the sum of the homogenous and nonhomogenous solution

For first-order linear differential equations, a property states that the solution will be a sum of two solutions:

$$y=y_c+y_p$$

Where y_c is a solution of the associated homogenous equation and y_p is a particular solution of the nonhomogenous equation.

(More about this later in the course...)

Procedure for finding the solution to a first-order, linear differential equation

(If you are interested in the derivation of this procedure, I've posted this in a separate PDF - it is 5 pages long and quite involved, but take a look if interested :)

1) Put the equation in standard form: $\frac{dy}{dx}+P(x)y=f(x)$

2) Identify *P(x)* and use it to compute something called the **integrating factor** (I.F.):

$$I.F.=e^{\int P(x)dx}$$

3) Multiply both sides of the DE by the integrating factor. The left-hand side of the resulting equation is automatically the derivative of the integrating factor and *y*:

$$\frac{d}{dx}\left[e^{\int P(x)dx}y\right]=e^{\int P(x)dx}f(x)$$

4) Now integrate both sides of this equation to obtain the solution.

An example

Solve $x\frac{dy}{dx} - 4y = x^6 e^x$

1) Put the equation in standard form: $\frac{dy}{dx} - \frac{4}{x}y = x^5 e^x$

2) Identify *P(x)* and use it to compute something called the **integrating factor** (I.F.):

$$P(x) = -4\frac{1}{x} \qquad I.F. = e^{\int P(x)dx} = e^{-4\int \frac{1}{x}dx}$$

$$I.F. = e^{-4\ln|x|} = e^{\ln x^{-4}} = x^{-4} \qquad x > 0$$

3) Multiply both sides of the DE by the integrating factor. The left-hand side of the resulting equation is automatically the derivative of the integrating factor and *y*:

$$x^{-4}\left(\frac{dy}{dx} - \frac{4}{x}y\right) = x^{-4}\left(x^5 e^x\right)$$

$$x^{-4}\left(\frac{dy}{dx} - \frac{4}{x}y\right) = xe^x$$

$$\frac{d}{dx}\left[x^{-4}y\right] = xe^x$$

4) Now integrate both sides of this equation to obtain the solution.

$$\frac{d}{dx}\left[x^{-4}y\right] = xe^x$$

$$\int \frac{d}{dx}\left[x^{-4}y\right]dx = \int xe^x\,dx$$

$$(by\ parts)\quad u = x \qquad dv = e^x\,dx$$

$$\frac{du}{dx} = 1 \qquad \int dv = \int e^x\,dx$$

$$du = dx \qquad v = e^x$$

$$uv - \int v\,du$$

$$x^{-4}y = xe^x - \int e^x\,dx$$

$$x^{-4}y = xe^x - e^x + C$$

$$y = x^4\left(xe^x - e^x + C\right)$$

$$\boxed{y = x^5 e^x - x^4 e^x + Cx^4}$$

You try this one...

Solve $\frac{dy}{dx} - 3y = 0$

Let's work HW #13 together...

#13) Solve $x^2 y' + x(x+2)y = e^x$

Transient terms...

In our solution to #13, the 2nd term 'fades out' as x increases towards infinity...

$$y = \frac{e^x}{2x^2} + C\frac{e^{-x}}{x^2}$$

(the first term does not, although this is harder to show...we need to evaluate the limit and use L'Hopital's Rule)

$$\lim_{x \to \infty} \frac{e^x}{2x^2} = \frac{\infty}{\infty}$$

$$\lim_{x \to \infty} \frac{e^x}{4x} = \frac{\infty}{\infty}$$

$$\lim_{x \to \infty} \frac{e^x}{4} = \infty$$

If a term approaches zero as the independent variable approaches infinity, the terms is called a **transient term** (which means it affects the solution for awhile, but the effect is 'transient'). Eventually the solution will 'settle' to the solution including only the non-transient terms.

Initial value problems

Just as with any differential equation, if we are provided with an initial condition, we can solve for the value of the constant to obtain the particular solution.

