DiffEq - Ch 8 - Extra Practice

8.1

#1b. Write the linear system in matrix form:

$$\frac{dx}{dt} = 4x - 7y$$

$$\frac{dy}{dt} = 5x$$

$$\boxed{\vec{X}' = \begin{bmatrix} 4 & -7 \\ 5 & 0 \end{bmatrix} \vec{X}}$$

$$\left(\vec{X} = \begin{bmatrix} x \\ y \end{bmatrix}\right)$$

#3b. Write the linear system in matrix form:

$$\frac{dx}{dt} = -3x + 4y + e^{-t} \sin 2t$$

$$\frac{dy}{dt} = 5x + 9z + 4e^{-t} \cos 2t$$

$$\frac{dz}{dt} = y + 6z - e^{-t}$$

$$\vec{X}' = \begin{bmatrix} -3 & 4 & 0 \\ 5 & 0 & 9 \\ 0 & 1 & 6 \end{bmatrix} \vec{X} + \vec{F}$$

$$\vec{X} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \vec{F} = \begin{bmatrix} e^{-t} \sin 2t \\ 4e^{-t} \cos 2t \\ -e^{-t} \end{bmatrix}$$

#2b. Write the linear system in matrix form:

$$\frac{dx}{dt} = x - y$$

$$\frac{dy}{dt} = x + 2z$$

$$\frac{dz}{dt} = -x + z$$

$$\boxed{\vec{X}' = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & 2 \\ -1 & 0 & 1 \end{bmatrix} \vec{X}}$$

$$\left(\vec{X} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}\right)$$

#4b. Write the given system without the use of matrices:

$$\vec{X}' = \begin{bmatrix} 7 & 5 & -9 \\ 4 & 1 & 1 \\ 0 & -2 & 3 \end{bmatrix} \vec{X} + \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} e^{5t} + \begin{bmatrix} 8 \\ 0 \\ 3 \end{bmatrix} e^{-2t}$$

$$\frac{dx}{dt} = 7x + 5y - 9z + 8e^{-2t}$$

$$\frac{dy}{dt} = 4x + y + z + 2e^{5t}$$

$$\frac{dz}{dt} = -2y + 3z + e^{5t} + 3e^{-2t}$$

#5b. Write the given system without the use of matrices:

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & -7 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 4 \\ 8 \end{bmatrix}\sin t + \begin{bmatrix} t-4 \\ 2t+1 \end{bmatrix}e^{4t}$$

$$\boxed{\begin{aligned} \frac{dx}{dt} &= 3x - 7y + 4\sin t + (t-4)e^{4t} \\ \frac{dy}{dt} &= x + y + 8\sin t + (2t+1)e^{4t} \end{aligned}}$$

#6b. Verify that the vector $\vec{X}$ is a solution of the given system:

$$\frac{dx}{dt} = -2x + 5y \qquad \vec{X} = \begin{bmatrix} 5\cos t \\ 3\cos t - \sin t \end{bmatrix}e^t$$

$$\frac{dy}{dt} = -2x + 4y$$

$x = 5\cos t e^t$

$y = 3\cos t e^t - \sin t e^t$

$\frac{dx}{dt} = (5\cos t)(e^t) + (e^t)(-5\sin t)$

$= 5e^t\cos t - 5e^t\sin t$

$\frac{dy}{dt} = (3\cos t)(e^t) + (e^t)(-3\sin t)$

$- [(\sin t)(e^t) + (e^t)(\cos t)]$

$= 2e^t\cos t - 4e^t\sin t$

$\frac{dx}{dt} \stackrel{?}{=} -2x + 5y$

$5e^t\cos t - 5e^t\sin t \stackrel{?}{=} -2(5e^t\cos t) + 5(3e^t\cos t - e^t\sin t)$

$5e^t\cos t - 5e^t\sin t = 5e^t\cos t - 5e^t\sin t$ ✓

$\frac{dy}{dt} \stackrel{?}{=} -2x + 4y$

$2e^t\cos t - 4e^t\sin t \stackrel{?}{=} -2(5e^t\cos t) + 4(3e^t\cos t - e^t\sin t)$

$2e^t\cos t - 4e^t\sin t = 2e^t\cos t - 4e^t\sin t$ ✓

#7b. Verify that the vector $\vec{X}$ is a solution of the given system:

$$\vec{X}' = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}\vec{X}; \qquad \vec{X} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}e^t + \begin{bmatrix} 4 \\ -4 \end{bmatrix}te^t$$

$x = e^t + 4te^t$

$\frac{dx}{dt} = e^t + 4te^t + e^t \cdot 4$

$= 5e^t + 4te^t$

$y = 3e^t - 4te^t$

$\frac{dy}{dt} = 3e^t - [4te^t + e^t \cdot 4]$

$= -e^t - 4te^t$

$\frac{dx}{dt} \stackrel{?}{=} 2x + y$

$5e^t + 4te^t \stackrel{?}{=} 2(e^t + 4te^t) + (3e^t - 4te^t)$

$5e^t + 4te^t = 5e^t + 4te^t$ ✓

$\frac{dy}{dt} \stackrel{?}{=} -x$

$-e^t - 4te^t \stackrel{?}{=} -(e^t + 4te^t)$

$-e^t - 4te^t = -e^t - 4te^t$ ✓

#8b. Verify that the vector $\vec{X}$ is a solution of the given system:

$$\vec{X}' = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ -2 & 0 & -1 \end{bmatrix}\vec{X}; \qquad \vec{X} = \begin{bmatrix} \sin t \\ -\frac{1}{2}\sin t - \frac{1}{2}\cos t \\ -\sin t + \cos t \end{bmatrix}$$

$x = \sin t \qquad y = -\frac{1}{2}\sin t - \frac{1}{2}\cos t \qquad z = -\sin t + \cos t$

