Differential Equations ** Practice Final Exam **

Solve using the integrating factor method.

#1a. $\frac{dy}{dx}+\frac{2}{x}y=\ln x$

$P(x)=\frac{2}{x}$

$IF=e^{\int P(x)dx}=e^{\int \frac{2}{x}dx}=e^{2\int \frac{1}{x}dx}=e^{2\ln x}=e^{\ln x^2}=x^2$

$x^2y=\int x^2\ln x\,dx$ $\quad u=\ln x \quad dv=x^2dx$

by parts: $\quad du=\frac{1}{x}dx \quad v=\int x^2dx=\frac{1}{3}x^3$

$uv-\int v\,du$

$\frac{1}{3}x^3\ln x-\int \frac{1}{3}x^3\frac{1}{x}dx$

$-\frac{1}{3}\int x^2dx$

$-\frac{1}{9}x^3$

$x^2y=\frac{1}{3}x^3\ln x-\frac{1}{9}x^3+C$

$\boxed{y=\frac{1}{3}x\ln x-\frac{1}{9}x+Cx^{-2}}$

#1b. $\frac{dy}{dx}-\frac{y}{x}=x^3$

$P(x)=-\frac{1}{x}$

$IF=e^{\int -\frac{1}{x}dx}=e^{-\ln x}=e^{\ln x^{-1}}=x^{-1}=\frac{1}{x}$

$\frac{1}{x}y=\int x^3\frac{1}{x}dx=\int x^2dx$

$\frac{1}{x}\cdot y=\frac{1}{3}x^3+C$

$\boxed{y=\frac{1}{3}x^4+Cx}$

Solve using the auxiliary equation method

#2a. $12y''-5y'-2y=0 \qquad y(0)=2 \qquad y'(0)=\frac{49}{12}$

$12m^2-5m-2=0$

$m=\frac{5\pm\sqrt{25+4(12)(2)}}{2(12)}=\frac{5\pm\sqrt{121}}{24}=\frac{5\pm 11}{24}=\frac{2}{3},\ -\frac{1}{4}$

$y=c_1e^{2/3x}+c_2e^{-1/4x} \rightarrow y'=\frac{2}{3}c_1e^{2/3x}-\frac{1}{4}c_2e^{-1/4x}$

$y(0)=2 \qquad y'(0)=49/12$

$2=c_1e^{2/3(0)}+c_2e^{-1/4(0)} \qquad \frac{49}{12}=\frac{2}{3}c_1e^{2/3(0)}-\frac{1}{4}c_2e^{-1/4(0)}$

$c_1+c_2=2 \qquad \frac{2}{3}c_1-\frac{1}{4}c_2=\frac{49}{12}$

$\begin{cases} c_1+c_2=2 \\ \frac{2}{3}c_1-\frac{1}{4}c_2=\frac{49}{12}\end{cases} \quad \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 2/3 & -1/4 & 49/12\end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & -3\end{array}\right]$

$c_1=5,\ c_2=-3$

$\boxed{y=5e^{2/3x}-3e^{-1/4x}}$

#2b. $y''+8y'+16y=0 \qquad y(0)=4 \qquad y'(0)=-15$

$m^2+8m+16=0$

$(m+4)(m+4)=0$

$m=-4$ repeated

$y=c_1e^{-4x}+c_2xe^{-4x} \rightarrow y'=-4c_1e^{-4x}+c_2x(-4e^{-4x})+e^{-4x}(c_2)$

$y(0)=4 \qquad y'(0)=-15$

$4=c_1e^{-4(0)}+c_2(0)e^{-4(0)} \qquad -15=-4c_1e^{-4(0)}-4c_2(0)e^{-4(0)}+c_2e^{-4(0)}$