Ex) Solve $\frac{dy}{dx} + y = x \quad y(1) = 4$

Singular Points

Since we begin the procedure by dividing by $a_1(x)$ if there are any values of *x* which make this first term zero, the standard form DE would be undefined.

Values for which this occurs are called **singular points** and we have to be cognizant of the fact that the solution is not defined at these values.

What if we can't integrate?

Consider this DE: $\frac{dy}{dx} = e^{-x^2} \quad y(3) = 5$

If we try to express this in linear, standard form, we get:

$$\frac{dy}{dx} + (0)y = e^{-x^2}$$

$$P(x) = 0 \qquad I.F. = e^{\int 0\,dx} = e^0 = 1$$

$$1\left(\frac{dy}{dx}\right) = 1\left(e^{-x^2}\right)$$

$$\frac{dy}{dx} = e^{-x^2}$$

...and we're just back to the original DE. But this one is actually separable:

$$dy = e^{-x^2}\,dx$$

$$\int dy = \int e^{-x^2}\,dx$$

$$y = \int e^{-x^2}\,dx$$

The problem is, we can't integrate the RHS directly.

But if we have an initial condition, we can make progress using the FTC part I: $\frac{d}{dx}\left[\int_a^x g(t)\,dt\right] = g(x)$

$$\frac{dy}{dx} = e^{-x^2} \quad y(3) = 5$$

$$\int_3^x \frac{dy}{dt}\,dt = \int_3^x e^{-t^2}\,dt$$

$$\left[y(t)\right]_3^x = \int_3^x e^{-t^2}\,dt$$

$$y(x) - y(3) = \int_3^x e^{-t^2}\,dt$$

$$y(x) = \int_3^x e^{-t^2}\,dt + y(3)$$

The Error Function

$$y(x)=\int_3^x e^{-t^2}\,dt+y(3)$$ is the particular solution to the DE $\frac{dy}{dx}=e^{-x^2}$

subject to the initial condition $y(3)=5$

This solution is expressed using an integral. This particular integral occurs frequently in natural systems so it has been given a name: the error function.

> The **Error Function** is defined to be: $$erf(x)=\frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt$$
>
> and the **Complementary Error Function** is defined to be: $$erfc(x)=\frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}\,dt$$

Using the error function, we could write our solution:

$$y(x)=\frac{\sqrt{\pi}}{2}erf(x)+y(3)$$

2.4: Solving First-Order Exact Equations

First-order, exact equations

If have a differential equation such as $\frac{dy}{dx} = -\frac{y}{x}$ sometimes the equation can be rearranged

by separating the differentials like this: $y\,dx + x\,dy = 0$

A differential equation written this way is referred to as in **differential form**.

> If a differential equation can be written in differential form with a zero on the RHS:
>
> $$M(x,y)\,dx + N(x,y)\,dy = 0$$
>
> and it is also true that: $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$
>
> The differential equation is called an **exact equation**.

There is a specific procedure for solving exact equations (derived in a separate posted PDF).

Procedure for finding the solution to a first-order, exact differential equation

1) Put the equation into differential form: $M(x,y)dx + N(x,y)dy = 0$

2) Identify *M(x,y)* and *N(x,y)* and use partial derivatives to verify this is an exact equation:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

3) Postulate a solution of the form $f(x,y) = c$ and initially set $\frac{\partial f}{\partial x} = M(x,y)$

then integrate both sides with respect to *x* to find an initial, partially complete, form of the solution:

$$f(x,y) = \int M(x,y)dx + g(y)$$

(note that the integration constant is instead some unknown function of y).

4) Now differentiate both sides of the result with respect to *y*. This produces an expression for $\frac{\partial f}{\partial y}$

5) Set this expression equal to *N(x,y)*, and solve this algebraically for *g'(y)*.

6) Integrate the resulting *g'(y)* expression with respect to *y* to find *g(y)*.

7) Finally, fill in *g(y)* to the partially completed solution *f(x,y)* from step 3.