$\frac{dx}{dt} = \cos t \qquad \frac{dy}{dt} = -\frac{1}{2}\cos t + \frac{1}{2}\sin t \qquad \frac{dz}{dt} = -\cos t - \sin t$

$\frac{dx}{dt} \overset{?}{=} x + z$

$\cos t \overset{?}{=} \sin t + (-\sin t + \cos t)$

$\cos t = \cos t$ ✓

$\frac{dy}{dt} \overset{?}{=} x + y$

$-\frac{1}{2}\cos t + \frac{1}{2}\sin t \overset{?}{=} \sin t + (-\frac{1}{2}\sin t - \frac{1}{2}\cos t)$

$-\frac{1}{2}\cos t + \frac{1}{2}\sin t = -\frac{1}{2}\cos t + \frac{1}{2}\sin t$ ✓

$\frac{dz}{dt} \overset{?}{=} -2x - z$

$-\cos t - \sin t \overset{?}{=} -2(\sin t) - (-\sin t + \cos t)$

$-\cos t - \sin t = -\cos t - \sin t$ ✓

#9b. The given vectors are solutions of a system $\vec{X}' = \vec{A}\vec{X}$. Determine whether the vectors form a fundamental set on the interval $(-\infty, \infty)$:

$$\vec{X}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^t, \quad \vec{X}_2 = \begin{bmatrix} 2 \\ 6 \end{bmatrix} e^t + \begin{bmatrix} 8 \\ -8 \end{bmatrix} te^t$$

$$W = \begin{vmatrix} e^t & 2e^t + 8te^t \\ -e^t & 6e^t - 8te^t \end{vmatrix}$$

$e^t(6e^t - 8te^t) - (-e^t)(2e^t + 8te^t)$

$6e^{2t} - 8te^{2t} + 2e^{2t} + 8te^{2t}$

$8e^{2t} \neq 0$ So these do form a fundamental set.
$(-\infty, \infty)$

#10b. Verify that the vector $\vec{X}_p$ is a particular solution of the given system:

$$\vec{X}' = \begin{bmatrix} 1 & 2 & 3 \\ -4 & 2 & 0 \\ -6 & 1 & 0 \end{bmatrix}\vec{X} + \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix}\sin 3t; \qquad \vec{X}_p = \begin{bmatrix} \sin 3t \\ 0 \\ \cos 3t \end{bmatrix}$$

$x = \sin 3t \qquad y = 0 \qquad z = \cos 3t$

$\frac{dx}{dt} = 3\cos 3t \qquad \frac{dy}{dt} = 0 \qquad \frac{dz}{dt} = -3\sin 3t$

$\frac{dx}{dt} \overset{?}{=} x + 2y + 3z - \sin 3t$

$3\cos 3t \overset{?}{=} (\sin 3t) + 2(0) + 3(\cos 3t) - \sin 3t$

$3\cos 3t = 3\cos 3t$ ✓

$\frac{dy}{dt} \overset{?}{=} -4x + 2y + 4\sin 3t$

$0 \overset{?}{=} -4(\sin 3t) + 2(0) + 4\sin 3t$

$0 = 0$ ✓

$\frac{dz}{dt} \overset{?}{=} -6x + y + 3\sin 3t$

$-3\sin 3t \overset{?}{=} -6(\sin 3t) + (0) + 3\sin 3t$

$-3\sin 3t = -3\sin 3t$ ✓

8.2 day 1

#1b. Find the general solution of the system:

$$\frac{dx}{dt} = 2x + 2y$$

$$\frac{dy}{dt} = x + 3y$$

$\vec{X}' = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}\vec{X}$

$\det(\vec{A} - \lambda\vec{I}) = 0$

$\begin{vmatrix} 2-\lambda & 2 \\ 1 & 3-\lambda \end{vmatrix} = 0$

$(2-\lambda)(3-\lambda) - 2 = 0$

$6 - 5\lambda + \lambda^2 - 2 = 0$

$\lambda^2 - 5\lambda + 4 = 0$

$(\lambda-1)(\lambda-4) = 0$

$\lambda = 1 \quad \lambda = 4$

now, $(\vec{A} - \lambda\vec{I})\vec{K} = \vec{0}$

$\lambda = 1$

$\begin{bmatrix} 2-1 & 2 \\ 1 & 3-1 \end{bmatrix}\begin{bmatrix} K_1 \\ K_2 \end{bmatrix} = \vec{0}$

$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 1 & 2 & 0 \end{array}\right]$

rref

$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$K_1 + 2K_2 = 0$

$K_1 = -2K_2$

$(-2K_2, K_2) \Rightarrow (-2, 1)$

$\lambda = 1 : \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$\lambda = 4$

$\begin{bmatrix} 2-4 & 2 \\ 1 & 3-4 \end{bmatrix}\begin{bmatrix} K_1 \\ K_2 \end{bmatrix} = \vec{0}$

$\left[\begin{array}{cc|c} -2 & 2 & 0 \\ 1 & -1 & 0 \end{array}\right]$

rref

$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$K_1 - K_2 = 0$

$K_1 = K_2$

$(K_2, K_2) \Rightarrow (1, 1)$

$\lambda = 4 : \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\boxed{\vec{X} = C_1\begin{bmatrix} -2 \\ 1 \end{bmatrix}e^{t} + C_2\begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{4t}}$

