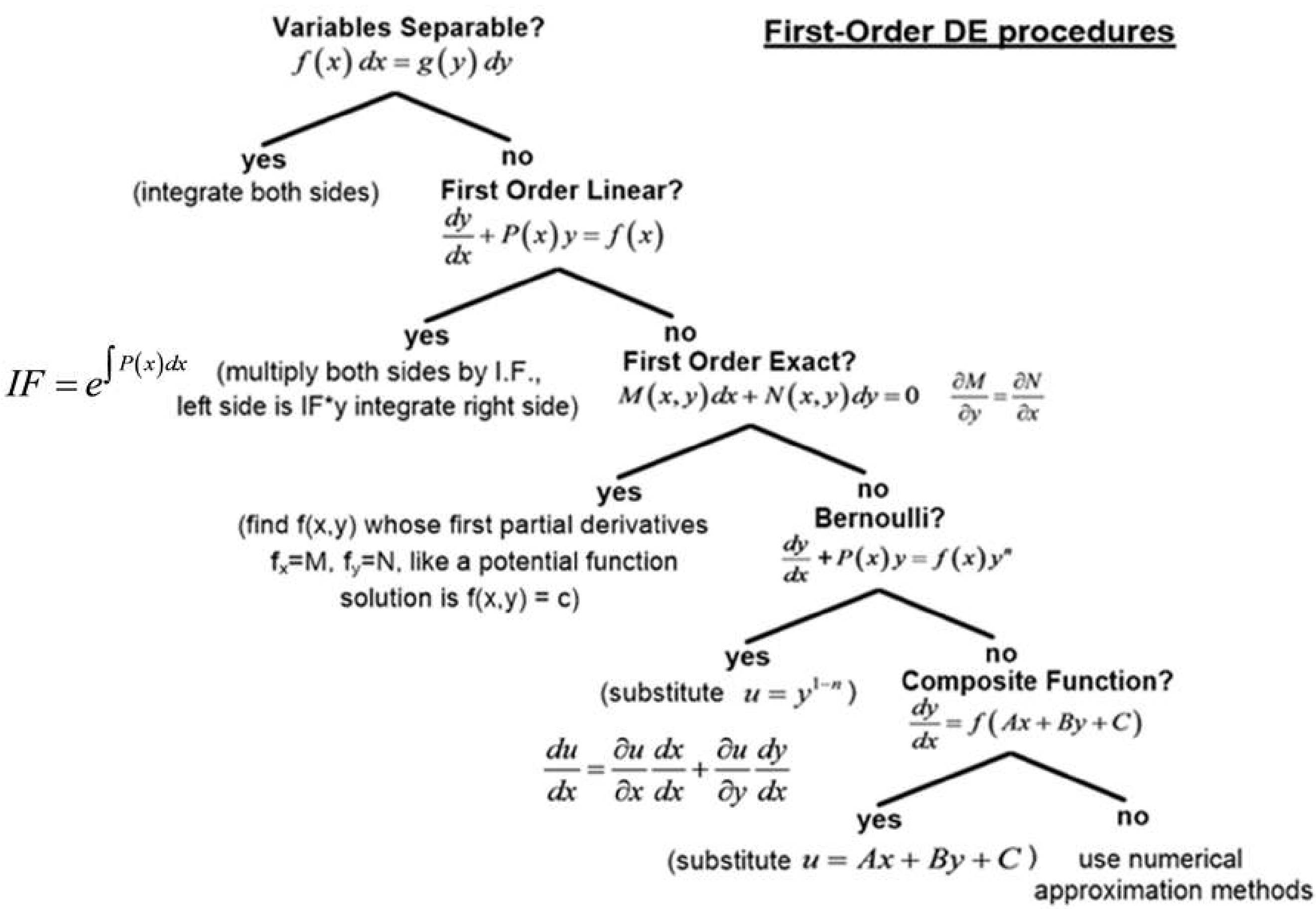
Formulas/Tables for Differential Equations Final Exam

Ch 1/2: 1st order solving methods



Ch 4: Higher-order solving methods

Higher-Order DE procedures
for solving homogeneous DEs

2nd-order linear, have one solution and need another?