The final solution is: $f(x,y) = c$ where the constant *c* is set by initial conditions for a particular solution.

An example

Solve $\frac{dy}{dx} = -\frac{e^{2y} - y\cos xy}{2xe^{2y} - x\cos xy + 2y}$

1) Put the equation in differential form: $\underbrace{\left(e^{2y} - y\cos xy\right)}_{M}dx + \underbrace{\left(2xe^{2y} - x\cos xy + 2y\right)}_{N}dy = 0$

2) Identify *M* and *N* and take partial derivatives to confirm this is an exact equation: $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$$M = e^{2y} - y\cos xy \qquad M_y = 2e^{2y} - \left(y\frac{\partial}{\partial y}[\cos xy] + \cos xy\frac{\partial}{\partial y}[y]\right)$$
$$= 2e^{2y} - \left(y(-x\sin xy) + \cos xy(1)\right)$$
$$= 2e^{2y} + xy\sin xy - \cos xy$$

$$N = 2xe^{2y} - x\cos xy + 2y \quad M_x = 2e^{2y} - \left(x\frac{\partial}{\partial x}[\cos xy] + \cos xy\frac{\partial}{\partial x}[x]\right) + 0$$
$$= 2e^{2y} - \left(x(-y\sin xy) + \cos xy(1)\right)$$
$$= 2e^{2y} + xy\sin xy - \cos xy$$

3) Postulate a solution of the form $f(x,y) = c$ and initially set $\frac{\partial f}{\partial x} = M(x,y)$

then integrate both sides with respect to *x* to find an initial, partially complete, form of the solution:

$$f(x,y) = \int M(x,y)\,dx + g(y)$$

(note that the integration constant is instead some unknown function of y).

$$\frac{\partial f}{\partial x} = e^{2y} - y\cos xy$$
$$f(x,y) = \int e^{2y} - y\cos xy\,dx = \int e^{2y}dx - \int y\cos xy\,dx$$
$$u = xy \quad du = ydx$$
$$= xe^{2y} - \int \cos u\,du$$
$$= xe^{2y} - \sin xy$$
$$f(x,y) = xe^{2y} - \sin xy + g(y)$$

(integration constant can be a function of y)

4) Now differentiate both sides of the result with respect to *y*. This produces an expression for $\frac{\partial f}{\partial y}$

$$f(x,y) = xe^{2y} - \sin xy + g(y)$$
$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left[xe^{2y} - \sin xy + g(y)\right]$$
$$= 2xe^{2y} - x\cos xy + g'(y)$$

5) Set this expression equal to *N(x,y)*, and solve this algebraically for *g'(y)*.

$$\frac{\partial f}{\partial y} = N(x,y)$$
$$\underline{2xe^{2y} - x\cos xy} + \underline{g'(y)} = \underline{2xe^{2y} - x\cos xy} + \underline{2y}$$
$$g'(y) = 2y$$

6) Integrate the resulting *g'(y)* expression with respect to *y* to find *g(y)*.

$$g'(y) = 2y$$
$$g(y) = \int 2y\, dy = y^2$$

7) Finally, fill in *g(y)* to the partially completed solution *f(x,y)* from step 3.

The final solution is: $f(x,y) = c$ where the constant *c* is set by initial conditions for a particular solution.

$$f(x,y) = xe^{2y} - \sin xy + g(y)$$
$$f(x,y) = xe^{2y} - \sin xy + y^2$$

solution...

$$xe^{2y} - \sin xy + y^2 = c$$

(Note: the solution is not the *f(x,y)* it is *f(x,y)* equals a constant. This is the general solution and the constant, *c*, would be established if we had an initial condition.)

You try this one... Solve $(2xy)\, dx + (x^2 - 1)\, dy = 0$

Let's work an inital value problem together...