#2b. Find the general solution of the system:

$$\frac{dx}{dt} = -\frac{5}{2}x + 2y$$

$$\frac{dy}{dt} = \frac{3}{4}x - 2y$$

$\vec{X}' = \begin{bmatrix} -5/2 & 2 \\ 3/4 & -2 \end{bmatrix}\vec{X}$

$\begin{vmatrix} -5/2-\lambda & 2 \\ 3/4 & -2-\lambda \end{vmatrix} = 0$

$(-5/2-\lambda)(-2-\lambda) - \frac{3}{2} = 0$

$5 + \frac{9}{2}\lambda + \lambda^2 - \frac{3}{2} = 0$

$\lambda^2 + 9/2\lambda + 7/2 = 0$

$2\lambda^2 + 9\lambda + 7 = 0$

$\frac{(2\lambda+2)(2\lambda+7)}{2 \cdot 1} = 0$

$\begin{array}{c|c} M & A \\ 14 & 9 \\ (2)(7) & 9 \end{array}$

$(\lambda+1)(2\lambda+7) = 0$

$\lambda = -1 \quad \lambda = -\frac{7}{2}$

$\lambda = -1$

$\begin{bmatrix} -5/2+1 & 2 \\ 3/4 & -2+1 \end{bmatrix}\begin{bmatrix} K_1 \\ K_2 \end{bmatrix} = \vec{0}$

$\left[\begin{array}{cc|c} -3/2 & 2 & 0 \\ 3/4 & -1 & 0 \end{array}\right]$ rref $\left[\begin{array}{cc|c} 1 & -4/3 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$K_1 - 4/3K_2 = 0, \; K_1 = 4/3K_2, \quad (4/3K_2, K_2) \; (4, 3)$

$\lambda = -1 \quad \begin{bmatrix} 4 \\ 3 \end{bmatrix}$

$\lambda = -7/2$

$\begin{bmatrix} -5/2+7/2 & 2 \\ 3/4 & -2+7/2 \end{bmatrix}\begin{bmatrix} K_1 \\ K_2 \end{bmatrix} = \vec{0}$

$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 3/4 & 3/2 & 0 \end{array}\right]$ rref $\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$K_1 + 2K_2 = 0, \; K_1 = -2K_2, \; (-2K_2, K_2) \; (-2, 1)$

$\lambda = -7/2 \quad \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$\boxed{\vec{X} = C_1\begin{bmatrix} 4 \\ 3 \end{bmatrix}e^{-t} + C_2\begin{bmatrix} -2 \\ 1 \end{bmatrix}e^{-7/2t}}$

#3b. Find the general solution of the system:

$$\vec{X}' = \begin{bmatrix} -6 & 2 \\ -3 & 1 \end{bmatrix} \vec{X}$$

$$\begin{vmatrix} -6-\lambda & 2 \\ -3 & 1-\lambda \end{vmatrix} = 0$$

$$(-6-\lambda)(1-\lambda) + 6 = 0$$

$$-6 + 5\lambda + \lambda^2 + 6 = 0$$

$$\lambda^2 + 5\lambda = 0$$

$$\lambda(\lambda+5) = 0$$

$$\underline{\lambda = 0}, \ \underline{\lambda = -5}$$

$\underline{\lambda = 0}$

$$\begin{bmatrix} -6-0 & 2 \\ -3 & 1-0 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \vec{0}$$

$$\left[\begin{array}{cc|c} -6 & 2 & 0 \\ -3 & 1 & 0 \end{array}\right] \text{rref} \left[\begin{array}{cc|c} 1 & -1/3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$k_1 - \frac{1}{3}k_2 = 0$, $k_1 = \frac{1}{3}k_2$, $(\frac{1}{3}k_2, k_2)$ $(1,3)$

$\lambda = 0$ $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$\underline{\lambda = -5}$

$$\begin{bmatrix} -6+5 & 2 \\ -3 & 1+5 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \vec{0}$$

$$\left[\begin{array}{cc|c} -1 & 2 & 0 \\ -3 & 6 & 0 \end{array}\right] \text{rref} \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$k_1 - 2k_2 = 0$, $k_1 = 2k_2$, $(2k_2, k_2)$ $(2,1)$

$\lambda = -5$ $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$$\vec{x} = c_1\begin{bmatrix} 1 \\ 3 \end{bmatrix}\underbrace{e^{0t}}_{1} + c_2\begin{bmatrix} 2 \\ 1 \end{bmatrix}e^{-5t}$$

$$\boxed{\vec{X} = c_1\begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2\begin{bmatrix} 2 \\ 1 \end{bmatrix}e^{-5t}}$$

#4b. Find the general solution of the system:

$$\frac{dx}{dt} = 2x - 7y$$

$$\frac{dy}{dt} = 5x + 10y + 4z$$

$$\frac{dz}{dt} = 5y + 2z$$

$$\vec{X}' = \begin{bmatrix} 2 & -7 & 0 \\ 5 & 10 & 4 \\ 0 & 5 & 2 \end{bmatrix}\vec{X}$$

$$\begin{array}{ccc} + & - & + \end{array}$$

$$\begin{vmatrix} 2-\lambda & -7 & 0 \\ 5 & 10-\lambda & 4 \\ 0 & 5 & 2-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)\begin{vmatrix} 10-\lambda & 4 \\ 5 & 2-\lambda \end{vmatrix} - (-7)\begin{vmatrix} 5 & 4 \\ 0 & 2-\lambda \end{vmatrix} + (0)\begin{vmatrix} 5 & 10-\lambda \\ 0 & 5 \end{vmatrix}$$

$$(2-\lambda)\left[(10-\lambda)(2-\lambda) - 20\right] + 7\left[5(2-\lambda) - 0\right] + 0\,\cancel{[\;]}$$

$$(2-\lambda)(10-\lambda)(2-\lambda) - 20(2-\lambda) + 35(2-\lambda) = 0$$

$$(2-\lambda)\left[(10-\lambda)(2-\lambda) - 20 + 35\right] = 0$$

$$(2-\lambda)\left[20 - 12\lambda + \lambda^2 + 15\right) = 0$$

$$(2-\lambda)(\lambda^2 - 12\lambda + 35) = 0$$

$$(2-\lambda)(\lambda - 7)(\lambda - 5) = 0$$

$$\lambda = 2 \quad \lambda = 7 \quad \lambda = 5$$

$\lambda = 2$

$$\begin{bmatrix} 2-2 & -7 & 0 \\ 5 & 10-2 & 4 \\ 0 & 5 & 2-2 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \vec{0}$$

$$\left[\begin{array}{ccc|c} 0 & -7 & 0 & 0 \\ 5 & 8 & 4 & 0 \\ 0 & 5 & 0 & 0 \end{array}\right]$$

rref

$$\left[\begin{array}{ccc|c} 1 & 0 & 4/5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$k_1 + 4/5 k_3 = 0 \rightarrow k_1 = 4/5 k_3$

