DiffEq - Ch 7 - Required Practice **Name:** full solutions

7.1

#1. Use the integral definition of the Laplace Transform to find $\mathscr{L}\{f(t)\}$

$$f(t)=\begin{cases}-1, & 0\le t<1\\ 1, & t\ge 1\end{cases}$$

$$\mathcal{L}\{f(t)\} = \int_0^1 e^{-st}(-1)\,dt + \int_1^\infty e^{-st}(1)\,dt$$

$$= -\int_0^1 e^{-st} + \lim_{b\to\infty}\int_1^b e^{-st}\,dt$$

$$= -\frac{1}{(-s)}\left[e^{-st}\right]_0^1 + \lim_{b\to\infty}\frac{1}{(-s)}\left[e^{-st}\right]_1^b$$

$$= \left[\frac{1}{s}e^{-s(1)} - \frac{1}{s}e^{0}\right] + \lim_{b\to\infty}\left[-\frac{1}{s}e^{-sb} + \frac{1}{s}e^{-s}\right]$$

$$= \frac{1}{s}e^{-s} - \frac{1}{s} + \lim_{b\to\infty}\left(-\frac{1}{s}\right)e^{-sb} + \frac{1}{s}e^{-s}$$

$$= \frac{2}{s}e^{-s} - \frac{1}{s} - \frac{1}{s}\cancel{\frac{1}{e^{s(\infty)}}}^{\,0}$$

$$= \boxed{\frac{2}{s}e^{-s} - \frac{1}{s}}$$

#2. Use the integral definition of the Laplace Transform to find $\mathscr{L}\{f(t)\}$

$$f(t)=\begin{cases}t, & 0\le t<1\\ 1, & t\ge 1\end{cases}$$

$$\mathcal{L}\{f(t)\} = \int_0^1 e^{-st}t\,dt + \int_1^\infty e^{-st}(1)\,dt$$

by parts

$u=t \quad dv=e^{-st}dt$

$\frac{du}{dt}=1 \quad \int dv=\int e^{-st}dt$

$du=dt \quad v=-\frac{1}{s}e^{-st}$

$uv-\int v\,du$

$$\left[t\left(-\frac{1}{s}\right)e^{-st} - \int -\frac{1}{s}e^{-st}dt\right]_0^1$$

$$\left[-\frac{1}{s}te^{-st} + \frac{1}{s}\int e^{-st}dt\right]_0^1$$

$$\left[-\frac{1}{s}te^{-st} - \frac{1}{s^2}e^{-st}\right]_0^1$$

$$+\lim_{b\to\infty}\int_1^b e^{-st}\,dt$$

$$+\lim_{b\to\infty}\left(-\frac{1}{s}\right)\left[e^{-st}\right]_1^b$$

$$\left[-\frac{1}{s}te^{-st} - \frac{1}{s^2}e^{-st}\right]_0^1 + \lim_{b\to\infty}\left(-\frac{1}{s}\right)\left[e^{-st}\right]_1^b$$

$$\left[-\frac{1}{s}(1)e^{-s} - \frac{1}{s^2}e^{-s}\right] - \left[-\cancel{\frac{1}{s}(0)e^{0}} - \frac{1}{s^2}e^{0}\right]$$

$$+\lim_{b\to\infty}\left[-\frac{1}{s}e^{-sb} + \frac{1}{s}e^{-s(1)}\right]$$

$$= -\cancel{\frac{1}{s}e^{-s}} - \frac{1}{s^2}e^{-s} + \frac{1}{s^2} + \lim_{b\to\infty}\left(-\frac{1}{s}\right)e^{-sb} + \cancel{\frac{1}{s}e^{-s}}$$

$$-\frac{1}{s}\cancel{\frac{1}{e^{s(\infty)}}}^{\,0}$$

$$= \boxed{-\frac{1}{s^2}e^{-s} + \frac{1}{s^2}}$$

#3. Use the integral definition of the Laplace Transform to find $\mathscr{L}\{f(t)\}$. Then use the table to verify your answer.

$f(t) = e^{t+7}$

$$\mathcal{L}\{e^{t+7}\} = \int_0^\infty e^{-st} e^{(t+7)} dt = \int_0^\infty e^{(1-s)t} e^{7} dt$$

$$= e^7 \int_0^\infty e^{(1-s)t} dt = e^7 \left[\frac{e^{(1-s)t}}{1-s}\right]_0^\infty$$

$$= \lim_{b\to\infty} \frac{e^7}{1-s} e^{(1-s)b} - \frac{e^7}{1-s} e^{(1-s)0}$$

(for $s \geq 1$)

$$= 0 - \frac{e^7}{1-s}$$

$$= \boxed{\frac{e^7}{s-1}}$$

table: $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$

$$\mathcal{L}\{e^{(t+7)}\}$$

$$= \mathcal{L}\{e^t e^7\}$$

$$= e^7 \mathcal{L}\{e^t\} = e^7 \frac{1}{s-1} = \boxed{\frac{e^7}{s-1}}$$

$a=1$

#4. Use the integral definition of the Laplace Transform to find $\mathscr{L}\{f(t)\}$. Then use the table to verify your answer.

$f(t) = te^{4t}$

$$\mathcal{L}\{te^{4t}\} = \int_0^\infty e^{-st} te^{4t} dt = \int_0^\infty te^{(4-s)t} dt$$

$$= \lim_{b\to\infty} \left[\frac{1}{4-s} te^{(4-s)t} - \frac{1}{(4-s)^2} e^{(4-s)t}\right]_0^b$$

$$= \lim_{b\to\infty} \left[\frac{1}{4-s} be^{(4-s)b} - \frac{1}{(4-s)^2} e^{(4-s)b}\right] - \left[\frac{1}{4-s}(0)e^0 - \frac{1}{(4-s)^2} e^0\right]$$

(for $s > 4$)

$$\frac{1}{4-s}(0 \cdot 0) - (0) - 0 + \frac{1}{(4-s)^2} = \boxed{\frac{1}{(4-s)^2}}$$

by parts: $\int te^{(4-s)t} dt$ $\quad u = t \quad dv = e^{(4-s)t} dt$

$uv - \int v\,du$ $\quad \frac{du}{dt} = 1 \quad \int dv = \int e^{(4-s)t} dt$

$\quad du = dt \quad v = \frac{e^{(4-s)t}}{4-s}$

$$\frac{1}{4-s} te^{(4-s)t} - \frac{1}{4-s}\int e^{(4-s)t} dt$$

$$\frac{1}{4-s} te^{(4-s)t} - \frac{1}{(4-s)} \frac{e^{(4-s)t}}{(4-s)}$$

$$\frac{1}{4-s} te^{(4-s)t} - \frac{1}{(4-s)^2} e^{(4-s)t}$$

by table: $\mathcal{L}\{te^{at}\} = \frac{1}{(s-a)^2}$

$$\mathcal{L}\{te^{4t}\} = \boxed{\frac{1}{(s-4)^2}}$$

$(a=4)$

$$\left(\begin{array}{l}(s-4) = -(4-s) \\ (s-4)^2 = (-(4-s))^2 \\ (s-4)^2 = (4-s)^2\end{array}\right)$$

#5. Use theorems and the table to find $\mathscr{L}\{f(t)\}$

$f(t)=2t^4$

$\mathcal{L}\{2t^4\}$

$2\mathcal{L}\{t^4\}$ table: $\mathcal{L}\{t^n\}=\frac{n!}{s^{n+1}}$

$n=4$

$2\left(\frac{4!}{s^5}\right)$

$\boxed{\frac{48}{s^5}}$

#8. Use theorems and the table to find $\mathscr{L}\{f(t)\}$

$f(t)=(1+e^{2t})^2 = 1+2e^{2t}+e^{4t}$

$\mathcal{L}\{1\}+2\mathcal{L}\{e^{2t}\}+\mathcal{L}\{e^{4t}\}$

table: $\left(\frac{1}{s}\right)$ $\left(\frac{1}{s-a}\right)$ $\left(\frac{1}{s-a}\right)$

$a=2$ $a=4$

$\boxed{\frac{1}{s}+2\frac{1}{s-2}+\frac{1}{s-4}}$

#6. Use theorems and the table to find $\mathscr{L}\{f(t)\}$

$f(t)=t^2+6t-3$

$\mathcal{L}\{t^2\}+6\mathcal{L}\{t\}-3\mathcal{L}\{1\}$

table: $(n=2)$ table: table:

$\mathcal{L}\{t^n\}$ $\mathcal{L}\{t\}$ $\mathcal{L}\{1\}$

$=\left(\frac{n!}{s^{n+1}}\right)$ $=\left(\frac{1}{s^2}\right)$ $\left(\frac{1}{s}\right)$

$\frac{2!}{s^3}+6\frac{1}{s^2}-3\frac{1}{s}$

$\boxed{2\frac{1}{s^3}+6\frac{1}{s^2}-3\frac{1}{s}}$

#9. Use theorems and the table to find $\mathscr{L}\{f(t)\}$

$f(t)=4t^2-5\sin(3t)$

$4\mathcal{L}\{t^2\}-5\mathcal{L}\{\sin(3t)\}$

table: $\left(\frac{n!}{s^{n+1}}\right)$ $\left(\frac{k}{s^2+k^2}\right)$

$(n=2)$ $(k=3)$

$\boxed{4\frac{2!}{s^3}-5\frac{3}{s^2+9}}$

#7. Use theorems and the table to find $\mathscr{L}\{f(t)\}$

$f(t)=1+e^{4t}$

$\mathcal{L}\{1\}+\mathcal{L}\{e^{4t}\}$

table: $\left(\frac{1}{s}\right)$ $\left(\frac{1}{s-a}\right)$

$a=4$

$\boxed{\frac{1}{s}+\frac{1}{s-4}}$

7.2 day 1

#1. Use theorems and the table to find $\mathscr{L}^{-1}\left\{\frac{1}{s^3}\right\}$

table: $\mathscr{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\} = t^n$

$n=2$ $\quad \frac{1}{2!}\mathscr{L}^{-1}\left\{\frac{2!}{s^3}\right\} = \boxed{\frac{1}{2}t^2}$