Ex) Solve $\dfrac{dy}{dx} = \dfrac{xy^2 - \cos x \sin x}{y\left(1-x^2\right)}$ $\quad y(0)=2$

Sometimes, you can use an integrating factor to turn a DE into an exact equation

1) If an equation in differential form is not exact:

$$M(x,y)dx + N(x,y)dy = 0 \qquad \text{but} \qquad \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

2) Identify *M* and *N*, and form the following two quotients with partial derivatives:

$$\frac{M_x - N_x}{N} \text{ and } \frac{N_x - M_x}{M}$$

3) Algebraically simplify each quotient and if either simplifies to be a function of only one of the independent variables, then that quotient can be used to compute an integrating factor:

$$I.F. = e^{\int \frac{M_x - N_x}{N} dx} \qquad \text{or} \qquad I.F. = e^{\int \frac{N_x - M_x}{M} dx}$$

4) Multiply the original DE by the I.F. to obtain an exact equation.

(Note: this procedure only works in some cases, but if the quotient produces a function of only one variable, the resulting DE will be exact.)

An example using an I.F. to make an equation exact

Ex) Solve $(xy)\,dx + (2x^2 + 3y^2 - 20)\,dy = 0$

2.5: Solutions by Substitutions

Sometimes you can do clever variable substitutions to get a form you can solve

It is sometimes possible to start with a differential equation which doesn't match any forms you know how to solve, but by making a specific variable substitution, the resulting differential equation is now in a form you can solve.

There are two specific substitutions we will include here:

1) Bernoulli Equations
2) Reduction to Separation of Variables

Substitution to convert a Bernoulli DE into a separable or linear DE

A differential equation which can be written in the following form...

$$\frac{dy}{dx} + P(x)y = f(x)y^n \quad \textit{for any real number } n$$

...is called a **Bernoulli Equation**.

For Bernoulli equations, substituting $u = y^{1-n}$

results in a separable DE which can be solved by known methods.

Bernoulli Equation example

Solve $x\frac{dy}{dx} + y = x^2 y^2$

Substitution for Reduction to Separation of Variables

For a differential equation which can be written in the following form...

$$\frac{dy}{dx} = f\left(Ax + By + C\right)$$

(a composite function where the 'inside' function is linear in x and y)

...substituting the 'inside' function: $u = Ax + By + C$

results in a separable DE which can be solved by integrating both sides.

Reduction to Separation of Variables example

Solve $\frac{dy}{dx} = \left(-2x + y\right)^2 - 7 \quad y\left(0\right) = 0$

One more example...

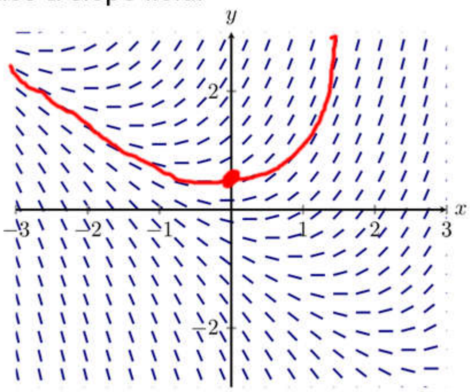
Solve $\frac{dy}{dx} - y = e^{x} y^{2}$

2.6: A Numerical Method

Reminder about terminology for first-order differential equations

We are given a differential equation and for first-order DEs, this would be of the form: $\frac{dy}{dx} = f(x, y)$

Plugging values into the *f(x,y)* gives the 'slope' of the solution at that *(x,y)*. This could be graphed using lineal elements to produce a slope field:



The solution to a differential equation is a function: $y = \phi(x)$

...and graphing the solution, produces the **solution curve** (for a particular initial condition), which follows the pattern of the slope field.

We can't always find the solution function analytically

We now know some ways to find the solution curve function $y = \phi(x)$ analytically.