$k_2 = 0$

$(4/5 k_3, 0, k_3)$

$(4, 0, 5)$

$$\lambda = 2 \begin{bmatrix} 4 \\ 0 \\ 5 \end{bmatrix}$$

$\lambda = 7$

$$\begin{bmatrix} 2-7 & -7 & 0 \\ 5 & 10-7 & 4 \\ 0 & 0 & 2-7 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \vec{0}$$

$$\left[\begin{array}{ccc|c} -5 & -7 & 0 & 0 \\ 5 & 3 & 4 & 0 \\ 0 & 5 & -5 & 0 \end{array}\right]$$

rref

$$\left[\begin{array}{ccc|c} 1 & 0 & 7/5 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$k_1 + 7/5 k_3 = 0 \rightarrow k_1 = -7/5 k_3$

$k_2 - k_3 = 0 \rightarrow k_2 = k_3$

$(-7/5 k_3, k_3, k_3)$

$(-7, 5, 5)$

$$\lambda = 7 \begin{bmatrix} -7 \\ 5 \\ 5 \end{bmatrix}$$

$\lambda = 5$

$$\begin{bmatrix} 2-5 & -7 & 0 \\ 5 & 10-5 & 4 \\ 0 & 5 & 2-5 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \vec{0}$$

$$\left[\begin{array}{ccc|c} -3 & -7 & 0 & 0 \\ 5 & 5 & 4 & 0 \\ 0 & 5 & -3 & 0 \end{array}\right]$$

rref

$$\left[\begin{array}{ccc|c} 1 & 0 & 7/5 & 0 \\ 0 & 1 & -3/5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$k_1 + 7/5 k_3 = 0, \rightarrow k_1 = -7/5 k_3$

$k_2 - 3/5 k_3 = 0 \rightarrow k_2 = 3/5 k_3$

$(-7/5 k_3, 3/5 k_3, k_3)$

$(-7, 3, 5)$

$$\lambda = 5 \begin{bmatrix} -7 \\ 3 \\ 5 \end{bmatrix}$$

$$\boxed{X = c_1\begin{bmatrix} 4 \\ 0 \\ 5 \end{bmatrix}e^{2t} + c_2\begin{bmatrix} -7 \\ 5 \\ 5 \end{bmatrix}e^{7t} + c_3\begin{bmatrix} -7 \\ 3 \\ 5 \end{bmatrix}e^{5t}}$$

#5b. Find the general solution of the system:

$$\vec{X}' = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \vec{X}$$

$$\begin{vmatrix} 1-\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)\left[(1-\lambda)(1-\lambda)-0\right] - 0\left[\sim\right] + 1\left[0-(1-\lambda)\right] = 0$$

$$(1-\lambda)(1-\lambda)(1-\lambda) - (1-\lambda) = 0$$

$$(1-\lambda)\left[(1-\lambda)(1-\lambda)-1\right] = 0$$

$$(1-\lambda)\left[1-2\lambda+\lambda^2-1\right] = 0$$

$$(1-\lambda)(\lambda^2-2\lambda) = 0, \quad (1-\lambda)\lambda(\lambda-2) = 0$$

$$\lambda = 1 \quad \lambda = 0 \quad \lambda = 2$$

$\lambda = 1$

$$\begin{bmatrix} 1-0 & 0 & 1 \\ 0 & 1-1 & 0 \\ 1 & 0 & 1-1 \end{bmatrix}\begin{bmatrix} K_1 \\ K_2 \\ K_3 \end{bmatrix} = \vec{0}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right]$$

rref

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$K_1 = 0$

$K_3 = 0$

$(0, K_2, 0)$

$(0, 1, 0)$

$$\lambda = 1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$\lambda = 0$

$$\begin{bmatrix} 1-0 & 0 & 1 \\ 0 & 1-0 & 0 \\ 1 & 0 & 1-0 \end{bmatrix}\begin{bmatrix} K_1 \\ K_2 \\ K_3 \end{bmatrix} = \vec{0}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right]$$

rref

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$K_1 + K_3 = 0 \rightarrow K_1 = -K_3$

$K_2 = 0$

$(-K_3, 0, K_3)$

$(-1, 0, 1)$

$$\lambda = 0 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$\lambda = 2$

$$\begin{bmatrix} 1-2 & 0 & 1 \\ 0 & 1-2 & 0 \\ 1 & 0 & 1-2 \end{bmatrix}\begin{bmatrix} K_1 \\ K_2 \\ K_3 \end{bmatrix} = \vec{0}$$

$$\left[\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \end{array}\right]$$

rref

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$K_1 - K_3 = 0 \rightarrow K_1 = K_3$

$K_2 = 0$

$(K_3, 0, K_3)$

$(1, 0, 1)$

$$\lambda = 2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\vec{X} = C_1\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}e^{t} + C_2\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}e^{0t} + C_3\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}e^{2t}$$

$$\boxed{\vec{X} = C_1\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}e^{t} + C_2\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + C_3\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}e^{2t}}$$