$4=c_1 \qquad -15=-4(4)+c_2, \quad c_2=-15+16=1$

$\boxed{y=4e^{-4x}+xe^{-4x}}$

#2c. $y''-4y'+5y=0 \qquad y(0)=-2 \qquad y'(0)=-1$

$m^2-4m+5=0$

$m=\frac{4\pm\sqrt{16-4(1)(5)}}{2}=\frac{4\pm\sqrt{-4}}{2}=\frac{4\pm 2i}{2}=2\pm i$

$y=c_1e^{2x}\cos x+c_2e^{2x}\sin x \rightarrow y'=c_1e^{2x}(-\sin x)+\cos x(2c_1e^{2x})+c_2e^{2x}(\cos x)+\sin x(2c_2e^{2x})$

$y(0)=-2 \qquad y'(0)=-1$

$-2=c_1e^{2(0)}\cos(0)+c_2e^{2(0)}\sin(0) \qquad -1=-c_1e^{2(0)}\sin(0)+2c_1e^{2(0)}\cos(0)+c_2e^{2(0)}\cos(0)+2c_2e^{2(0)}\sin(0)$

$-2=c_1 \qquad -1=2c_1+c_2=2(-2)+c_2$

$c_2=-1+4=3$

$\boxed{y=-2e^{2x}\cos x+3e^{2x}\sin x}$

#3a. Solve $y'' - y' + \frac{1}{4}y = 3 + e^{\frac{1}{2}x}$ using the table method for RHS

y_c: $m^2 - m + \frac{1}{4} = 0$

$(m-\frac{1}{2})(m-\frac{1}{2}) = 0$

$m = 1/2$ repeated

$y_c = C_1 e^{1/2x} + C_2 x e^{1/2x}$

y_p: $y_p = A + \underset{\sim}{B}e^{1/2x}$ absorbed

$A + \underset{\sim}{Bx}e^{1/2x}$ absorbed

$y = A + \underline{Bx^2e^{1/2x}}$

$y' = Bx^2(\frac{1}{2}e^{1/2x}) + e^{1/2x}(2Bx)$

$y'' = Bx^2(\frac{1}{4}e^{1/2x}) + \frac{1}{2}e^{1/2x}(2Bx) + e^{1/2x}(2B) + 2Bx(\frac{1}{2}e^{1/2x})$

plug into DE:

$\left[\frac{1}{4}Bx^2e^{1/2x} + Bxe^{1/2x} + 2Be^{1/2x} + Bxe^{1/2x}\right] - \left[\frac{1}{2}Bx^2e^{1/2x} + 2Bxe^{1/2x}\right] + \frac{1}{4}\left[A + Bx^2e^{1/2x}\right] = 3 + e^{1/2x}$

combine like terms:

$(\underbrace{\frac{1}{4}B - \frac{1}{2}B + \frac{1}{4}B}_{0})x^2e^{1/2x} + (\underbrace{B + B - 2B}_{0})xe^{1/2x} + (2B)e^{1/2x} + (\frac{1}{4}A) = 3 + e^{1/2x}$

System $\begin{cases} 2B = 1 \\ \frac{1}{4}A = 3 \end{cases}$ $\quad B = 1/2$, $A = 12$

so $y_p = 12 + \frac{1}{2}x^2e^{1/2x}$

and $\boxed{y = C_1e^{1/2x} + C_2xe^{1/2x} + 12 + \frac{1}{2}x^2e^{1/2x}}$

#3b. Solve $y'' - 36y = 4e^{6x}$

$\underline{y_c}$: $m^2 - 36 = 0$

$(m-6)(m+6) = 0$

$m = 6 \quad m = -6$

$y_c = C_1 e^{6x} + C_2 e^{-6x}$

choose ~~one of these~~ to proceed...

~~table~~ — or — $\underline{\text{uvarsity}}$

$y_p = Ae^{6x}$ absorbed

$y = Axe^{6x}$

$y' = Ax(6e^{6x}) + e^{6x}A$

$y'' = Ax(36e^{6x}) + 6e^{6x}(A) + 6Ae^{6x}$

$= 36Axe^{6x} + 12Ae^{6x}$

plug in!