$$a(x)y'' + b(x)y' + c(x)y = 0$$

- **yes**
  Reduction of Order
  $$y_2(x) = u(x)\,y_1(x)$$
  (take derivatives and substitute w=u', solve resulting 1st-order DE, then integrate to get u).
- **no**
  homogeneous with constant coefficients?
  $$ay'' + by' + cy = 0$$
  - **yes**
    use auxiliary equation
    $$am^2 + bm + c = 0$$

single root:	$y = C_1 e^{mx}$
repeated roots:	$y = C_1 e^{mx} + C_2 x e^{mx} + C_3 x^2 e^{mx} \ldots$
complex pairs:	$y = C_1 e^{\alpha x}\cos\beta x + C_2 e^{\alpha x}\sin\beta x$

  - **no**
    2nd-order, Cauchy-Euler form?
    $$ax^2 y'' + bxy' + cy = 0$$
    - **yes**
      use auxiliary equation
      $$am^2 + (b-a)m + c = 0$$

distinct roots:	$y = C_1 x^{m_1} + C_2 x^{m_2}$
repeated roots:	$y = C_1 x^m + C_2 x^m \ln x$
complex pairs:	$y = C_1 x^\alpha \cos(\beta \ln x) + C_2 x^\alpha \sin(\beta \ln x)$

    - **no**
      We'll need methods from future chapters...

Check if solutions are Linearly Independent (form a fundamental solution set) using Wronskian:

$$W\left(f_1(x), f_2(x) \ldots f_n(x)\right) = \begin{vmatrix} f_1 & f_2 & \ldots & f_n \\ f_1' & f_2' & \ldots & f_n' \\ \ldots & \ldots & \ldots & \ldots \\ f_1^{(n-1)} & f_2^{(n-1)} & \ldots & f_n^{(n-1)} \end{vmatrix}$$

Linearly Independent if Wronskian is non-zero:

Linear non-homogeneous DE with constant coefficients?

$$ay'' + by' + cy = g(x)$$

(can be higher order too)

yes

Undetermined Coefficients

use table to find form...

$g(x)$	Form of y_p
1. 1 (any constant)	A
2. $5x+7$	$Ax+B$
3. $3x^2-2$	Ax^2+Bx+C
4. x^3-x+1	Ax^3+Bx^2+Cx+E
5. $\sin 4x$	$A\cos 4x + B\sin 4x$
6. $\cos 4x$	$A\cos 4x + B\sin 4x$
7. e^{5x}	Ae^{5x}
8. $(9x-2)e^{5x}$	$(Ax+B)e^{5x}$
9. x^2e^{5x}	$(Ax^2+Bx+C)e^{5x}$
10. $e^{3x}\sin 4x$	$Ae^{3x}\cos 4x + Be^{3x}\sin 4x$
11. $5x^2\sin 4x$	$(Ax^2+Bx+C)\cos 4x + (Ex^2+Fx+G)\sin 4x$
12. $xe^{3x}\cos 4x$	$(Ax+B)e^{3x}\cos 4x + (Cx+E)e^{3x}\sin 4x$

if a y_p term matches a y_c term (use smallest *n* that isn't absorbed) = 'absorbed' so multiply term by x^n

...take derivatives, plug into DE and solve for constants, then: $y = y_C + y_P$

no

Variation of Parameters

assume terms have same form as homogeneous solution multiplied by *u* functions:

$$y_C = C_1y_1 + C_2y_2$$

$$y_P = u_1(x)y_1 + u_2(x)y_2$$

Divide by leading term to get $f(x) = \dfrac{g(x)}{a_n(x)}$

then find *u* function derivatives with Cramer's rule:

$$u_1' = \frac{\begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}}, \quad u_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}}$$

then integrate to find *u* functions, and solution:

$$u_1 = \int u_1'\,dx, \qquad u_2 = \int u_2'\,dx$$

$$y_P = u_1y_1 + u_2y_2$$

$$y = y_C + y_P$$

Ch 5: Laplace methods

Derivatives…

$$\mathcal{L}\{0\} = 0$$

$$\mathcal{L}\{y(t)\} = Y(s)$$

$$\mathcal{L}\{y'(t)\} = sY(s) - y(0)$$

$$\mathcal{L}\{y''(t)\} = s^2Y(s) - sy(0) - y'(0)$$

$$\mathcal{L}\{y'''(t)\} = s^3Y(s) - s^2y(0) - sy'(0) - y(0)$$

$$\mathcal{L}\{u(t-a)\} = \frac{e^{-as}}{s}$$

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}\left[F(s)\right]$$

$$\mathcal{L}\{\delta(t-t_o)\} = e^{-st_0}$$

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a)$$

$$\mathcal{L}^{-1}\{F(s-a)\} = e^{at}f(t)$$

$$\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}F(s)$$

$$\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)u(t-a)$$

$$\mathcal{L}\{g(t)u(t-a)\} = e^{-as}\mathcal{L}\{g(t+a)\}$$

$f(t)$	$\mathscr{L}\{f(t)\} = F(s)$
1. 1	$\dfrac{1}{s}$
2. t	$\dfrac{1}{s^2}$
3. t^n	$\dfrac{n!}{s^{n+1}}$, n a positive integer
4. $t^{-1/2}$	$\sqrt{\dfrac{\pi}{s}}$
5. $t^{1/2}$	$\dfrac{\sqrt{\pi}}{2s^{3/2}}$
6. t^α	$\dfrac{\Gamma(\alpha + 1)}{s^{\alpha+1}}$, $\alpha > -1$
7. $\sin kt$	$\dfrac{k}{s^2 + k^2}$
8. $\cos kt$	$\dfrac{s}{s^2 + k^2}$
9. $\sin^2 kt$	$\dfrac{2k^2}{s(s^2 + 4k^2)}$
10. $\cos^2 kt$	$\dfrac{s^2 + 2k^2}{s(s^2 + 4k^2)}$
11. e^{at}	$\dfrac{1}{s - a}$
12. $\sinh kt$	$\dfrac{k}{s^2 - k^2}$
13. $\cosh kt$	$\dfrac{s}{s^2 - k^2}$
14. $\sinh^2 kt$	$\dfrac{2k^2}{s(s^2 - 4k^2)}$
15. $\cosh^2 kt$	$\dfrac{s^2 - 2k^2}{s(s^2 - 4k^2)}$
16. te^{at}	$\dfrac{1}{(s - a)^2}$
17. $t^n e^{at}$	$\dfrac{n!}{(s - a)^{n+1}}$, n a positive integer
18. $e^{at} \sin kt$	$\dfrac{k}{(s - a)^2 + k^2}$
19. $e^{at} \cos kt$	$\dfrac{s - a}{(s - a)^2 + k^2}$

$f(t)$	$\mathscr{L}\{f(t)\} = F(s)$
20. $e^{at} \sinh kt$	$\dfrac{k}{(s-a)^2 - k^2}$
21. $e^{at} \cosh kt$	$\dfrac{s-a}{(s-a)^2 - k^2}$
22. $t \sin kt$	$\dfrac{2ks}{(s^2 + k^2)^2}$
23. $t \cos kt$	$\dfrac{s^2 - k^2}{(s^2 + k^2)^2}$
24. $\sin kt + kt \cos kt$	$\dfrac{2ks^2}{(s^2 + k^2)^2}$
25. $\sin kt - kt \cos kt$	$\dfrac{2k^3}{(s^2 + k^2)^2}$
26. $t \sinh kt$	$\dfrac{2ks}{(s^2 - k^2)^2}$
27. $t \cosh kt$	$\dfrac{s^2 + k^2}{(s^2 - k^2)^2}$
28. $\dfrac{e^{at} - e^{bt}}{a - b}$	$\dfrac{1}{(s-a)(s-b)}$
29. $\dfrac{ae^{at} - be^{bt}}{a - b}$	$\dfrac{s}{(s-a)(s-b)}$
30. $1 - \cos kt$	$\dfrac{k^2}{s(s^2 + k^2)}$
31. $kt - \sin kt$	$\dfrac{k^3}{s^2(s^2 + k^2)}$
32. $\dfrac{a \sin bt - b \sin at}{ab(a^2 - b^2)}$	$\dfrac{1}{(s^2 + a^2)(s^2 + b^2)}$
33. $\dfrac{\cos bt - \cos at}{a^2 - b^2}$	$\dfrac{s}{(s^2 + a^2)(s^2 + b^2)}$
34. $\sin kt \sinh kt$	$\dfrac{2k^2 s}{s^4 + 4k^4}$
35. $\sin kt \cosh kt$	$\dfrac{k(s^2 + 2k^2)}{s^4 + 4k^4}$
36. $\cos kt \sinh kt$	$\dfrac{k(s^2 - 2k^2)}{s^4 + 4k^4}$
37. $\cos kt \cosh kt$	$\dfrac{s^3}{s^4 + 4k^4}$
38. $J_0(kt)$	$\dfrac{1}{\sqrt{s^2 + k^2}}$