#6b. Solve the initial-value problem:

$$\vec{X}' = \begin{bmatrix} 1 & 1 & 4 \\ 0 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix}\vec{X}, \quad \vec{X}(0) = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$$

$$\begin{array}{c} + \\ - \\ + \end{array}\begin{vmatrix} 1-\lambda & 1 & 4 \\ 0 & 2-\lambda & 0 \\ 1 & 1 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)[(2-\lambda)(1-\lambda)-0] - 0[\sim] + 1[0-4(2-\lambda)] \Rightarrow$$

$$(1-\lambda)(2-\lambda)(1-\lambda) - 4(2-\lambda) \Rightarrow$$

$$(2-\lambda)[(1-\lambda)(1-\lambda)-4] = (2-\lambda)[1-2\lambda^2+\lambda^2-4] = (2-\lambda)(\lambda^2-2\lambda^2-3) \Rightarrow$$

$$(2-\lambda)(\lambda-3)(\lambda+1) = 0$$

$\lambda=2 \quad \lambda=3 \quad \lambda=-1$

$\lambda=2$

$$\begin{bmatrix} 1-2 & 1 & 4 \\ 0 & 2-2 & 0 \\ 1 & 1 & 1-2 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \vec{0}$$

$$\left[\begin{array}{ccc|c} -1 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 \end{array}\right]$$

rref

$$\left[\begin{array}{ccc|c} 1 & 0 & -5/2 & 0 \\ 0 & 1 & 3/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$k_1 - \frac{5}{2}k_3 = 0 \Rightarrow k_1 = \frac{5}{2}k_3$

$k_2 + \frac{3}{2}k_3 = 0 \Rightarrow k_2 = -\frac{3}{2}k_3$

$(\frac{5}{2}k_3, -\frac{3}{2}k_3, k_3)$

$(5, -3, 2)$

$\lambda=2 \begin{bmatrix} 5 \\ -3 \\ 2 \end{bmatrix}$

$\lambda=3$

$$\begin{bmatrix} 1-3 & 1 & 4 \\ 0 & 2-3 & 0 \\ 1 & 1 & 1-3 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \vec{0}$$

$$\left[\begin{array}{ccc|c} -2 & 1 & 4 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right]$$

rref

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$k_1 - 2k_3 = 0 \Rightarrow k_1 = 2k_3$

$k_2 = 0$

$(2k_3, 0, k_3)$

$(2, 0, 1)$

$\lambda=3 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$

$\lambda=-1$

$$\begin{bmatrix} 1+1 & 1 & 4 \\ 0 & 2+1 & 0 \\ 1 & 1 & 1+1 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \vec{0}$$

$$\left[\begin{array}{ccc|c} 2 & 1 & 4 & 0 \\ 0 & 3 & 0 & 0 \\ 1 & 1 & 2 & 0 \end{array}\right]$$

rref

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$k_1 + 2k_3 = 0 \Rightarrow k_1 = -2k_3$

$k_2 = 0$

$(-2k_3, 0, k_3)$

$(-2, 0, 1)$

$\lambda=-1 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$

$$\underline{\underline{\vec{X} = c_1\begin{bmatrix} 5 \\ -3 \\ 2 \end{bmatrix}e^{2t} + c_2\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}e^{3t} + c_3\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}e^{-t}}}$$

now use $\vec{X}(0) = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} = c_1\begin{bmatrix} 5 \\ -3 \\ 2 \end{bmatrix}e^0 + c_2\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}e^0 + c_3\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}e^0$$

$$\begin{cases} 5c_1 + 2c_2 - 2c_3 = 1 \\ -3c_1 = 3 \\ 2c_1 + c_2 + c_3 = 0 \end{cases}$$

$$\left[\begin{array}{ccc|c} 5 & 2 & -2 & 1 \\ -3 & 0 & 0 & 3 \\ 2 & 1 & 1 & 0 \end{array}\right] \text{rref} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 5/2 \\ 0 & 0 & 1 & -1/2 \end{array}\right] \begin{array}{l} =c_1 \\ =c_2 \\ =c_3 \end{array}$$

$$\boxed{\vec{X} = -1\begin{bmatrix} 5 \\ -3 \\ 2 \end{bmatrix}e^{2t} + \frac{5}{2}\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}e^{3t} - \frac{1}{2}\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}e^{-t}}$$

8.2 day 2

#1b. Find the general solution of the system:

$$\frac{dx}{dt} = -6x + 5y \qquad \vec{X}' = \begin{bmatrix} -6 & 5 \\ -5 & 4 \end{bmatrix} \vec{X}$$

$$\frac{dy}{dt} = -5x + 4y \qquad \begin{vmatrix} -6-\lambda & 5 \\ -5 & 4-\lambda \end{vmatrix} = 0$$

$(-6-\lambda)(4-\lambda) + 25 = 0$

$\lambda^2 + 2\lambda - 24 + 25 = 0$

$\lambda^2 + 2\lambda + 1 = 0$

$(\lambda+1)(\lambda+1) = 0$

$\underline{\lambda = -1}$ (repeated)

$\underline{\underline{\lambda = -1}}$ $\begin{bmatrix} -6+1 & 5 \\ -5 & 4+1 \end{bmatrix}\begin{bmatrix} K_1 \\ K_2 \end{bmatrix} = \vec{0}$

$\left[\begin{array}{cc|c} -5 & 5 & 0 \\ -5 & 5 & 0 \end{array}\right]$

rref

$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$K_1 - K_2 = 0 \rightarrow K_1 = K_2$

(K_2, K_2) $(1,1)$ $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ for $\underline{\lambda = -1}$

<u>repeat</u>

$\left[\begin{array}{cc|c} -5 & 5 & 1 \\ -5 & 5 & 1 \end{array}\right]$

rref

$\left[\begin{array}{cc|c} 1 & -1 & -1/5 \\ 0 & 0 & 0 \end{array}\right]$

$K_1 - K_2 = -1/5 \rightarrow K_1 = K_2 - 1/5$

$(K_2 - 1/5, K_2)$

$(-1/5, 0)$ $\begin{bmatrix} -1/5 \\ 0 \end{bmatrix}$

$$\boxed{\vec{X} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t} + c_2\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} te^{-t} + \begin{bmatrix} -1/5 \\ 0 \end{bmatrix} e^{-t}\right)}$$