#2. Use theorems and the table to find $\mathscr{L}^{-1}\left\{\frac{1}{s^2}-\frac{48}{s^5}\right\}$

$\mathscr{L}^{-1}\left\{\frac{1}{s^2}\right\} - 2\mathscr{L}^{-1}\left\{\frac{4!}{s^5}\right\}$

$\boxed{t - 2t^4}$

#3. Use theorems and the table to find $\mathscr{L}^{-1}\left\{\frac{(s+1)^3}{s^4}\right\}$

$(s+1)^3 = 1(s)^3(1)^0 + 3(s)^2(1)^1 + 3(s)^1(1)^2 + 1(s)^0(1)^3$

$\mathscr{L}^{-1}\left\{\frac{s^3}{s^4} + 3\frac{s^2}{s^4} + \frac{3s}{s^4} + \frac{1}{s^4}\right\}$

1
1 1
1 2 1
1 3 3 1

$\mathscr{L}^{-1}\left\{\frac{1}{s}\right\} + 3\mathscr{L}^{-1}\left\{\frac{1}{s^2}\right\} + 3\mathscr{L}^{-1}\left\{\frac{1}{s^3}\right\} + \mathscr{L}^{-1}\left\{\frac{1}{s^4}\right\}$

$1 + 3t + 3\frac{1}{2!}\mathscr{L}^{-1}\left\{\frac{2!}{s^3}\right\} + \frac{1}{3!}\mathscr{L}^{-1}\left\{\frac{3!}{s^4}\right\}$

$\boxed{1 + 3t + \frac{3}{2}t^2 + \frac{1}{6}t^3}$

#4. Use theorems and the table to find $\mathscr{L}^{-1}\left\{\frac{1}{s^2}-\frac{1}{s}+\frac{1}{s-2}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{s^2}\right\} - \mathscr{L}^{-1}\left\{\frac{1}{s}\right\} + \mathscr{L}^{-1}\left\{\frac{1}{s-2}\right\}$

$\boxed{t - 1 + e^{2t}}$

#5. Use theorems and the table to find $\mathscr{L}^{-1}\left\{\frac{1}{4s+1}\right\} = \mathscr{L}^{-1}\left\{\frac{(1/4)}{s+(\frac{1}{4})}\right\} = \frac{1}{4}\mathscr{L}^{-1}\left\{\frac{1}{s+(1/4)}\right\}$

$\boxed{\frac{1}{4}e^{-\frac{1}{4}t}}$

#6. Use theorems and the table to find

$\mathscr{L}^{-1}\left\{\frac{5}{s^2+49}\right\} = \frac{5}{7}\mathscr{L}^{-1}\left\{\frac{7}{s^2+49}\right\}$

$\boxed{\frac{5}{7}\sin(7t)}$

#7. Use theorems and the table to find

$\mathscr{L}^{-1}\left\{\frac{4s}{4s^2+1}\right\} = \mathscr{L}^{-1}\left\{\frac{s}{s^2+1/4}\right\}$

$\boxed{\cos(\frac{1}{2}t)}$

#8. Use theorems and the table to find

$\mathscr{L}^{-1}\left\{\frac{2s-6}{s^2+9}\right\}$ ~~Not ok to factor and cancel~~, instead: $2\mathscr{L}^{-1}\left\{\frac{s}{s^2+9}\right\} - 2\mathscr{L}^{-1}\left\{\frac{3}{s^2+9}\right\}$

$\boxed{2\cos(3t) - 2\sin(3t)}$

#9. Use theorems and the table to find

$\mathscr{L}^{-1}\left\{\frac{1}{s^2+3s}\right\}$ $\frac{1}{s^2+3s} = \frac{1}{s(s+3)} = \frac{A}{s} + \frac{B}{s+3}$

$= \frac{1}{3}\frac{1}{s} - \frac{1}{3}\frac{1}{s+3}$

$A(s+3) + B(s) = 1$, $As + 3A + Bs = 1$

$(A+B)s + (3A) = (0)s + (1)$

$\begin{cases} A+B=0 \\ 3A=1 \end{cases}$ $A = 1/3$, $B = -1/3$

$\frac{1}{3}\mathscr{L}^{-1}\left\{\frac{1}{s}\right\} - \frac{1}{3}\mathscr{L}^{-1}\left\{\frac{1}{s+3}\right\}$

$\frac{1}{3}(1) - \frac{1}{3}e^{-3t} = \boxed{\frac{1}{3} - \frac{1}{3}e^{-3t}}$

#10. Use theorems and the table to find

$\mathscr{L}^{-1}\left\{\frac{s}{s^2+2s-3}\right\}$ $\frac{s}{(s+3)(s-1)} = \frac{A}{s+3} + \frac{B}{s-1}$

$= \frac{3}{4}\frac{1}{s+3} + \frac{1}{4}\frac{1}{s-1}$

$A(s-1) + B(s+3) = s$, $As - A + Bs + 3B = s$

$(A+B)s + (-A+3B) = (1)s + (0)$

$\begin{cases} A+B=1 \\ -A+3B=0 \end{cases}$ $\left[\begin{array}{cc|c} 1 & 1 & 1 \\ -1 & 3 & 0 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{cc|c} 1 & 0 & 3/4 \\ 0 & 1 & 1/4 \end{array}\right]$

$\frac{3}{4}\mathscr{L}^{-1}\left\{\frac{1}{s+3}\right\} + \frac{1}{4}\mathscr{L}^{-1}\left\{\frac{1}{s-1}\right\}$

$\boxed{\frac{3}{4}e^{-3t} + \frac{1}{4}e^{t}}$

#11. Use theorems and the table to find

$\mathscr{L}^{-1}\left\{\dfrac{0.9s}{(s-0.1)(s+0.2)}\right\}$ $\dfrac{0{,}9s}{(s-{,}1)(s+{,}2)} = \dfrac{A}{s-{,}1} + \dfrac{B}{s+{,}2}$ $A(s+{,}2)+B(s-{,}1)$, $As+{,}2A+Bs-{,}1B = 0{,}9s$

$(A+B)s + ({,}2A-{,}1B) = ({,}9)s + (0)$

$\begin{cases} A+B = {,}9 \\ {,}2A - {,}1B = 0 \end{cases}$ $\left[\begin{array}{cc|c} 1 & 1 & {,}9 \\ {,}2 & -{,}1 & 0 \end{array}\right]$

$= 0{,}3\left(\frac{1}{s-{,}1}\right) + 0{,}6\left(\frac{1}{s+{,}2}\right)$

rref

$\left[\begin{array}{cc|c} 1 & 0 & 0{,}3 \\ 0 & 1 & 0{,}6 \end{array}\right]$

$0{,}3\,\mathscr{L}^{-1}\left\{\frac{1}{s-0{,}1}\right\} + 0{,}6\,\mathscr{L}^{-1}\left\{\frac{1}{s+0{,}2}\right\}$

$\boxed{0{,}3e^{0{,}1t} + 0{,}6\,e^{-0{,}2t}}$

#12. Use theorems and the table to find

$\mathscr{L}^{-1}\left\{\dfrac{s}{(s-2)(s-3)(s-6)}\right\}$ $\dfrac{A}{s-2} + \dfrac{B}{s-3} + \dfrac{C}{s-6}$

online PFE:

$= \frac{1}{2}\left(\frac{1}{s-2}\right) - \left(\frac{1}{s-3}\right) + \frac{1}{2}\left(\frac{1}{s-6}\right)$

$\frac{1}{2}\mathscr{L}^{-1}\left\{\frac{1}{s-2}\right\} - \mathscr{L}^{-1}\left\{\frac{1}{s-3}\right\} + \frac{1}{2}\mathscr{L}^{-1}\left\{\frac{1}{s-6}\right\}$

$\boxed{\frac{1}{2}e^{2t} - e^{3t} + \frac{1}{2}e^{6t}}$

#13. Use theorems and the table to find

$\mathscr{L}^{-1}\left\{\dfrac{1}{s^3+5s}\right\}$ online PFE: $\dfrac{1}{s(s^2+5)} = \frac{1}{5}\frac{1}{s} - \frac{1}{5}\frac{s}{s^2+5}$

$\frac{1}{5}\mathscr{L}^{-1}\left\{\frac{1}{s}\right\} - \frac{1}{5}\mathscr{L}^{-1}\left\{\frac{s}{s^2+(\sqrt{5})^2}\right\}$

$\frac{1}{5}(1) - \frac{1}{5}\cos(\sqrt{5}t)$

$\boxed{\frac{1}{5} - \frac{1}{5}\cos(\sqrt{5}t)}$

#14. Use theorems and the table to find

$$\mathscr{L}^{-1}\left\{\frac{2s-4}{(s^2+s)(s^2+1)}\right\}$$

online PFE: $-4\frac{1}{s}+3\frac{1}{s+1}+\frac{s+3}{s^2+1}$

$$-4\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}+3\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}+\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\}+3\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$-4(1)+3e^{-t}+\cos(t)+3\sin(t)$$

$$\boxed{-4+3e^{-t}+\cos(t)+3\sin(t)}$$

#15. Use theorems and the table to find

$$\mathscr{L}^{-1}\left\{\frac{1}{(s^2+1)(s^2+4)}\right\}$$

online PFE: $\frac{1}{3}\frac{1}{s^2+1}-\frac{1}{3}\frac{1}{s^2+4}$

$$\frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}-\frac{1}{6}\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\}$$

$$\boxed{\frac{1}{3}\sin(t)-\frac{1}{6}\sin(2t)}$$

7.2 day 2

#1. Use the Laplace transform to solve the initial-value problem:

$$\frac{dy}{dt} - y = 1, \quad y(0) = 0$$

$[sY(s) - y(0)] - Y(s) = \frac{1}{s}$

$sY(s) - 0 - Y(s) = \frac{1}{s}$

$(s-1)Y(s) = \frac{1}{s}$

$Y(s) = \frac{1}{s(s-1)}$

online PFE: $\frac{1}{s(s-1)} = -\frac{1}{s} + \frac{1}{s-1}$

$Y(s) = -\frac{1}{s} + \frac{1}{s-1}$

$\mathcal{L}^{-1}$:

$y(t) = -\mathcal{L}^{-1}\{\frac{1}{s}\} + \mathcal{L}^{-1}\{\frac{1}{s-1}\}$

$y(t) = -(1) + e^{t}$

$\boxed{y(t) = -1 + e^{t}}$

#2. Use the Laplace transform to solve the initial-value problem:

$$y' + 6y = e^{4t}, \quad y(0) = 2$$

$[sY(s) - y(0)] + 6Y(s) = \frac{1}{s-4}$

$(s+6)Y(s) - 2 = \frac{1}{s-4}$

$(s+6)Y(s) = 2 + \frac{1}{s-4}$

$Y(s) = \frac{2}{(s+6)} + \frac{1}{(s-4)(s+6)}$

online PFE: $\frac{1}{(s-4)(s+6)} = \frac{1}{10}\frac{1}{s-4} - \frac{1}{10}\frac{1}{s+6}$

$Y(s) = 2\frac{1}{s+6} + \frac{1}{10}\frac{1}{s-4} - \frac{1}{10}\frac{1}{s+6}$

$Y(s) = \frac{19}{10}\frac{1}{s+6} + \frac{1}{10}\frac{1}{s-4}$

$\mathcal{L}^{-1}$:

$\boxed{y(t) = \frac{19}{10}e^{-6t} + \frac{1}{10}e^{4t}}$

#3. Use the Laplace transform to solve the initial-value problem:

$$y' - y = 2\cos(5t), \quad y(0) = 0$$

$[sY(s) - y_0] - Y(s) = \mathcal{L}\{2\cos(5t)\}$

$sY(s) - 0 - Y(s) = 2\frac{s}{s^2+25}$

$(s-1)Y(s) = 2\frac{s}{s^2+25}$

online PFE: $\frac{2s}{(s-1)(s^2+25)} = \frac{1}{13}\frac{1}{s-1} + \frac{-s+25}{13(s^2+25)}$

$Y(s) = \frac{2s}{(s-1)(s^2+25)} = \frac{1}{13}\frac{1}{s-1} - \frac{1}{13}\frac{s}{s^2+25} + \frac{25}{13}\frac{1}{s^2+25}$

$Y(s) = \frac{1}{13}\frac{1}{s-1} - \frac{1}{13}\frac{s}{s^2+25} + \frac{5}{13}\frac{5}{s^2+25}$

$\mathcal{L}^{-1}$:

$\boxed{y(t) = \frac{1}{13}e^{t} - \frac{1}{13}\cos(5t) + \frac{5}{13}\sin(5t)}$

#4. Use the Laplace transform to solve the initial-value problem:

$y''+5y'+4y=0, \quad y(0)=1, \quad y'(0)=0$

$[s^2Y(s)-sy(0)-y'(0)]+5[sY(s)-y(0)]+4Y(s)=0$

$s^2Y(s)-s(1)-0+5sY(s)-5(1)+4Y(s)=0$

$(s^2+5s+4)Y(s)-s-5=0$

$(s^2+5s+4)Y(s)=s+5$

$Y(s)=\frac{s+5}{s^2+5s+4}=\frac{4}{3}\frac{1}{s+1}-\frac{1}{3}\frac{1}{s+4}$

Online PFE:

$\frac{s+5}{s^2+5s+4}=\frac{4}{3}\frac{1}{s+1}-\frac{1}{3}\frac{1}{s+4}$

$\mathcal{L}^{-1}$:

$\boxed{y(t)=\frac{4}{3}e^{-t}-\frac{1}{3}e^{-4t}}$

#5. Use the Laplace transform to solve the initial-value problem:

$y''+y=\sqrt{2}\sin(\sqrt{2}\,t), \quad y(0)=10, \quad y'(0)=0$

$[s^2Y(s)-sy(0)-y'(0)]+Y(s)=\mathcal{L}\{\sqrt{2}\sin(\sqrt{2}t)\}$

$s^2Y(s)-s(10)-0+Y(s)=\sqrt{2}\frac{\sqrt{2}}{s^2+2}$

$(s^2+1)Y(s)-10s=\frac{2}{s^2+2}$

$Y(s)=\frac{10s}{(s^2+1)}+\frac{2}{(s^2+1)(s^2+2)}$

$Y(s)=10\frac{s}{s^2+1}+2\frac{1}{s^2+1}-\frac{2}{\sqrt{2}}\frac{\sqrt{2}}{s^2+2}$

Online PFE:

$\frac{2}{(s^2+1)(s^2+2)}=\frac{2}{s^2+1}-\frac{2}{s^2+2}$

$\mathcal{L}^{-1}$:

$\boxed{y(t)=10\cos(t)+2\sin(t)-\frac{2}{\sqrt{2}}\sin(\sqrt{2}t)}$

#6. Use the Laplace transform to solve the initial-value problem:

$2y''' + 3y'' - 3y' - 2y = e^{-t}$

$y(0) = 0, \quad y'(0) = 0, \quad y''(0) = 1$

$$2[s^3Y(s) - s^2y(0) - sy'(0) - y''(0)] + 3[s^2Y(s) - sy(0) - y'(0)] - 3(sY(s) - y(0)] - 2Y(s) = \mathcal{L}\{e^{-t}\}$$

$$2s^3Y(s) - 2s^2(0) - 2s(0) - 2(1) + 3s^2Y(s) - 3s(0) + 3(0) - 3sY(s) + 3(0) - 2Y(s) = \frac{1}{s+1}$$

$$(2s^3 + 3s^2 - 3s - 2)Y(s) - 2 = \frac{1}{s+1}$$

$$(2s^3 + 3s^2 - 3s - 2)Y(s) = 2 + \frac{1}{s+1}$$

$$Y(s) = \frac{2}{2s^3+3s^2-3s-2} + \frac{1}{(s+1)(2s^3+3s^2-3s-2)}$$

online PFE:

$$Y(s) = \left(\frac{2}{9}\frac{1}{s-1} - \frac{8}{9}\frac{1}{2s+1} + \frac{2}{9}\frac{1}{s+2}\right) + \left(\frac{1}{2}\frac{1}{s+1} + \frac{1}{18}\frac{1}{s-1} - \frac{8}{9}\frac{1}{2s+1} - \frac{1}{9}\frac{1}{s+2}\right)$$

$$Y(s) = \frac{1}{2}\frac{1}{s+1} + \frac{5}{18}\frac{1}{s-1} - \frac{16}{9}\frac{1}{2s+1} + \frac{1}{9}\frac{1}{s+2}$$

$$Y(s) = \frac{1}{2}\frac{1}{s+1} + \frac{5}{18}\frac{1}{s-1} - \frac{16}{18}\frac{1}{s+1/2} + \frac{1}{9}\frac{1}{s+2}$$

$\left(\frac{8}{9}\right)$

$\mathcal{L}^{-1}$:

$$\boxed{y(t) = \frac{1}{2}e^{-t} + \frac{5}{18}e^{t} - \frac{8}{9}e^{-1/2t} + \frac{1}{9}e^{-2t}}$$

7.3 day 1

#1. Evaluate the Laplace transform to find $F(s)$

$\mathscr{L}\{te^{10t}\}$ $\quad e^{10t}$: shift $s \to s-10$

$\mathscr{L}\{t\} = \frac{1}{s^2}$

$$\boxed{\mathscr{L}\{te^{10t}\} = \frac{1}{(s-10)^2}}$$

#2. Evaluate the Laplace transform to find $F(s)$

$\mathscr{L}\{t^3e^{-2t}\}$ $\quad e^{-2t}$: shift $s \to s+2$

$\mathscr{L}\{t^3\} = \frac{3!}{s^4}$

$$\boxed{\mathscr{L}\{t^3e^{-2t}\} = \frac{3!}{(s+2)^4}}$$

#3. Evaluate the Laplace transform to find $F(s)$

$\mathscr{L}\{t(e^t + e^{2t})^2\} = \mathscr{L}\{t(e^{2t}+2e^{3t}+e^{4t})\}$

$= \mathscr{L}\{te^{2t}\} + 2\mathscr{L}\{te^{3t}\} + \mathscr{L}\{te^{4t}\}$

e^{2t}: shift $s \to s-2$ $\quad e^{3t}$: shift $s \to s-3$ $\quad e^{4t}$: shift $s \to s-4$

$\mathscr{L}\{t\} = \frac{1}{s^2}$

$$\boxed{\frac{1}{(s-2)^2} + 2\frac{1}{(s-3)^2} + \frac{1}{(s-4)^2}}$$

#4. Evaluate the Laplace transform to find $F(s)$

$\mathscr{L}\{e^t \sin(3t)\}$ $\quad e^t$: shift, $s \to s-1$

$\mathscr{L}\{\sin(3t)\} = \frac{3}{s^2+9}$

$$\boxed{\mathscr{L}\{e^t\sin(3t)\} = \frac{3}{(s-1)^2+9}}$$

#5. Evaluate the Laplace transform to find $F(s)$

$\mathscr{L}\{(1-e^t+3e^{-4t})\cos(5t)\}$

$\mathscr{L}\{\cos(5t)\} - \mathscr{L}\{e^t\cos(5t)\} + 3\mathscr{L}\{e^{-4t}\cos(5t)\}$

$\frac{s}{s^2+25}$ $\quad e^t$: shift $s \to s-1$ $\quad e^{-4t}$: shift $s \to s+4$

$$\boxed{\frac{s}{s^2+25} - \frac{(s-1)}{(s-1)^2+25} + 3\frac{(s+4)}{(s+4)^2+25}}$$

#6. Evaluate the Inverse Laplace transform to find $f(t)$: $\mathscr{L}^{-1}\left\{\dfrac{1}{(s+2)^3}\right\}$

$s+2$: shift e^{-2t}, $\mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\}$

$\frac{1}{2!}\mathcal{L}^{-1}\left\{\frac{2!}{s^3}\right\}$

$\frac{1}{2}t^2$

$\boxed{\frac{1}{2}e^{-2t}t^2}$

#9. Evaluate the Inverse Laplace transform to find $f(t)$: $\mathscr{L}^{-1}\left\{\dfrac{s}{(s+1)^2}\right\}$