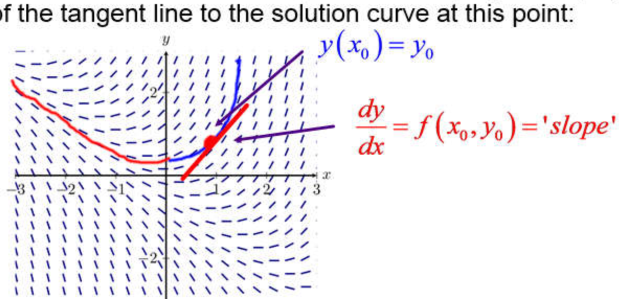
But not all differential equations will be in the forms we know how to solve. We may still need to know something about the solution, though - need to know the y for given x values in the solution.

When we can't find a solution to a differential equation analytically, we can resort to using a **numerical method** to find the approximate solution - an approximate *y* value for any given value *x* for the solution curve, for a specific solution curve (that is, we need to be given an initial condition).

The method we are going to present here is called **Euler's Method**. This method takes advantage of the fact that with a first-order differential equation of the form...

$\frac{dy}{dx} = f(x, y)$ ...with an initial condition given: $y(x_0) = y_0$

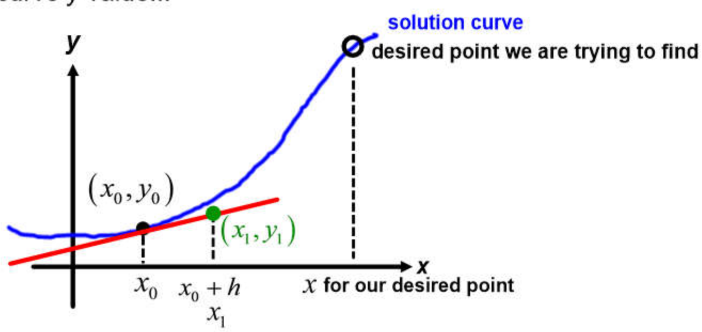
...we know the initial condition is on the solution curve, and we can plug in this (x_0, y_0) into the differential equation to get the 'slope' of the tangent line to the solution curve at this point:



Euler's Method

Given: $\frac{dy}{dx} = f(x,y)$ ...with an initial condition given: $y(x_0) = y_0$

We can move a distance *h* away in x from the initial condition towards the *x* value we wish to know the solution curve *y* value...

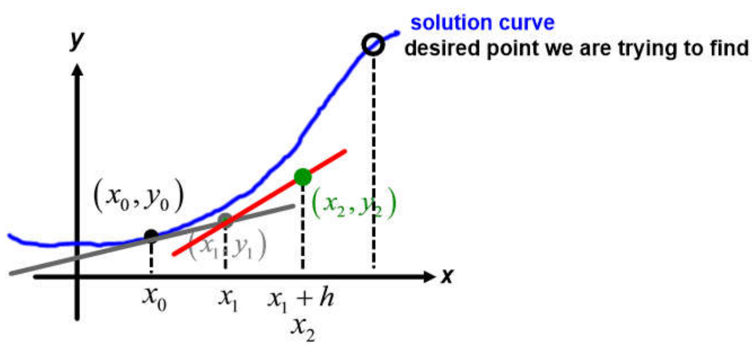


...and use the slope to compute an updated *y* value:

$$\frac{dy}{dx} = f(x_0, y_0) = 'slope' = \frac{\Delta y}{\Delta x} = \frac{y_1 - y_0}{h}$$

$$so \quad y_1 = y_0 + hf(x_0, y_0)$$

We then use this new point (x_1, y_1) to establish the slope for another estimate moving closer to the desired point by plugging this point into the original DE *f(x,y)* function to establish the next slope:



This process continues iteratively until we are at the desired x value, and we then have an estimate of the *(x,y)* on the solution curve at the desired *x* value.