#2b. Find the general solution of the system:

$$\vec{X}' = \begin{bmatrix} 12 & -9 \\ 4 & 0 \end{bmatrix} \vec{X} \qquad \begin{vmatrix} 12-\lambda & -9 \\ 4 & 0-\lambda \end{vmatrix}$$

$(12-\lambda)(-\lambda) + 36 = 0$

$\lambda^2 - 12\lambda + 36 = 0$

$(\lambda - 6)(\lambda - 6) = 0$

$\underline{\lambda = 6}$ (repeated)

$\underline{\underline{\lambda = 6}}$ $\begin{bmatrix} 12-6 & -9 \\ 4 & 0-6 \end{bmatrix}\begin{bmatrix} K_1 \\ K_2 \end{bmatrix} = \vec{0}$

$\left[\begin{array}{cc|c} 6 & -9 & 0 \\ 4 & -6 & 0 \end{array}\right]$

rref

$\left[\begin{array}{cc|c} 1 & -3/2 & 0 \\ 0 & 0 & 0 \end{array}\right]$ $K_1 - 3/2K_2 = 0$, $K_1 = 3/2K_2$

$(3/2K_2, K_2)$ $(3, 2)$ $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ for $\lambda = 6$

<u>repeated</u>

$\left[\begin{array}{cc|c} 6 & -9 & 3 \\ 4 & -6 & 2 \end{array}\right]$

rref

$\left[\begin{array}{cc|c} 1 & -3/2 & 1/2 \\ 0 & 0 & 0 \end{array}\right]$ $K_1 - 3/2K_2 = 1/2$, $K_1 = 3/2K_2 + 1/2$

$(3/2K_2 + 1/2, K_2)$

$(1/2, 0)$ $\begin{bmatrix} 1/2 \\ 0 \end{bmatrix}$

$$\boxed{\vec{X} = c_1 \begin{bmatrix} 3 \\ 2 \end{bmatrix} e^{6t} + c_2\left(\begin{bmatrix} 3 \\ 2 \end{bmatrix} te^{6t} + \begin{bmatrix} 1/2 \\ 0 \end{bmatrix} e^{6t}\right)}$$

#3b. Find the general solution of the system:

$$\frac{dx}{dt} = 3x+2y+4z$$

$$\frac{dy}{dt} = 2x+2z$$

$$\frac{dz}{dt} = 4x+2y+3z$$

$$\vec{X}' = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix} \vec{X}$$

$$\begin{array}{ccc} + & - & + \end{array}$$

$$\begin{vmatrix} 3-\lambda & 2 & 4 \\ 2 & 0-\lambda & 2 \\ 4 & 2 & 3-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)\left[-\lambda(3-\lambda)-4\right]-(2)\left[2(3-\lambda)-8\right]+(4)\left[4+4\lambda\right]=0$$

$$-\lambda(9-6\lambda+\lambda^2)-12+4\lambda-12+4\lambda+16+16+16\lambda=0$$

$$-9\lambda+6\lambda^2-\lambda^3-12+4\lambda-12+4\lambda+16+16+16\lambda=0$$

$$-\lambda^3+6\lambda^2+15\lambda+8=0$$

$$-(\lambda^3-6\lambda^2-15\lambda-8)=0$$

try -1

$$\begin{array}{r|rrrr} -1 & 1 & -6 & -15 & -8 \\ & & -1 & 7 & 8 \\ \hline & 1 & -7 & -8 & 0 \end{array}$$

$$(\lambda+1)(\lambda^2-7\lambda-8)=0$$

$$(\lambda+1)(\lambda-8)(\lambda+1)=0$$

$\underline{\underline{\lambda=-1}}$ (repeated) $\quad \underline{\underline{\lambda=8}}$

$\underline{\underline{\lambda=8}}$

$$\left[\begin{array}{ccc|c} 3-8 & 2 & 4 & 0 \\ 2 & 0-8 & 2 & 0 \\ 4 & 2 & 3-8 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} -5 & 2 & 4 & 0 \\ 2 & -8 & 2 & 0 \\ 4 & 2 & -5 & 0 \end{array}\right]$$

rref

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$k_1 - k_3 = 0 \to k_1 = k_3$

$k_2 - \frac{1}{2}k_3 = 0 \to k_2 = \frac{1}{2}k_3$

$(k_3, \frac{1}{2}k_3, k_3)$

$(2, 1, 2)$

$$\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$

for $\lambda=8$

$\underline{\underline{\lambda=-1}}$

$$\left[\begin{array}{ccc|c} 3+1 & 2 & 4 & 0 \\ 2 & 0+1 & 2 & 0 \\ 4 & 2 & 3+1 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 4 & 2 & 4 & 0 \\ 2 & 1 & 2 & 0 \\ 4 & 2 & 4 & 0 \end{array}\right]$$

rref

$$\left[\begin{array}{ccc|c} 1 & 1/2 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$k_1 + \frac{1}{2}k_2 + k_3 = 0$

$k_1 = -\frac{1}{2}k_2 - k_3$

$(-\frac{1}{2}k_2 - k_3, k_2, k_3)$

choose $k_2=-2, k_3=1$

$(\overset{1-1}{0}, -2, 1)$

choose $k_2=-2, k_3=0$

$(\overset{1-0}{1}, -2, 0)$

$$\begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} \& \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$$

both w/ $\lambda=-1$

$$\boxed{\vec{X} = c_1\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}e^{8t} + c_2\begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}e^{-t} + c_3\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}e^{-t}}$$