$\mathcal{L}^{-1}\left\{\frac{s+1-1}{(s+1)^2}\right\}$

$\mathcal{L}^{-1}\left\{\frac{(s+1)}{(s+1)^2}\right\}-\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\}$

$s+1$: shift, e^{-t}

$\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}$ $\quad$ $\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}$

(1) $\quad$ (t)

$\boxed{e^{-t}-te^{-t}}$

#7. Evaluate the Inverse Laplace transform to find $f(t)$: $\mathscr{L}^{-1}\left\{\dfrac{1}{s^2-6s+10}\right\}$

$s^2-6s+9+10-9$ $\quad$ $\mathcal{L}^{-1}\left\{\frac{1}{(s-3)^2+1}\right\}$

$(s-3)^2+1$ $\quad$ $s-3$: shift e^{3t}

$\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}=\sin(t)$

$\boxed{e^{3t}\sin(t)}$

#10. Use the Laplace transform to solve the initial-value problem

$y'+4y=e^{-4t}, \quad y(0)=2$

$[sY(s)-y(0)]+4Y(s)=\mathcal{L}\{e^{-4t}\}$

e^{-4t}: shift $s\to s+4$

$\mathcal{L}\{1\}=\frac{1}{s}$

$(s+4)Y(s)-2=\frac{1}{s+4}$

$Y(s)=\frac{2}{s+4}+\frac{1}{(s+4)^2}$

$\mathcal{L}^{-1}$: $\quad$ $s+4$: shift, e^{-4t}

$2\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}+\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}$

$2e^{-4t}(1)+e^{-4t}t$

$\boxed{y(t)=2e^{-4t}+te^{-4t}}$

#8. Evaluate the Inverse Laplace transform to find $f(t)$: $\mathscr{L}^{-1}\left\{\dfrac{s}{s^2+4s+5}\right\}$

$s^2+4s+4+5-4$ $\quad$ $\mathcal{L}^{-1}\left\{\frac{s}{(s+2)^2+1}\right\}=\mathcal{L}^{-1}\left\{\frac{s+2-2}{(s+2)^2+1}\right\}$

$(s+2)^2+1$

$\mathcal{L}^{-1}\left\{\frac{(s+2)}{(s+2)^2+1}\right\}-2\mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2+1}\right\}$

$s+2$; shift e^{-2t} $\quad$ $\cos(t)-2\sin(t)$

$\boxed{e^{-2t}(\cos t-2\sin t)}$

$\boxed{e^{-2t}\cos t-2e^{-2t}\sin t}$

#11. Use the Laplace transform to solve the initial-value problem

$y''+2y'+y=0, \qquad y(0)=1, \qquad y'(0)=1$

$$[s^2Y(s)-sy(0)-y'(0)]+2[sY(s)-y(0)]+Y(s)=0$$

$$s^2Y(s)-s(1)-1+2sY(s)-2(1)+Y(s)=0$$

$$(s^2+2s+1)Y(s)-s-3=0$$

$$Y(s)=\frac{s+3}{s^2+2s+1} \qquad s^2+2s+\underline{1}+1-\underline{1}$$

$$(s+1)^2+0$$

$$Y(s)=\frac{s+3}{(s+1)^2}=\frac{s+1-1+3}{(s+1)^2}=\frac{(s+1)}{(s+1)^2}+\frac{2}{(s+1)^2}$$

$$y(t)=\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}+2\mathcal{L}\left\{\frac{1}{(s+1)^2}\right\}$$

$s+1$: shift e^{-t} $\qquad \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}$

$\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} \qquad t$

(1)

$$\boxed{y(t)=e^{-t}+2te^{-t}}$$

#12. Use the Laplace transform to solve the initial-value problem

$y''-6y'+9y=t, \qquad y(0)=0, \qquad y'(0)=1$

$$[s^2Y(s)-sy(0)-y'(0)]-6[sY(s)-y(0)]+9Y(s)=\mathcal{L}\{t\}$$

$$s^2Y(s)-s(0)-1-6sY(s)+6(0)+9Y(s)=\frac{1}{s^2}$$

$$(s^2-6s+9)Y(s)-1=\frac{1}{s^2} \qquad s^2-6s+\cancel{9}+9-9$$

$$(s-3)^2+0$$

$$Y(s)=\frac{1}{(s^2-6s+9)}+\frac{1}{s^2(s^2-6s+9)}$$

$$Y(s)=\frac{1}{(s-3)^2}+\frac{1}{s^2(s-3)^2}=\frac{1}{(s-3)^2}+\frac{2}{27}\frac{1}{s}+\frac{1}{9}\frac{1}{s^2}-\frac{2}{27}\frac{1}{(s-3)}+\frac{1}{9}\frac{1}{(s-3)^2}$$

$$Y(s)=\frac{10}{9}\frac{1}{(s-3)^2}+\frac{2}{27}\frac{1}{s}+\frac{1}{9}\frac{1}{s^2}-\frac{2}{27}\frac{1}{(s-3)}$$

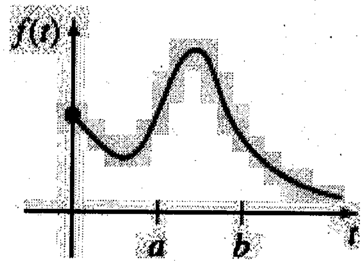
$$y(t)=\frac{10}{9}\mathcal{L}^{-1}\left\{\frac{1}{(s-3)^2}\right\}+\frac{2}{27}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}+\frac{1}{9}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}-\frac{2}{27}\mathcal{L}^{-1}\left\{\frac{1}{(s-3)}\right\}$$

$s-3$: shift e^{3t}

$$\boxed{y(t)=\frac{10}{9}te^{3t}+\frac{2}{27}+\frac{1}{9}t-\frac{2}{27}e^{3t}}$$

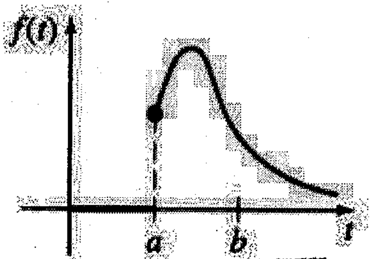
7.3 day 2

For #1-6, the function $f(t)$ is given by the graph:



Given the graph of a modified version of this function, use a combination of terms with the original function, time shifted original function and the unit step function to write a new function that produces the function in the problem's graph.

#1.

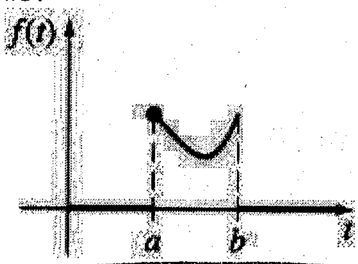


$= \boxed{f(t)u(t-a)}$

#2.

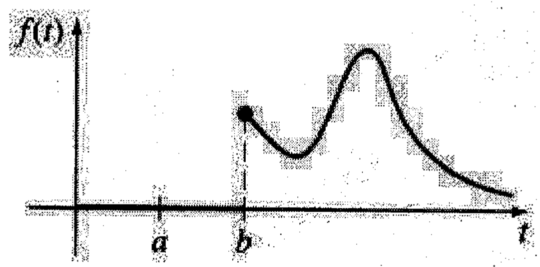
$= \boxed{f(t)u(t-a) - f(t)u(t-b)}$

#3.



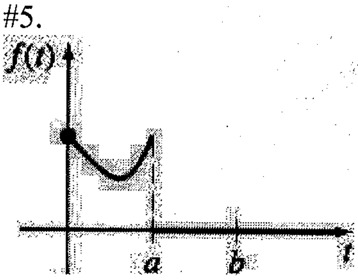
$= \boxed{f(t-a)u(t-a) - f(t-a)u(t-b)}$

#4.



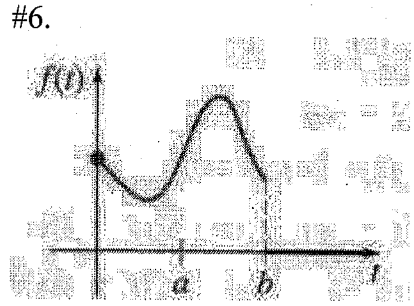
$= \boxed{f(t-b)u(t-b)}$

#5.



$= \boxed{f(t) - f(t)u(t-a)}$

#6.



$= \boxed{f(t) - f(t)u(t-b)}$

7.3 day 3

#1. Evaluate the Laplace transform to find $F(s)$

$\mathcal{L}\{(t-1)\,u(t-1)\}$

$\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}F(s)$, $a=1$

$\mathcal{L}\{t\} = \frac{1}{s^2}$

$e^{-1s}\frac{1}{s^2}$

$\boxed{e^{-s}\frac{1}{s^2}}$

#2. Evaluate the Laplace transform to find $F(s)$

$\mathcal{L}\{t\,u(t-2)\}$

$\mathcal{L}\{g(t)u(t-a)\} = e^{-as}\mathcal{L}\{g(t+a)\}$ $a=2$

$e^{-2s}\mathcal{L}\{(t+2)\} = e^{-2s}\mathcal{L}\{t\}+2e^{-2s}\mathcal{L}\{1\}$

$\boxed{e^{-2s}\frac{1}{s^2}+2e^{-2s}\frac{1}{s}}$

#3. Evaluate the Laplace transform to find $F(s)$

$\mathcal{L}\{\cos(2t)\,u(t-\pi)\}$

$\mathcal{L}\{g(t)u(t-a)\} = e^{-as}\mathcal{L}\{g(t+a)\}$ $a=\pi$

$e^{-\pi s}\mathcal{L}\{\cos(2(t+\pi))\} = e^{-\pi s}\mathcal{L}\{\cos(2t+2\pi)\}$

$= e^{-\pi s}\mathcal{L}\{\cos(2t)\}$

$= \boxed{e^{-\pi s}\frac{s}{s^2+4}}$

#4. Evaluate the Inverse Laplace transform to find

$f(t)$: $\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^3}\right\}$ e^{-2s}: shift $t\to t-2$

$\frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^3}\right\} = \frac{1}{2}t^2$