$f(t)$	$\mathscr{L}\{f(t)\} = F(s)$
39. $\dfrac{e^{bt} - e^{at}}{t}$	$\ln \dfrac{s-a}{s-b}$
40. $\dfrac{2(1 - \cos kt)}{t}$	$\ln \dfrac{s^2 + k^2}{s^2}$
41. $\dfrac{2(1 - \cosh kt)}{t}$	$\ln \dfrac{s^2 - k^2}{s^2}$
42. $\dfrac{\sin at}{t}$	$\arctan\left(\dfrac{a}{s}\right)$
43. $\dfrac{\sin at \cos bt}{t}$	$\dfrac{1}{2}\arctan\dfrac{a+b}{s} + \dfrac{1}{2}\arctan\dfrac{a-b}{s}$
44. $\dfrac{1}{\sqrt{\pi t}} e^{-a^2/4t}$	$\dfrac{e^{-a\sqrt{s}}}{\sqrt{s}}$
45. $\dfrac{a}{2\sqrt{\pi t^3}} e^{-a^2/4t}$	$e^{-a\sqrt{s}}$
46. $\operatorname{erfc}\left(\dfrac{a}{2\sqrt{t}}\right)$	$\dfrac{e^{-a\sqrt{s}}}{s}$
47. $2\sqrt{\dfrac{t}{\pi}}\, e^{-a^2/4t} - a \operatorname{erfc}\left(\dfrac{a}{2\sqrt{t}}\right)$	$\dfrac{e^{-a\sqrt{s}}}{s\sqrt{s}}$
48. $e^{ab} e^{b^2 t} \operatorname{erfc}\left(b\sqrt{t} + \dfrac{a}{2\sqrt{t}}\right)$	$\dfrac{e^{-a\sqrt{s}}}{\sqrt{s}(\sqrt{s} + b)}$
49. $-e^{ab} e^{b^2 t} \operatorname{erfc}\left(b\sqrt{t} + \dfrac{a}{2\sqrt{t}}\right) + \operatorname{erfc}\left(\dfrac{a}{2\sqrt{t}}\right)$	$\dfrac{be^{-a\sqrt{s}}}{s(\sqrt{s} + b)}$
50. $e^{at} f(t)$	$F(s - a)$
51. $\mathscr{U}(t - a)$	$\dfrac{e^{-as}}{s}$
52. $f(t - a)\mathscr{U}(t - a)$	$e^{-as}F(s)$
53. $g(t)\mathscr{U}(t - a)$	$e^{-as}\mathscr{L}\{g(t + a)\}$
54. $f^{(n)}(t)$	$s^n F(s) - s^{(n-1)} f(0) - \cdots - f^{(n-1)}(0)$
55. $t^n f(t)$	$(-1)^n \dfrac{d^n}{ds^n} F(s)$
56. $\displaystyle\int_0^t f(\tau) g(t - \tau)\, d\tau$	$F(s)G(s)$
57. $\delta(t)$	1
58. $\delta(t - t_0)$	e^{-st_0}