$[36Axe^{6x} + 12Ae^{6x}] - 36[Axe^{6x}] = 4e^{6x}$

$(\cancel{36A} - 36A)xe^{6x} + (12A)e^{6x} = 4e^{6x}$

$12A = 4$

$A = 4/12 = 1/3$

so $y_p = \frac{1}{3}xe^{6x}$

and $\boxed{y = C_1 e^{6x} + C_2 e^{-6x} + \frac{1}{3}xe^{6x}}$

↑ (recommend using table if possible)

$y_p = u_1 e^{6x} + u_2 e^{-6x}$

$$u_1' = \frac{\begin{vmatrix} 0 & e^{-6x} \\ 4e^{6x} & -6e^{-6x} \end{vmatrix}}{\begin{vmatrix} e^{6x} & e^{-6x} \\ 6e^{6x} & -6e^{-6x} \end{vmatrix}} = \frac{0-4}{-6-6} = \frac{-4}{-12} = \frac{1}{3}$$

$u_1 = \int \frac{1}{3}\,dx = \frac{1}{3}x$

$$u_2' = \frac{\begin{vmatrix} e^{6x} & 0 \\ 6e^{6x} & 4e^{6x} \end{vmatrix}}{\begin{vmatrix} e^{6x} & e^{-6x} \\ 6e^{6x} & -6e^{-6x} \end{vmatrix}} = \frac{4e^{12x} - 0}{-12} = -\frac{1}{3}e^{12x}$$

$u_2 = \int -\frac{1}{3}e^{12x}\,dx = -\frac{1}{3}\frac{e^{12x}}{12} = -\frac{1}{36}e^{12x}$

$y_p = u_1 e^{6x} + u_2 e^{-6x}$

$= \frac{1}{3}xe^{6x} - \frac{1}{36}e^{12x}e^{-6x}$

$= \frac{1}{3}xe^{6x} - \frac{1}{36}e^{6x}$

and $y = C_1 e^{6x} + C_2 e^{-6x} + \frac{1}{3}xe^{6x} - \frac{1}{36}e^{6x}$

(combine terms)

$\boxed{y = C_3 e^{6x} + C_2 e^{-6x} + \frac{1}{3}xe^{6x}}$

Find the Inverse Laplace Transform:

#4a. $L^{-1}\left\{\frac{e^{-2s}}{s^3}\right\} = \mathcal{L}^{-1}\{\underline{e^{-2s}}\frac{1}{s^3}\} = \frac{1}{2}\mathcal{L}^{-1}\{e^{-2s}\frac{2}{s^3}\} = \boxed{\frac{1}{2}(t-2)^2 u(t-2)}$

shift $a=2$
$t \to t-2$
$(u(t-2))$

$\frac{1}{s^3} \to$

table:
$t^n \Leftrightarrow \frac{n!}{s^{n+1}}$ $\quad t^2 \Leftrightarrow \frac{2!}{s^3}$
$n=2$

#4b. $L^{-1}\left\{\frac{e^{-\pi s}}{s^2+36}\right\} = \mathcal{L}^{-1}\{\underline{e^{-\pi s}}\frac{1}{s^2+36}\} = \frac{1}{6}\mathcal{L}^{-1}\{e^{-\pi s}\frac{6}{s^2+36}\}$

shift $a=\pi$
$t \to t-\pi$
(u)

$= \boxed{\frac{1}{6}\sin 6(t-\pi)\, u(t-\pi)}$

table:
$\sin kt \Leftrightarrow \frac{k}{s^2+k^2}$

#4c. $L^{-1}\left\{\frac{s}{\underline{s^2+4s+5}}\right\} = \mathcal{L}^{-1}\left\{\frac{s}{(s+2)^2+1}\right\} = \mathcal{L}^{-1}\left\{\frac{(s+2)-2}{(s+2)^2+1}\right\}$

$= \mathcal{L}^{-1}\left\{\frac{(s+2)}{(s+2)^2+1}\right\} - 2\mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2+1}\right\}$

s^2+4s+5
doesn't factor
so complete the square!