$\boxed{\frac{1}{2}(t-2)^2u(t-2)}$

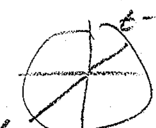
#5. Evaluate the Inverse Laplace transform to find

$f(t)$: $\mathcal{L}^{-1}\left\{\frac{e^{-\pi s}}{s^2+1}\right\}$ $e^{-\pi s}$: shift $t\to t-\pi$

$\boxed{\sin(t-\pi)u(t-\pi)}$

or

$\boxed{-\sin(t)\,u(t-\pi)}$



$\sin(t-\pi) = -\sin(t)$

#6. Evaluate the Inverse Laplace transform to find

$f(t)$: $\mathcal{L}^{-1}\left\{\frac{e^{-s}}{s(s+1)}\right\}$ e^{-s}: shift $t\to t-1$

$\mathcal{L}^{-1}\left\{\frac{1}{s(s+1)}\right\}$

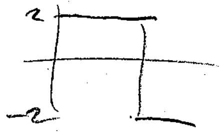
$\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}$

$(1) - e^{-t}$

$\boxed{1u(t-1) - e^{-(t-1)}u(t-1)}$

#7. Write the function in terms of unit step functions, then find the Laplace transform of the given function:

$$f(t)=\begin{cases}2, & 0\le t<3\\ -2, & t\ge 3\end{cases}$$



$\boxed{f(t)=2-4u(t-3)}$

$2\mathcal{L}\{1\}-4\mathcal{L}\{1u(t-3)\}$

$\mathcal{L}\{g(t)u(t-a)\}=e^{-as}\mathcal{L}\{g(t+a)\}$

$a=3$

$2\mathcal{L}\{1\}-4e^{-3s}\mathcal{L}\{1\}$

$\boxed{2\frac{1}{s}-4e^{-3s}\frac{1}{s}}$

#8. Write the function in terms of unit step functions, then find the Laplace transform of the given function:

$$f(t)=\begin{cases}0, & 0\le t<1\\ t^2, & t\ge 1\end{cases}$$

$\boxed{f(t)=t^2u(t-1)}$

$\mathcal{L}\{t^2u(t-1)\}$

$\mathcal{L}\{g(t)u(t-a)\}=e^{-as}\mathcal{L}\{g(t+a)\}$

$a=1$

$e^{-s}\mathcal{L}\{(t+1)^2\}$

$e^{-s}\mathcal{L}\{t^2+2t+1\}$

$\boxed{e^{-s}\frac{2}{s^3}+2e^{-s}\frac{1}{s^2}+e^{-s}\frac{1}{s}}$

#9. Write the function in terms of unit step functions, then find the Laplace transform of the given function:

$$f(t)=\begin{cases}t, & 0\le t<2\\ 0, & t\ge 2\end{cases}$$

$\boxed{f(t)=t-tu(t-2)}$

$\mathcal{L}\{t\}-\mathcal{L}\{tu(t-2)\}$

$\mathcal{L}\{g(t)u(t-a)\}=e^{-as}\mathcal{L}\{g(t+a)\}$

$a=2$

$\frac{1}{s^2}-e^{-2s}\mathcal{L}\{(t+2)\}$

$\boxed{\frac{1}{s^2}-e^{-2s}\frac{1}{s^2}-2e^{-2s}\frac{1}{s}}$

#10. Use the Laplace transform to solve the initial-value problem:

$y'+y=f(t),\quad y(0)=0$

where $f(t)=\begin{cases}0, & 0\le t<1\\ 5, & t\ge 1\end{cases}$ $f(t)=5u(t-1)$

$[sY(s)-y(0)]+Y(s)=\mathcal{L}\{5u(t-1)\}$

$\mathcal{L}\{g(t)u(t-a)\}=e^{-as}\mathcal{L}\{g(t+a)\}$

$a=1$

$\mathcal{L}\{5\}$

$sY(s)-0+Y(s)=e^{-s}5\frac{1}{s}$

$(s+1)Y(s)=5e^{-s}\frac{1}{s}$

$Y(s)=5e^{-s}\frac{1}{s(s+1)}=5e^{-s}\left(\frac{1}{s}-\frac{1}{s+1}\right)$

$y(t)=5\mathcal{L}^{-1}\{e^{-s}\frac{1}{s}\}-5\mathcal{L}^{-1}\{e^{-s}\frac{1}{s+1}\}$

$(\mathcal{L}^{-1}\{e^{-as}F(s)\}=f(t-a)u(t-a))$

$t\to t-1$

$y(t)=5(1)u(t-1)-5e^{-(t-1)}u(t-1)$

$\boxed{y(t)=5u(t-1)-5e^{-(t-1)}u(t-1)}$

#11. Use the Laplace transform to solve the initial-value problem:

$y' + 2y = f(t), \quad y(0) = 0$

where $f(t) = \begin{cases} t, & 0 \le t < 1 \\ 0, & t \ge 1 \end{cases}$ $\quad f(t) = t - t\,u(t-1)$

$[sY(s) - y(0)] + 2Y(s) = \mathcal{L}\{t\} - \mathcal{L}\{t\,u(t-1)\}$

$\mathcal{L}\{g(t)u(t-a)\} = e^{-as}\mathcal{L}\{g(t+a)\}$

$a = 1$

$sY(s) - 0 + 2Y(s) = \frac{1}{s^2} - e^{-s}\mathcal{L}\{(t+1)\}$

$(s+2)Y(s) = \frac{1}{s^2} - e^{-s}\left(\frac{1}{s^2} + \frac{1}{s}\right) = \frac{1}{s^2} - e^{-s}\frac{1}{s^2} - e^{-s}\frac{1}{s}$

$Y(s) = \frac{1}{s^2(s+2)} - e^{-s}\frac{1}{s^2(s+2)} - e^{-s}\frac{1}{s(s+2)}$

$Y(s) = \left(-\frac{1}{4}\frac{1}{s} + \frac{1}{2}\frac{1}{s^2} + \frac{1}{4}\frac{1}{s+2}\right) - e^{-s}\left(-\frac{1}{4}\frac{1}{s} + \frac{1}{2}\frac{1}{s^2} + \frac{1}{4}\frac{1}{s+2}\right) - e^{-s}\left(\frac{1}{2}\frac{1}{s} - \frac{1}{2}\frac{1}{s+2}\right)$

$Y(s) = -\frac{1}{4}\frac{1}{s} + \frac{1}{2}\frac{1}{s^2} + \frac{1}{4}\frac{1}{s+2} - \frac{1}{4}e^{-s}\frac{1}{s} - \frac{1}{2}e^{-s}\frac{1}{s^2} + \frac{1}{4}e^{-s}\frac{1}{s+2}$

$y(t) = -\frac{1}{4}\mathcal{L}^{-1}\{\frac{1}{s}\} + \frac{1}{2}\mathcal{L}^{-1}\{\frac{1}{s^2}\} + \frac{1}{4}\mathcal{L}^{-1}\{\frac{1}{s+2}\} - \frac{1}{4}\mathcal{L}^{-1}\{e^{-s}\frac{1}{s}\} - \frac{1}{2}\mathcal{L}^{-1}\{e^{-s}\frac{1}{s^2}\} + \frac{1}{4}\mathcal{L}^{-1}\{e^{-s}\frac{1}{s+2}\}$

$\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\,u(t-a)$

$a = 1$

$y(t) = -\frac{1}{4}(1) + \frac{1}{2}t + \frac{1}{4}e^{-2t} - \frac{1}{4}(1)u(t-1) - \frac{1}{2}(t-1)u(t-1) + \frac{1}{4}e^{-2(t-1)}u(t-1)$

$$\boxed{y(t) = -\frac{1}{4} + \frac{1}{2}t + \frac{1}{4}e^{-2t} - \frac{1}{4}u(t-1) - \frac{1}{2}(t-1)u(t-1) + \frac{1}{4}e^{-2(t-1)}u(t-1)}$$

#12. Use the Laplace transform to solve the initial-value problem:

$y'' + 4y = \sin t\, \mathcal{U}(t - 2\pi)$,

$y(0) = 1, \quad y'(0) = 0$

$[s^2Y(s) - sy(0) - y'(0)] + 4Y(s) = \mathcal{L}\{\sin t\, u(t-2\pi)\}$ $\quad (\mathcal{L}\{g(t)u(t-a)\} = e^{-sa}\mathcal{L}\{g(t+a)\})$ $\quad a = 2\pi$

$s^2Y(s) - s(1) - 0 + 4Y(s) = e^{-2\pi s}\mathcal{L}\{\sin(t+2\pi)\}$

$(s^2+4)Y(s) - s = e^{-2\pi s}\mathcal{L}\{\sin t\}$

$(s^2+4)Y(s) = s + e^{-2\pi s}\frac{1}{s^2+1}$

$Y(s) = \frac{s}{s^2+4} + e^{-2\pi s}\frac{1}{(s^2+4)(s^2+1)} = \frac{s}{s^2+4} + e^{-2\pi s}\left(\frac{1}{3}\frac{1}{s^2+1} - \frac{1}{3}\frac{1}{s^2+4}\right)$

$y(t) = \mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} + \frac{1}{3}\mathcal{L}^{-1}\left\{e^{-2\pi s}\frac{1}{s^2+1}\right\} - \frac{1}{6}\mathcal{L}^{-1}\left\{e^{-2\pi s}\frac{2}{s^2+4}\right\}$

$\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)u(t-a) \quad a = 2\pi$

$y(t) = \cos(2t) + \frac{1}{3}\underbrace{\sin(t-2\pi)}_{\sin t}u(t-2\pi) - \frac{1}{6}\sin(2(t-2\pi))\,u(t-2\pi)$

$\sin(2t - 8\pi)$

$\sin(2t)$

$$\boxed{y(t) = \cos(2t) + \frac{1}{3}\sin t\, u(t-2\pi) - \frac{1}{6}\sin(2t)\, u(t-2\pi)}$$

#13. Use the Laplace transform to solve the initial-value problem:

$y'' + y = f(t), \quad y(0) = 0, \quad y'(0) = 1$

$$\text{where } f(t) = \begin{cases} 0, & 0 \le t < \pi \\ 1, & \pi \le t < 2\pi \\ 0, & t \ge 2\pi \end{cases} \qquad f(t) = 1u(t-\pi) - 1u(t-2\pi)$$

$$[s^2Y(s) - sy(0) - y'(0)] + Y(s) = \underbrace{\mathcal{L}\{1u(t-\pi)\} - \mathcal{L}\{1u(t-2\pi)\}}_{\mathcal{L}\{g(t)u(t-a)\} = e^{-as}\mathcal{L}\{g(t+a)\}}$$

$$s^2Y(s) - s(0) - 1 + Y(s) = e^{-\pi s}\frac{1}{s} - e^{-2\pi s}\frac{1}{s}$$

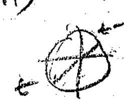
$$(s^2+1)Y(s) = 1 + e^{-\pi s}\frac{1}{s} - e^{-2\pi s}\frac{1}{s}$$

$$Y(s) = \frac{1}{s^2+1} + e^{-\pi s}\frac{1}{s(s^2+1)} - e^{-2\pi s}\frac{1}{s(s^2+1)}$$

$$Y(s) = \frac{1}{s^2+1} + e^{-\pi s}\left(\frac{1}{s} - \frac{s}{s^2+1}\right) - e^{-2\pi s}\left(\frac{1}{s} - \frac{s}{s^2+1}\right)$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} + \underbrace{\mathcal{L}^{-1}\left\{e^{-\pi s}\frac{1}{s}\right\} - \mathcal{L}^{-1}\left\{e^{-\pi s}\frac{s}{s^2+1}\right\} - \mathcal{L}^{-1}\left\{e^{-2\pi s}\frac{1}{s}\right\} + \mathcal{L}^{-1}\left\{e^{-2\pi s}\frac{s}{s^2+1}\right\}}_{\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)u(t-a)}$$

$$y(t) = \sin(t) + (1)u(t-\pi) - \cos(t-\pi)u(t-\pi) - (1)u(t-2\pi) + \underbrace{\cos(t-2\pi)}_{=\cos t}u(t-2\pi)$$



$\cos(t-\pi) = -\cos t$

$$\boxed{y(t) = \sin t + u(t-\pi) + \cos t\, u(t-\pi) - u(t-2\pi) + \cos t\, u(t-2\pi)}$$

7.4

#1. Evaluate the Laplace transform to find $F(s)$

$\mathscr{L}\{te^{-10t}\}$

$\mathscr{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^2}[F(s)]$ $n=1$

$(-1)^1 \frac{d}{ds}\left[\frac{1}{s+10}\right]$

$(-1)\left(\frac{(s+10)(0)-1(1)}{(s+10)^2}\right)$

$= \boxed{\frac{1}{(s+10)^2}}$

#2. Evaluate the Laplace transform to find $F(s)$

$\mathscr{L}\{t\cos(2t)\}$

$\mathscr{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}[F(s)]$ $n=1$

$(-1)^1 \frac{d}{ds}\left[\frac{s}{s^2+4}\right]$

$(-1)\left[\frac{(s^2+4)(1)-s(2s)}{(s^2+4)^2}\right] = \frac{2s^2-s^2-4}{(s^2+4)^2}$

$= \boxed{\frac{s^2-4}{(s^2+4)^2}}$

#3. Evaluate the Laplace transform to find $F(s)$

$\mathscr{L}\{te^{2t}\sin(6t)\}$

$\mathscr{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^2}(F(s))$ $n=1$

$(-1)\frac{d}{ds}[\mathscr{L}\{e^{2t}\sin(6t)\}]$

$\mathscr{L}\{e^{at}f(t)\} = F(s-a)$ $a=2$

$(-1)\frac{d}{ds}\left[\frac{6}{(s-2)^2+36}\right]$

$(-1)\left[\frac{((s-2)^2+36)(0)-6(2(s-2)(1))}{((s-2)^2+36)^2}\right]$

$(-1)\frac{-12s+24}{((s-2)^2+36)^2} = \frac{12s-24}{((s-2)^2+36)^2} = \boxed{\frac{12(s-2)}{((s-2)^2+36)^2}}$

#4. Use the Laplace transform to solve the initial-value problem:

$y' + y = t\sin(t)$, $y(0)=0$

$[sY(s)-\cancel{y(0)}] + Y(s) = \mathscr{L}\{t\sin(t)\}$

0

$\mathscr{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}[F(s)]$ $n=1$

$(s+1)Y(s) =$ $\quad (-1)\frac{d}{ds}\left[\frac{1}{s^2+1}\right] = (-1)\left[\frac{(s^2+1)(0)-1(2s)}{(s^2+1)^2}\right] = \frac{2s}{(s^2+1)^2}$

$Y(s) = \frac{2s}{(s+1)(s^2+1)^2} = -\frac{1}{2}\frac{1}{s+1} + \frac{1}{2}\frac{s}{s^2+1} - \frac{1}{2}\frac{1}{s^2+1} + \frac{1}{2}\frac{2s}{(s^2+1)^2} + \frac{1}{2}\frac{\cancel{k}\,2}{(s^2+1)^2}$

extended table:

$t\sin kt \longleftrightarrow \frac{2ks}{(s^2+k^2)^2}$

$\sin kt - kt\cos kt \longleftrightarrow \frac{2k^3}{(s^2+k^2)^2}$

$y(t) = -\frac{1}{2}e^{-t} + \frac{1}{2}\cos(t) - \frac{1}{2}\sin(t) + \frac{1}{2}t\sin(t) + \frac{1}{2}(\sin(t) - t\cos(t))$

$\boxed{y(t) = -\frac{1}{2}e^{-t} + \frac{1}{2}\cos(t) + \frac{1}{2}t\sin(t) - \frac{1}{2}t\cos(t)}$

#5. Use the Laplace transform to solve the initial-value problem:

$y''+9y=\cos(3t),\quad y(0)=2,\quad y'(0)=5$

$[s^2Y(s)-sy(0)-y'(0)]+9Y(s)=\mathcal{L}\{\cos(3t)\}$

$s^2Y(s)-s(2)-5+9Y(s)=\frac{s}{s^2+9}$

$(s^2+9)Y(s)=2s+5+\frac{s}{s^2+9}$

extended table

$Y(s)=2\frac{s}{s^2+9}+\frac{5}{3}\frac{3}{s^2+9}+\frac{1}{6}\frac{6s}{(s^2+9)^2}$

$\mathcal{L}\{t\sin kt\}=\frac{2ks}{(s^2+k^2)^2}$

$y(t)=2\cos(3t)+\frac{5}{3}\sin(3t)+\frac{1}{6}(t\sin(3t))$

$\boxed{y(t)=2\cos(3t)+\frac{5}{3}\sin(3t)+\frac{1}{6}t\sin(3t)}$

#6. Use the Laplace transform to solve the initial-value problem:

$y''+16y=f(t),\quad y(0)=0,\quad y'(0)=1$

where $f(t)=\begin{cases}\cos(4t), & 0\le t<\pi\\ 0, & t\ge\pi\end{cases}$

$f(t)=\cos(4t)-\cos(4t)u(t-\pi)$

$[s^2Y(s)-sy(0)-y'(0)]+16Y(s)=\mathcal{L}\{\cos(4t)\}-\mathcal{L}\{\cos(4t)u(t-\pi)\}$

$\mathcal{L}\{g(t)u(t-a)\}=e^{-as}\mathcal{L}\{g(t+a)\}\quad a=\pi$

$s^2Y(s)-s(0)-1+16Y(s)=\frac{s}{s^2+16}-e^{-\pi s}\mathcal{L}\{\cos(4(t+\pi))\}$

$\cos(4t+4\pi)$

$\mathcal{L}\{\cos(4t)\}$

$(s^2+16)Y(s)=1+\frac{s}{s^2+16}-e^{-\pi s}\frac{s}{s^2+16}$

$Y(s)=\frac{1}{4}\frac{4}{s^2+16}+\frac{1}{8}\frac{8s}{(s^2+16)^2}-\frac{1}{8}e^{-\pi s}\frac{8s}{(s^2+16)^2}$

$y(t)=\frac{1}{4}\sin(4t)+\frac{1}{8}t\sin(4t)-\frac{1}{8}\mathcal{L}^{-1}\{e^{-\pi s}\frac{8s}{(s^2+16)^2}\}$

$\mathcal{L}^{-1}\{e^{-as}F(s)\}=f(t-a)u(t-a)\quad(a=\pi)$

$\boxed{y(t)=\frac{1}{4}\sin(4t)+\frac{1}{8}t\sin(4t)-\frac{1}{8}(t-\pi)\sin(4(t-\pi))u(t-\pi)}$

7.5

#1. Use the Laplace transform to solve the initial-value problem:

$y' - 3y = \delta(t-2), \quad y(0) = 0$

$[sY(s) - y(0)] - 3Y(s) = \mathcal{L}\{\delta(t-2)\}$

$(s-3)Y(s) = e^{-2s}$

$Y(s) = e^{-2s}\frac{1}{s-3}$

$y(t) = \mathcal{L}^{-1}\{e^{-2s}\frac{1}{s-3}\}$

$\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)u(t-a) \quad (a=2)$

$\boxed{y(t) = e^{3(t-2)}u(t-2)}$

#2. Use the Laplace transform to solve the initial-value problem:

$y'' + y = \delta(t-2\pi), \quad y(0) = 0, \quad y'(0) = 1$

$[s^2Y(s) - sy(0) - y'(0)] + Y(s) = \mathcal{L}\{t-2\pi\}$

$(s^2+1)Y(s) - 1 = e^{-2\pi s}$

$Y(s) = \frac{1}{s^2+1} + e^{-2\pi s}\frac{1}{s^2+1}$

$y(t) = \mathcal{L}^{-1}\{\frac{1}{s^2+1}\} + \mathcal{L}^{-1}\{e^{-2\pi s}\frac{1}{s^2+1}\}$

$\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)u(t-a) \quad (a = 2\pi)$

$y(t) = \sin(t) + \sin(t-2\pi)u(t-2\pi)$

$\boxed{y(t) = \sin t + \sin t\, u(t-2\pi)}$

#3. Use the Laplace transform to solve the initial-value problem:

$y'' + y = \delta\left(t - \frac{1}{2}\pi\right) + \delta\left(t - \frac{3}{2}\pi\right),$

$y(0) = 0, \quad y'(0) = 0$

$[s^2Y(s) - sy(0) - y'(0)] + Y(s) = \mathcal{L}\{\delta(t-\pi/2)\} + \mathcal{L}\{t - 3\pi/2\}$

$(s^2+1)Y(s) = e^{-\pi/2\, s} + e^{-3\pi/2\, s}$

$Y(s) = e^{-\pi/2}\frac{1}{s^2+1} + e^{-3\pi/2}\frac{1}{s^2+1} \quad (\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)u(t-a))$

$\boxed{y(t) = \sin(t-\pi/2)u(t-\pi/2) + \sin(t-3\pi/2)u(t-3\pi/2)}$

#4. Use the Laplace transform to solve the initial-value problem:

$y'' + 2y' = \delta(t-1), \quad y(0) = 0, \quad y'(0) = 1$

$[s^2Y(s) - sy(0) - y'(0)] + 2[sY(s) - y(0)] = \mathcal{L}\{\delta(t-1)\}$

$s^2Y(s) - s(0) - 1 + 2sY(s) - 2(0) = e^{-s}$

$(s^2+2s)Y(s) - 1 = e^{-s}$

$Y(s) = \frac{1}{s(s+2)} + e^{-s}\frac{1}{s(s+2)} = \frac{1}{2}\frac{1}{s} - \frac{1}{2}\frac{1}{s+2} + \frac{1}{2}e^{-s}\frac{1}{s} - \frac{1}{2}e^{-s}\frac{1}{s+2}$

$y(t) = \frac{1}{2}\mathcal{L}^{-1}\{\frac{1}{s}\} - \frac{1}{2}\mathcal{L}^{-1}\{\frac{1}{s+2}\} + \frac{1}{2}\mathcal{L}^{-1}\{e^{-s}\frac{1}{s}\} - \frac{1}{2}\mathcal{L}^{-1}\{e^{-s}\frac{1}{s+2}\}$ $\quad (\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)u(t-a), \ a=1)$

$y(t) = \frac{1}{2}(1) - \frac{1}{2}e^{-2t} + \frac{1}{2}(1)u(t-1) - \frac{1}{2}e^{-2(t-1)}u(t-1)$

$\boxed{y(t) = \frac{1}{2} - \frac{1}{2}e^{-2t} + \frac{1}{2}u(t-1) - \frac{1}{2}e^{-2(t-1)}u(t-1)}$

Ch7 Test Review

#1. Use theorems and the table to find $\mathscr{L}\{f(t)\}$

$f(t)=2t^4$

$2\mathscr{L}\{t^4\}$

$2\frac{4!}{s^5}$

$\boxed{\frac{48}{s^5}}$

#2. Use theorems and the table to find $\mathscr{L}\{f(t)\}$

$f(t)=t^2+6t-3$

$\mathscr{L}\{t^2\}+6\mathscr{L}\{t\}-3\mathscr{L}\{1\}$

$\frac{2!}{s^3}+6\frac{1}{s^2}-3\frac{1}{s}$

$\boxed{\frac{2}{s^3}+\frac{6}{s^2}-\frac{3}{s}}$

#3. Use theorems and the table to find $\mathscr{L}\{f(t)\}$

$f(t)=t^2-e^{-9t}+5$

$\mathscr{L}\{t^2\}-\mathscr{L}\{e^{-9t}\}+5\mathscr{L}\{1\}$

$\frac{2!}{s^3}-\frac{1}{s-(-9)}+5\frac{1}{s}$

$\boxed{\frac{2}{s^3}-\frac{1}{s+9}+\frac{5}{s}}$

#4. Use theorems and the table to find

$\mathscr{L}^{-1}\left\{\frac{\cancel{3}5}{(s-2)(s-3)(s-6)}\right\}$ (online PFE)

$=\mathscr{L}^{-1}\left\{\frac{1}{2}\frac{1}{s-2}-\frac{1}{s-3}+\frac{1}{2}\frac{1}{s-6}\right\}$

$\frac{1}{2}\mathscr{L}^{-1}\left\{\frac{1}{s-2}\right\}-\mathscr{L}^{-1}\left\{\frac{1}{s-3}\right\}+\frac{1}{2}\mathscr{L}^{-1}\left\{\frac{1}{s-6}\right\}$

$\boxed{\frac{1}{2}e^{2t}-e^{3t}+\frac{1}{2}e^{6t}}$

#5. Use theorems and the table to find

$\mathscr{L}^{-1}\left\{\frac{s^2+1}{s(s-1)(s+1)(s-2)}\right\}$

$\mathscr{L}^{-1}\left\{\frac{1}{2}\frac{1}{s}-\frac{1}{s-1}-\frac{1}{3}\frac{1}{s+1}+\frac{5}{6}\frac{1}{s-2}\right\}$

$\frac{1}{2}\mathscr{L}^{-1}\left\{\frac{1}{s}\right\}-\mathscr{L}^{-1}\left\{\frac{1}{s-1}\right\}-\frac{1}{3}\mathscr{L}^{-1}\left\{\frac{1}{s+1}\right\}+\frac{5}{6}\mathscr{L}^{-1}\left\{\frac{1}{s-2}\right\}$

$\frac{1}{2}(1)-e^{t}-\frac{1}{3}e^{-t}+\frac{5}{6}e^{2t}$

$\boxed{\frac{1}{2}-e^{t}-\frac{1}{3}e^{-t}+\frac{5}{6}e^{2t}}$

#6. Use the Laplace transform to solve the initial-value problem:

$y'+6y=e^{4t}, \quad y(0)=2$

$[sY(s)-y(0)]+6Y(s)=\mathcal{L}\{e^{4t}\}$

$(s+6)Y(s)-2=\frac{1}{s-4}$

$Y(s)=\frac{2}{s+6}+\frac{1}{(s+6)(s-4)}=\frac{2}{s+6}-\frac{1}{10}\frac{1}{s+6}+\frac{1}{10}\frac{1}{s-4}=\frac{19}{10}\frac{1}{s+6}+\frac{1}{10}\frac{1}{s-4}$

$y(t)=\frac{19}{10}\mathcal{L}^{-1}\{\frac{1}{s+6}\}+\frac{1}{10}\mathcal{L}^{-1}\{\frac{1}{s-4}\}$

$\boxed{y(t)=\frac{19}{10}e^{-6t}+\frac{1}{10}e^{4t}}$

#7. Use the Laplace transform to solve the initial-value problem:

$y''-4y'=6e^{3t}-3e^{-t}, \quad y(0)=1, \quad y'(0)=-1$

$[s^2Y(s)-sy(0)-y'(0)]-4[sY(s)-y(0)]=6\mathcal{L}\{e^{3t}\}-3\mathcal{L}\{e^{-t}\}$

$s^2Y(s)-s(1)+1-4sY(s)+4(1)=6\frac{1}{s-3}-3\frac{1}{s+1}$

$(s^2-4s)Y(s)=s-5+6\frac{1}{s-3}-3\frac{1}{s+1}$

$Y(s)=\frac{s}{s(s-4)}-5\frac{1}{s(s-4)}+6\frac{1}{s(s-4)(s-3)}-3\frac{1}{s(s-4)(s+1)}$

$Y(s)=\frac{1}{s-4}-5\left(-\frac{1}{4}\frac{1}{s}+\frac{1}{4}\frac{1}{s-4}\right)+6\left(\frac{1}{12}\frac{1}{s}+\frac{1}{4}\frac{1}{s-4}-\frac{1}{3}\frac{1}{s-3}\right)-3\left(-\frac{1}{4s}+\frac{1}{20}\frac{1}{s-4}+\frac{1}{5}\frac{1}{s+1}\right)$

$Y(s)=\frac{11}{10}\frac{1}{s-4}+\frac{5}{2}\frac{1}{s}-2\frac{1}{s-3}-\frac{3}{5}\frac{1}{s+1}$

$\boxed{y(t)=\frac{11}{10}e^{4t}+\frac{5}{2}-2e^{3t}-\frac{3}{5}e^{-t}}$

#8. Evaluate the Inverse Laplace transform to find $f(t)$: $\mathscr{L}^{-1}\left\{\frac{1}{s^2-6s+10}\right\}$

doesn't factor, so complete the square:

$s^2-6s+\frac{9}{}+10-\frac{9}{}$

$(s-3)^2+1$

$\mathcal{L}^{-1}\left\{\frac{1}{(s-3)^2+1}\right\}$

$\mathcal{L}^{-1}\{F(s-a)\}=e^{at}f(t)\ (a=3)$

$e^{3t}\mathcal{L}\left\{\frac{1}{s^2+1}\right\}$

$\boxed{e^{3t}\sin(t)}$

#9. Evaluate the Inverse Laplace transform to find $f(t)$: $\mathscr{L}^{-1}\left\{\frac{1}{s^2+2s+5}\right\}$

doesn't factor, so complete the square:

$s^2+2s+\underline{1}+5-\underline{1}$

$(s+1)^2+4$

$\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+4}\right\}$

$\mathcal{L}^{-1}\{F(s-a)\}=e^{at}f(t)\ (a=-1)$

$e^{-t}\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\}$ fix up constant

$\frac{1}{2}e^{-t}\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\}$

$\boxed{\frac{1}{2}e^{-t}\sin(2t)}$

#10. Evaluate the Inverse Laplace transform to find $f(t)$: $\mathscr{L}^{-1}\left\{\frac{s}{s^2+4s+5}\right\}$

doesn't factor, so complete the square:

$s^2+4s+\underline{4}+5-\underline{4}$

$(s+2)^2+1$

$\mathcal{L}^{-1}\left\{\frac{s}{(s+2)^2+1}\right\}$ shift? top must also be s+2

$\mathcal{L}^{-1}\left\{\frac{(s+2)-2}{(s+2)^2+1}\right\}$

$\mathcal{L}^{-1}\left\{\frac{(s+2)}{(s+2)^2+1}\right\}-2\mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2+1}\right\}$