#4b. Find the general solution of the system:

$$\vec{X}' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & -1 & 1 \end{bmatrix} \vec{X}$$

$$\begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 3-\lambda & 1 \\ 0 & -1 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)[(3-\lambda)(1-\lambda)+1] - (0)[\sim] + (0)(\sim) = 0$$

$$(1-\lambda)[3-4\lambda+\lambda^2+1] = 0$$

$$(1-\lambda)(\lambda^2-4\lambda+4) = 0$$

$$(1-\lambda)(\lambda-2)(\lambda-2) = 0$$

$\lambda = 1$, $\lambda = 2$ repeated

$\lambda = 1$

$$\left[\begin{array}{ccc|c} 1-1 & 0 & 0 & 0 \\ 0 & 3-1 & 1 & 0 \\ 0 & -1 & 1-1 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right]$$

rref

$$\left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$K_2 = 0$

$K_3 = 0$

$(K_1, 0, 0)$

$(1, 0, 0)$

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

for $\lambda = 1$

$\lambda = 2$

$$\left[\begin{array}{ccc|c} 1-2 & 0 & 0 & 0 \\ 0 & 3-2 & 1 & 0 \\ 0 & -1 & 1-2 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 \end{array}\right]$$

rref

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$K_1 = 0$

$K_2 + K_3 = 0 \rightarrow K_2 = -K_3$

$(0, -K_3, K_3)$

$(0, -1, 1)$

$$\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

for $\lambda = 2$

repeat for $\lambda = 2$

$$\left[\begin{array}{ccc|c} -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & -1 & -1 & 1 \end{array}\right]$$

rref

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$K_1 = 0$

$K_2 + K_3 = -1 \rightarrow K_2 = -1 - K_3$

$(0, -1-K_3, K_3)$

$(0, -1, 0)$

$$\begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$$

repeat for $\lambda = 2$

$$\boxed{\vec{X} = c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} e^{2t} + c_3 \left(\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} t e^{2t} + \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} e^{2t} \right)}$$

#5b. Find the general solution of the system:

$$\vec{X}' = \begin{bmatrix} 4 & 1 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 4 \end{bmatrix} \vec{X}$$

$$\begin{array}{ccc} + & - & + \end{array}$$
$$\begin{vmatrix} 4-\lambda & 1 & 0 \\ 0 & 4-\lambda & 1 \\ 0 & 0 & 4-\lambda \end{vmatrix} = 0$$

$$(4-\lambda)[(4-\lambda)(4-\lambda)-0] - (1)(0-0) + 0(\sim) = 0$$

$$(4-\lambda)(4-\lambda)(4-\lambda) = 0 \qquad \underline{\lambda = 4} \text{ multiplicity } 3$$

$\underline{\lambda = 4}$

$$\left[\begin{array}{ccc|c} 4-4 & 1 & 0 & 0 \\ 0 & 4-4 & 1 & 0 \\ 0 & 0 & 4-4 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

rref

$$\left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$K_2 = 0$

$K_3 = 0$

$(K_1, 0, 0)$

$(1, 0, 0)$

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

for $\lambda = 4$

$\underline{1^{st} \text{ repeat } \lambda = 4}$

$$\left[\begin{array}{ccc|c} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

rref

$$\left[\begin{array}{ccc|c} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$K_2 = 1$

$K_3 = 0$

$(K_1, 1, 0)$

$(0, 1, 0)$

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

1^{st} repeat for $\lambda = 4$

$\underline{2^{nd} \text{ repeat for } \lambda = 4}$

$$\left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

rref

$$\left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$K_2 = 0$

$K_3 = 1$

$(K_1, 0, 1)$

$(0, 0, 1)$

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

2^{nd} repeat for $\lambda = 4$

$$\boxed{\vec{X} = c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{4t} + c_2\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} te^{4t} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^{4t}\right) + c_3\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \frac{t^2}{2} e^{4t} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} te^{4t} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{4t}\right)}$$

#6b. Solve the initial-value problem:

$$\vec{X}' = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \vec{X}, \qquad \vec{X}(0) = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$$

$$\begin{array}{ccc} + & - & + \end{array}$$
$$\begin{bmatrix} -\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & -\lambda \end{bmatrix} = 0$$

$(-\lambda)[-\lambda(1-\lambda)-0] - 0(-) + 1[0-(1-\lambda)] = 0$

$\lambda^2(1-\lambda) - (1-\lambda) = 0$

$(1-\lambda)(\lambda^2-1) = 0$

$(1-\lambda)(\lambda-1)(\lambda+1) = 0$ $\quad \lambda = 1$ repeated, $\lambda = -1$

$\lambda = -1$

$$\left[\begin{array}{ccc|c} 0+1 & 0 & 1 & 0 \\ 0 & 1+1 & 0 & 0 \\ 1 & 0 & 0+1 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right]$$

rref

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$k_1 + k_3 = 0 \Rightarrow k_1 = -k_3$

$k_2 = 0$

$(-k_3, 0, k_3)$

$(-1, 0, 1)$

$\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ for $\lambda = -1$

$\lambda = 1$

$$\left[\begin{array}{ccc|c} 0-1 & 0 & 1 & 0 \\ 0 & 1-1 & 0 & 0 \\ 1 & 0 & 0-1 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 \end{array}\right]$$

rref

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$k_1 - k_3 = 0 \rightarrow k_1 = k_3$

(k_3, k_2, k_3)

$(1, 0, 1)$

could also choose

$(0, 1, 0)$

$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

for $\lambda = 1$

$$\vec{X} = c_1 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{t} + c_3 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^{t}$$

now, $\vec{X}(0) = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$

$x = -c_1 e^{-t} + c_2 e^{t}$

$1 = -c_1 e^{0} + c_2 e^{0}$

$1 = -c_1 + c_2$

$y = c_3 e^{t}$

$z = c_3 e^{0}$

$\underline{c_3 = 2}$

$z = c_1 e^{-t} + c_2 e^{t}$

$5 = c_1 e^{0} + c_2 e^{0}$

$5 = c_1 + c_2$

$$\begin{aligned} -c_1 + c_2 &= 1 \\ + \quad c_1 + c_2 &= 5 \\ \hline 2c_2 &= 6 \end{aligned}$$