An example

Use Euler's method to obtain a four-decimal approximation for $y(1.5)$

on the solution curve for $y' = 0.2xy \;\; with \;\; y(1) = 1$

Let's set *h* = 0.1, taking 5 iterations to reach from x=1 to x=1.5:

$$(x_0, y_0) = (1,1)$$

$$y_1 = 1 + 0.1(0.2(1)(1)) = 1.02 \qquad (x_1, y_1) = (1.1, 1.02)$$

$$y_2 = 1.02 + 0.1(0.2(1.1)(1.02)) = 1.04244 \qquad (x_2, y_2) = (1.2, 1.04244)$$

$$y_3 = 1.04244 + 0.1(0.2(1.2)(1.04244)) = 1.0674... \quad (x_3, y_3) = (1.3, 1.0674...)$$

$$y_4 = 1.0674.. + 0.1(0.2(1.3)(1.0674...)) = 1.0952... \quad (x_4, y_4) = (1.4, 1.0952...)$$

$$y_5 = 1.0952.. + 0.1(0.2(1.4)(1.0952...)) = 1.125878.. \quad (x_5, y_5) = (1.5, 1.125878...)$$

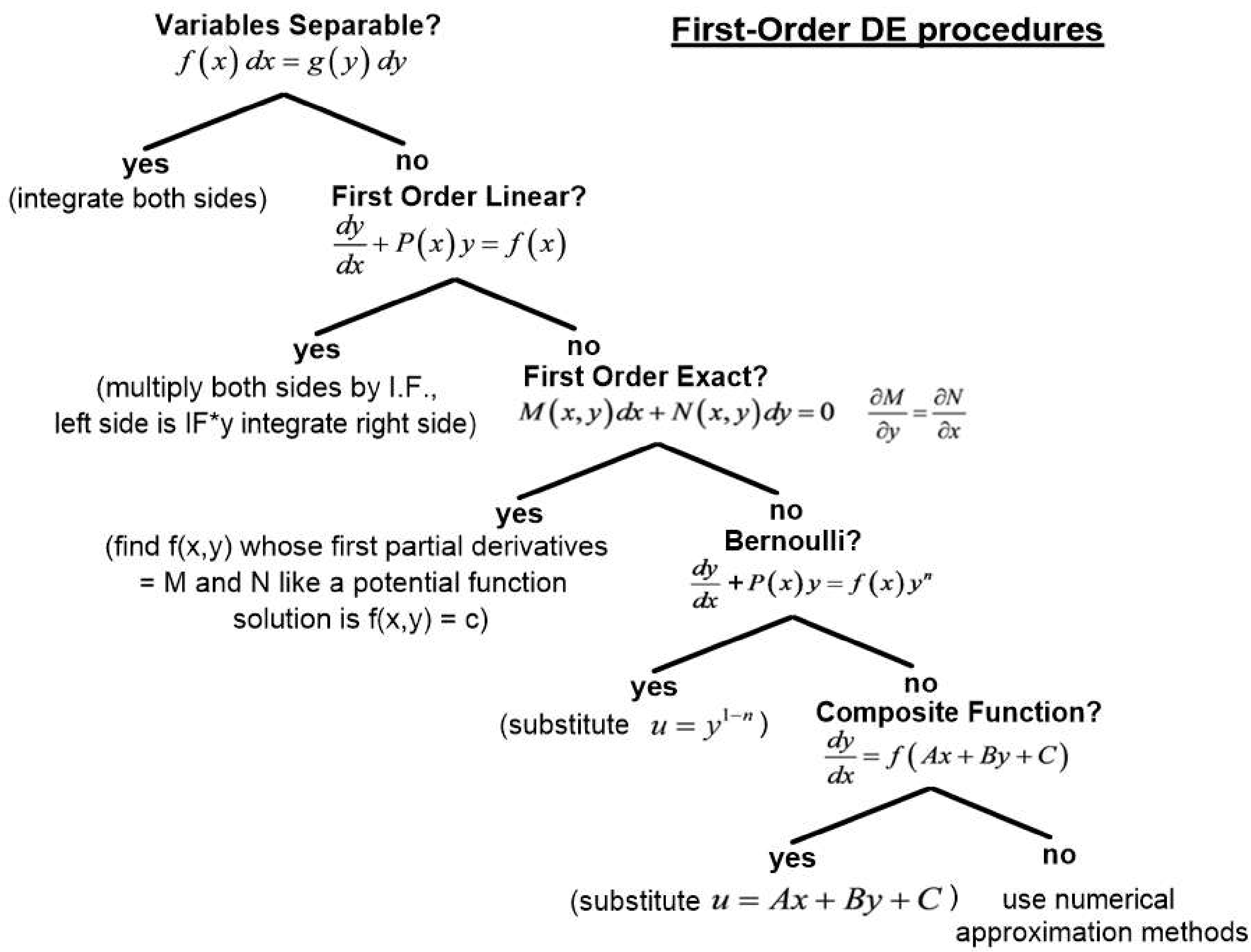
$$y(1.5) \approx 1.1259$$

There are many numerical methods

Euler's method is presented here because it is straightforward and is a good way to understand how numerical approximation methods work, but in practice other more complicated methods are often used instead.

One method that is used often is called the Runge-Kutta method. We will explore this topic further towards the end of the course. Computing approximations numerically can be tedious, so this is often done using a computer.

Later in the course, we will include an introduction to MATLAB programming (actually, we will use OCTAVE which is a free open-source version of MATLAB with the same syntax), and we will write simple programs which allow us to use more advanced methods, such as Runge-Kutta, to find numerical solutions to differential equations which are difficult or impossible to solve analytically.



First-order w/Separable variables? **Separate and integrate each side.**

First-order Linear?

1) Rewrite in std form: $\frac{dy}{dx} + P(x)y = f(x)$

2) Identify P(x) and Integrating Factor $I.F. = e^{\int P(x)dx}$

3) Multiply both sides of DE by I.F., LHS is derivative of I.F.y

4) Integrate both sides

First-order Exact? **Procedure for finding the solution to a first-order, exact differential equation**

1) Put the equation into differential form: $M(x,y)dx + N(x,y)dy = 0$