$s^2+4s+4+5-4$
$(s+2)^2+1$

table: $\frac{s}{s^2+1} \Leftrightarrow \cos t$ $\quad \frac{1}{s^2+1} \Leftrightarrow \sin t$

but shifted $s+2 = s-(-2)$
$a=-2$

so multiply by e^{-2t}

$= \boxed{e^{-2t}\cos t - 2e^{-2t}\sin t}$

Solve this Exact form DE: #5. Solve this Exact form DE.
(leave solution in implicit form – do not solve for y)

#5a. $\overset{M}{(5x+4y)}dx+\overset{N}{(4x-8y^3)}dy=0$

don't need to check if exact, problem says it is

$f_x = 5x+4y$

$f = \int f_x dx = \int (5x+4y)dx = \frac{5}{2}x^2 + 4xy + g(y)$

$f_y = \frac{\partial}{\partial y}\left[\frac{5}{2}x^2+4xy+g(y)\right] = 0 + 4x + g'(y) \overset{\text{must}}{=} 4x - 8y^3$

so $g'(y) = -8y^3$

$g(y) = \int g'(y)dy = \int -8y^3 dy = -8\frac{y^4}{4} = -2y^4$

so $f = \frac{5}{2}x^2 + 4xy - 2y^4$

and solution is:

$$\boxed{\frac{5}{2}x^2 + 4xy - 2y^4 = C}$$

#5b. $\overset{M}{(3x^2y+e^y)}dx+\overset{N}{(x^3+xe^y-2y)}dy=0$

$f_x = 3x^2y + e^y$

$f = \int (3x^2y + e^y)dx = x^3y + e^y x + g(y)$

$f_y = x^3 + e^y x + g'(y) \overset{\text{must}}{=} x^3 + xe^y - 2y$

so $g'(y) = -2y$

$g(y) = \int -2y\, dy = -y^2$

so $f = x^3y + xe^y - y^2$

and solution is: $\boxed{x^3y + xe^y - y^2 = C}$

#6. Solve the DE system using Eigenvalues/Eigenvectors: $\frac{dx}{dt} = -4x + 2y$, $\frac{dy}{dt} = -\frac{5}{2}x + 2y$

$\vec{X} = \begin{bmatrix} x \\ y \end{bmatrix}$ $\quad \vec{X}' = \begin{bmatrix} -4 & 2 \\ -5/2 & 2 \end{bmatrix} \vec{X}$

$\begin{vmatrix} -4-\lambda & 2 \\ -5/2 & 2-\lambda \end{vmatrix} = 0$

$(-4-\lambda)(2-\lambda) + 2(\frac{5}{2}) = 0$

$\lambda^2 + 2\lambda - 8 + 5 = 0$

$\lambda^2 + 2\lambda - 3 = 0$

$(\lambda - 1)(\lambda + 3) = 0$

$\lambda = 1 \quad \lambda = -3$

$\lambda = 1$

$\left[\begin{array}{cc|c} -5 & 2 & 0 \\ -5/2 & 1 & 0 \end{array}\right]$

rref

$\left[\begin{array}{cc|c} 1 & -2/5 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$K_1 - 2/5 K_2 = 0$

$K_1 = 2/5 K_2$

$(2/5 K_2, K_2) \quad \begin{bmatrix} 2 \\ 5 \end{bmatrix}$

$\lambda = -3$

$\left[\begin{array}{cc|c} -1 & 2 & 0 \\ -5/2 & 5 & 0 \end{array}\right]$

rref

$\left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$K_1 - 2K_2 = 0$

$K_1 = 2K_2$

$(2K_2, K_2) \quad \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$$\boxed{\vec{X} = c_1 \begin{bmatrix} 2 \\ 5 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-3t}}$$