Ch 8: Systems of Differential Equations

Homogeneous systems…

Finding eigenvalues: $\left| \vec{A} - \lambda \vec{I} \right| = 0$

Finding eigenvector for an eigenvalue: $\left(\vec{A} - \lambda \vec{I} \right) \vec{K} = \vec{0}$

Distinct real eigenvalues: $\vec{X}_C = C_1 \begin{bmatrix} k_{11} \\ k_{21} \end{bmatrix} e^{\lambda_1 t} + C_2 \begin{bmatrix} k_{12} \\ k_{22} \end{bmatrix} e^{\lambda_2 t}$

Repeated real eigenvalues: find 2nd eigenvalue using $\left(\vec{A} - \lambda \vec{I} \right) \vec{P} = \vec{K}$

$$\vec{X}_C = C_1 \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} e^{\lambda t} + C_2 \left(\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} t e^{\lambda t} + \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} e^{\lambda t} \right)$$

Complex conjugate eigenvalues:

use positive version $\lambda = \alpha + \beta i$ to find eigenvector $\vec{K} = \begin{bmatrix} a+bi \\ c+di \end{bmatrix}$

$$\vec{B}_1 = \operatorname{Re} \vec{K} = \begin{bmatrix} a \\ c \end{bmatrix} \qquad \vec{B}_2 = \operatorname{Im} \vec{K} = \begin{bmatrix} b \\ d \end{bmatrix}$$

$$\vec{X}_C = C_1 \left(\vec{B}_1 \cos \beta t - \vec{B}_2 \sin \beta t \right) e^{\alpha t} + C_2 \left(\vec{B}_2 \cos \beta t + \vec{B}_1 \sin \beta t \right) e^{\alpha t}$$

Non-Homogeneous systems…

$$\vec{X}' = \vec{A} \vec{X} + \vec{F}$$

<u>Method of Variation of Parameters</u>: $\vec{\Phi}$ = fundamental matrix (from $\vec{X}_C$) $\quad \vec{X}_P = \vec{\Phi} \int \vec{\Phi}^{-1} \vec{F} \, dt$

$$\vec{\Phi}^{-1} = \frac{1}{\det \vec{\Phi}} \vec{\Phi}^{T}$$

$$\vec{\Phi}^{T} = transpose = for\ 2x2\ reverse\ elements\ on\ diagonal,\ negate\ everything\ else$$

<u>Solving with initial condition</u>: $\vec{X} = \vec{\Phi}(t) \vec{\Phi}^{-1}(t_0) \vec{X}_0 + \vec{\Phi} \int_{t_0}^{t} \vec{\Phi}^{-1}(s) \vec{F}(s) \, ds$

Method of Undetermined Coefficients:

$$\vec{X}_p = \begin{bmatrix} \textit{function from table to match forms of terms of } F \\ \textit{function from table to match forms of terms of } F \end{bmatrix}$$

(must be the same form for all rows – so selected terms must cover all terms in all rows of F)

$g(x)$	Form of y_p
1. 1 (any constant)	A
2. $5x + 7$	$Ax + B$
3. $3x^2 - 2$	$Ax^2 + Bx + C$
4. $x^3 - x + 1$	$Ax^3 + Bx^2 + Cx + E$
5. $\sin 4x$	$A \cos 4x + B \sin 4x$
6. $\cos 4x$	$A \cos 4x + B \sin 4x$
7. e^{5x}	Ae^{5x}
8. $(9x - 2)e^{5x}$	$(Ax + B)e^{5x}$
9. x^2e^{5x}	$(Ax^2 + Bx + C)e^{5x}$
10. $e^{3x} \sin 4x$	$Ae^{3x} \cos 4x + Be^{3x} \sin 4x$
11. $5x^2 \sin 4x$	$(Ax^2 + Bx + C) \cos 4x + (Ex^2 + Fx + G) \sin 4x$
12. $xe^{3x} \cos 4x$	$(Ax + B)e^{3x} \cos 4x + (Cx + E)e^{3x} \sin 4x$

Then take derivative, plug into system of DEs, and solve for constants.

(Note: you need to multiply by extra *t*s if terms match any terms in X_C)