$(\mathcal{L}^{-1}\{F(s+a)\}=e^{-at}f(t),\ (a=-2))$

$e^{-2t}\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\}-2e^{-2t}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$

$\boxed{e^{-2t}\cos t-2e^{-2t}\sin t}$

#11. Evaluate the Inverse Laplace transform to find $f(t)$: $\mathscr{L}^{-1}\left\{\frac{2s+5}{s^2+6s+34}\right\}$

doesn't factor, complete the square:

$s^2+6s+\frac{9}{}+34-\frac{9}{}$

$(s+3)^2+25$

(factor out) $2\mathcal{L}^{-1}\left\{\frac{s+5/2}{(s+3)^2+25}\right\}$ shift? top must also be s+3

$2\mathcal{L}^{-1}\left\{\frac{(s+3)-3+5/2}{(s+3)^2+25}\right\}$

$2\mathcal{L}^{-1}\left\{\frac{(s+3)}{(s+3)^2+25}\right\}-\mathcal{L}^{-1}\left\{\frac{1}{(s+3)^2+25}\right\}$ ← fix up constant

$2\mathcal{L}^{-1}\left\{\frac{(s+3)}{(s+3)^2+25}\right\}-\frac{1}{5}\mathcal{L}^{-1}\left\{\frac{5}{(s+3)^2+25}\right\}$

shift: $\mathcal{L}^{-1}\{F(s-a)\}=e^{-at}f(t)\ (a=-3)$

$2e^{-3t}\mathcal{L}^{-1}\left\{\frac{s}{s^2+25}\right\}-\frac{1}{5}e^{-3t}\mathcal{L}^{-1}\left\{\frac{5}{s^2+25}\right\}$

$\boxed{2e^{-3t}\cos(5t)-\frac{1}{5}e^{-3t}\sin(5t)}$

#12. Use the Laplace transform to solve the initial-value problem

$y'' - 6y' + 9y = t, \qquad y(0) = 0, \qquad y'(0) = 1$

$[s^2Y(s) - sy(0) - y'(0)] - 6[sY(s) - y(0)] + 9Y(s) = \mathcal{L}\{t\}$

$s^2Y(s) - s(0) - 1 - 6sY(s) + 6(0) + 9Y(s) = \frac{1}{s^2}$

$(s^2 - 6s + 9)Y(s) = 1 + \frac{1}{s^2}$

$Y(s) = \frac{1}{s^2 - 6s + 9} + \frac{1}{s^2(s^2 - 6s + 9)} = \frac{1}{(s-3)^2} + \frac{2}{27}\frac{1}{s} + \frac{1}{9}\frac{1}{s^2} - \frac{2}{27}\frac{1}{s-3} + \frac{1}{9}\frac{1}{(s-3)^2}$

$Y(s) = \frac{10}{9}\frac{1}{(s-3)^2} + \frac{2}{27}\frac{1}{s} + \frac{1}{9}\frac{1}{s^2} - \frac{2}{27}\frac{1}{s-3}$

$\boxed{y(t) = \frac{10}{9}te^{3t} + \frac{2}{27}(1) + \frac{1}{9}t - \frac{2}{27}e^{3t}}$

#13. Evaluate the Inverse Laplace transform to find $f(t)$: $\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^3}\right\} = \mathcal{L}^{-1}\left\{e^{-2s}\frac{1}{s^3}\right\}$

$\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)u(t-a)$

$(a = 2)$

$\mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\}u(t-2)$

$\frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^3}\right\}u(t-2)$

$\boxed{\frac{1}{2}(t-2)^2u(t-2)}$

#14. Evaluate the Inverse Laplace transform to find $f(t)$: $\mathcal{L}^{-1}\left\{\frac{e^{-\pi s}}{s^2+1}\right\}$

$\mathcal{L}^{-1}\left\{e^{-\pi s}\frac{1}{s^2+1}\right\}$

$\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)u(t-a) \quad (a = \pi)$

$\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}u(t-\pi)$

$\boxed{\sin(t-\pi)\,u(t-\pi)}$

or, because $\sin(t-\pi) = -\sin t$

$\boxed{-\sin t\; u(t-\pi)}$

#15. Write the function in terms of unit step functions, then find the Laplace transform of the given function:

$$f(t)=\begin{cases}2, & 0\le t<3\\ -2, & t\ge 3\end{cases}$$

$\boxed{f(t)=2-4u(t-3)}$

$2\mathcal{L}\{1\}-4\mathcal{L}\{1u(t-3)\}$

$\mathcal{L}\{g(t)u(t-a)\}=e^{-as}\mathcal{L}\{g(t+a)\}$ $(a=3)$

$2\frac{1}{s}-4e^{-3s}\mathcal{L}\{1\}$

$\boxed{2\frac{1}{s}-4e^{-3s}\frac{1}{s}}$

#16. Write the function in terms of unit step functions, then find the Laplace transform of the given function:

$$f(t)=\begin{cases}t, & 0\le t<2\\ 0, & t\ge 2\end{cases}$$

$\boxed{f(t)=t-tu(t-2)}$

$\mathcal{L}\{t\}-\mathcal{L}\{t\,u(t-2)\}$

$\mathcal{L}\{g(t)u(t-a)\}=e^{-as}\mathcal{L}\{g(t+a)\}$ $(a=2)$

$\frac{1}{s^2}-e^{-2s}\mathcal{L}\{(t+2)\}$

$\boxed{\frac{1}{s^2}-e^{-2s}\frac{1}{s^2}-2e^{-2s}\frac{1}{s}}$

#17. Evaluate the Laplace transform to find $F(s)$

$\mathcal{L}\{te^{-10t}\}$

$\mathcal{L}\{t^n f(t)\}=(-1)^n\frac{d^n}{ds^n}[F(s)]$ $(n=1)$

$(-1)\frac{d}{ds}[\mathcal{L}\{e^{-10t}\}]$

$(-1)\frac{d}{ds}\left[\frac{1}{s+10}\right]$

$(-1)\left[\frac{(s+10)(0)-(1)(1)}{(s+10)^2}\right]$

$\boxed{\frac{1}{(s+10)^2}}$

#18. Evaluate the Laplace transform to find $F(s)$

$\mathcal{L}\{te^{2t}\sin(6t)\}$

$\mathcal{L}\{t^n f(t)\}=(-1)^n\frac{d^n}{ds^n}[F(s)]$ $(n=1)$

$(-1)\frac{d}{ds}[\mathcal{L}\{e^{2t}\sin(6t)\}]$

$\mathcal{L}\{e^{at}f(t)\}=F(s-a)$ $(a=2)$

$(-1)\frac{d}{ds}\left[\frac{6}{(s-2)^2+36}\right]$

$(-1)\left[\frac{((s-2)^2+36)(0)-6(2(s-2)^1(1))}{[(s-2)^2+36]^2}\right]$

$\boxed{\frac{12(s-2)}{[(s-2)^2+36]^2}}$

#19. Evaluate the Laplace transform to find $F(s)$

$\mathcal{L}\{te^{-3t}\cos(3t)\}$

$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}[F(s)]$ $(n=1)$

$(-1)\frac{d}{ds}[\mathcal{L}\{e^{-3t}\cos(3t)\}]$

$\mathcal{L}\{e^{-as}f(t)\} = F(s-a)$ $(a=-3)$

$(-1)\frac{d}{ds}\left[\frac{(s+3)}{(s+3)^2+9}\right]$

$(-1)\left(\frac{[(s+3)^2+9](1) - (s+3)(2(s+3)'(1))}{[(s+3)^2+9]^2}\right)$

$(-1)\dfrac{(s+3)^2+9-2(s+3)^2}{[(s+3)^2+9]^2} = \boxed{\dfrac{(s+3)^2-9}{[(s+3)^2+9]^2}}$

#20. Use the Laplace transform to solve the initial-value problem

$y'' - 2y' + y = e^t, \quad y(0) = 0, \quad y'(0) = 5$

$[s^2Y(s) - s\,y(0) - y'(0)] - 2[sY(s) - y(0)] + Y(s) = \mathcal{L}\{e^t\}$

$s^2Y(s) - s(0) - 5 - 2sY(s) + 2(0) + Y(s) = \frac{1}{s-1}$

$(s^2 - 2s + 1)Y(s) = 5 + \frac{1}{s-1}$

s^2-2s+1
$(s-1)^2$

$Y(s) = \frac{5}{s^2-2s+1} + \frac{1}{(s-1)(s^2-2s+1)}$

$Y(s) = \frac{5}{(s-1)^2} + \frac{1}{(s-1)^3}$

two possible approaches:

1) treat s-1 as a shift

$y(t) = 5\mathcal{L}^{-1}\{\frac{1}{(s-1)^2}\} + \mathcal{L}^{-1}\{\frac{1}{(s-1)^3}\}$

$\mathcal{L}^{-1}\{F(s-a)\} = e^{at}f(t)$ $(a=1)$

$y(t) = 5e^t\mathcal{L}^{-1}\{\frac{1}{s^2}\} + e^t\mathcal{L}^{-1}\{\frac{1}{s^3}\}$

fix up constant

$y(t) = 5e^t\mathcal{L}^{-1}\{\frac{1}{s^2}\} + \frac{1}{2}e^t\mathcal{L}^{-1}\{\frac{2}{s^3}\}$

$\boxed{y(t) = 5e^t t + \frac{1}{2}e^t t^2}$

or 2) use more advanced table entries

$y(t) = 5\mathcal{L}^{-1}\{\frac{1}{(s-1)^2}\} + \mathcal{L}^{-1}\{\frac{1}{(s-1)^3}\}$

table: $\mathcal{L}^{-1}\{\frac{1}{(s-a)^2}\} = te^{at}$ $(a=1)$ $\quad \mathcal{L}^{-1}\{\frac{n!}{(s-a)^{n+1}}\} = t^n e^{at}$

fix up constant

$5te^t + \frac{1}{2!}\mathcal{L}^{-1}\{\frac{2!}{(s-1)^3}\}$ $(a=1)$ $(n=2)$

$\boxed{5te^t + \frac{1}{2}t^2e^t}$