2) Identify *M(x,y)* and *N(x,y)* and use partial derivatives to verify this is an exact equation:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

3) Postulate a solution of the form $f(x,y) = c$ and initially set $\frac{\partial f}{\partial x} = M(x,y)$

then integrate both sides with respect to *x* to find an initial, partially complete, form of the solution:

$$f(x,y) = \int M(x,y)dx + g(y)$$

(note that the integration constant is instead some unknown function of y).

4) Now differentiate both sides of the result with respect to *y*. This produces an expression for $\frac{\partial f}{\partial y}$

5) Set this expression equal to *N(x,y)*, and solve this algebraically for *g'(y)*.

6) Integrate the resulting *g'(y)* expression with respect to *y* to find *g(y)*.

7) Finally, fill in *g(y)* to the partially completed solution *f(x,y)* from step 3.

The final solution is: $f(x,y) = c$ where the constant *c* is set by initial conditions for a particular solution.

First-order not exact but can be made exact using I.F.?

Sometimes, you can use an integrating factor to turn a DE into an exact equation

(Derivation for this procedure is in the published PDF)

1) If an equation in differential form is not exact:

$$M(x,y)dx + N(x,y)dy = 0 \quad \text{but} \quad \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

2) Identify *M* and *N*, and form the following two quotients with partial derivatives:

$$\frac{M_x - N_x}{N} \text{ and } \frac{N_x - M_x}{M}$$

3) Algebraically simplify each quotient and if either simplifies to be a function of only one of the independent variables, then that quotient can be used to compute an integrating factor:

$$I.F. = e^{\int \frac{M_x - N_x}{N} dx} \quad \text{or} \quad I.F. = e^{\int \frac{N_x - M_x}{M} dx}$$

4) Multiply the original DE by the I.F. to obtain an exact equation.

(Note: this procedure only works in some cases, but if the quotient produces a function of only one variable, the resulting DE will be exact.)

Bernoulli? (substitution) $\frac{dy}{dx} = P(x)y = f(x)y^n$ substitute: $u = y^{1-n}$

Composite function? (substitution) $\frac{dy}{dx} = f(Ax + By + C)$ substitute: $u = Ax + By + C$