$\underline{c_2 = 3}$

$c_1 + 3 = 5$

$\underline{c_1 = 2}$

$$\boxed{\vec{X} = 2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} e^{-t} + 3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{t} + 2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^{t}}$$

8.2 day 3

#1b. Find the general solution of the system:

$$\frac{dx}{dt} = x + y$$

$$\frac{dy}{dt} = -2x - y$$

$$\begin{vmatrix} 1-\lambda & 1 \\ -2 & -1-\lambda \end{vmatrix} = 0 \qquad (1-\lambda)(-1-\lambda)+2=0 \qquad \lambda^2+1=0 \qquad \lambda^2=-1 \qquad \lambda = \pm i = 0 \pm i$$

$\underline{\lambda = 0 + i}$

$$\left[\begin{array}{cc|c} 1-(0+i) & 1 & 0 \\ -2 & -1-(0+i) & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1-i & 1 & 0 \\ -2 & -1-i & 0 \end{array}\right]$$

$(1-i)K_1 + K_2 = 0$

$K_2 = -(1-i)K_1$

$(K_1, -(1-i)K_1)$

$(1, -1+i) \quad \begin{bmatrix} 1 \\ -1+i \end{bmatrix}$

$\vec{B}_1 = Re\begin{bmatrix} 1 \\ -1+i \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\vec{B}_2 = Im\begin{bmatrix} 1 \\ -1+i \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\vec{X}_1 = (\vec{B}_1 \cos\beta t - \vec{B}_2 \sin\beta t)e^{\alpha t}$

$= \left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\cos t - \begin{bmatrix} 0 \\ 1 \end{bmatrix}\sin t\right)\cancel{e^{0t}}\, 1$

$\vec{X}_2 = (\vec{B}_2 \cos\beta t + \vec{B}_1 \sin\beta t)e^{\alpha t}$

$= \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\cos t + \begin{bmatrix} 1 \\ -1 \end{bmatrix}\sin t\right)\cancel{e^{0t}}\, 1$

$\vec{X} = C_1\vec{X}_1 + C_2\vec{X}_2$

$$\boxed{\begin{array}{l} \vec{X} = C_1\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\cos t - \begin{bmatrix} 0 \\ 1 \end{bmatrix}\sin t\right) \\ \quad + C_2\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\cos t + \begin{bmatrix} 1 \\ -1 \end{bmatrix}\sin t\right) \\ \qquad -\text{or}- \\ \vec{X} = C_1\begin{bmatrix} \cos t \\ -\cos t - \sin t \end{bmatrix} + C_2\begin{bmatrix} \sin t \\ \cos t - \sin t \end{bmatrix} \end{array}}$$

—or—

$-2K_1 + (-1-i)K_2 = 0$

$K_1 = \left(\frac{-1-i}{2}\right)K_2$

$\left(\frac{-1-i}{2}K_2, K_2\right) \quad \begin{bmatrix} -1-i \\ 2 \end{bmatrix}$

$(-1-i, 2)$

$\vec{B}_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \quad \vec{B}_2 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

$\vec{X}_1 = \left(\begin{bmatrix} -1 \\ 2 \end{bmatrix}\cos t - \begin{bmatrix} -1 \\ 0 \end{bmatrix}\sin t\right)\cancel{e^{0t}}\, 1$

$\vec{X}_2 = \left(\begin{bmatrix} -1 \\ 0 \end{bmatrix}\cos t + \begin{bmatrix} -1 \\ 2 \end{bmatrix}\sin t\right)\cancel{e^{0t}}\, 1$

$$\boxed{\begin{array}{l} \vec{X} = C_1\left(\begin{bmatrix} -1 \\ 2 \end{bmatrix}\cos t - \begin{bmatrix} -1 \\ 0 \end{bmatrix}\sin t\right) \\ \quad + C_2\left(\begin{bmatrix} -1 \\ 0 \end{bmatrix}\cos t + \begin{bmatrix} -1 \\ 2 \end{bmatrix}\sin t\right) \\ \qquad -\text{or}- \\ \vec{X} = C_1\begin{bmatrix} -\cos t + \sin t \\ 2\cos t \end{bmatrix} + C_2\begin{bmatrix} -\cos t - \sin t \\ 2\sin t \end{bmatrix} \end{array}}$$

#2b. Find the general solution of the system:

$$\vec{X}' = \begin{bmatrix} 1 & -8 \\ 1 & -3 \end{bmatrix} \vec{X}$$

$$\begin{vmatrix} 1-\lambda & -8 \\ 1 & -3-\lambda \end{vmatrix} = 0 \qquad (1-\lambda)(-3-\lambda)+8=0$$

$$\lambda^2+2\lambda+5=0$$

$$\lambda = \frac{-2 \pm \sqrt{4-4(5)}}{2} = \frac{-2 \pm 4i}{2} = -1 \pm 2i$$

$\lambda = -1+2i$

$$\left[\begin{array}{cc|c} 1-(-1+2i) & -8 & 0 \\ 1 & -3-(-1+2i) & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 2-2i & -8 & 0 \\ 1 & -2-2i & 0 \end{array}\right]$$

$(2-2i)K_1 - 8K_2 = 0$

$K_2 = \frac{2-2i}{8}K_1 = \frac{1-i}{4}K_1$

$(K_1, \frac{1-i}{4}K_1) \quad (4, 1-i)$

$\begin{bmatrix} 4 \\ 1-i \end{bmatrix} \quad \vec{B}_1 = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \quad \vec{B}_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

$\vec{X}_1 = \left(\begin{bmatrix} 4 \\ 1 \end{bmatrix}\cos 2t - \begin{bmatrix} 0 \\ -1 \end{bmatrix}\sin 2t\right)e^{-t}$

$\vec{X}_2 = \left(\begin{bmatrix} 0 \\ -1 \end{bmatrix}\cos 2t + \begin{bmatrix} 4 \\ 1 \end{bmatrix}\sin 2t\right)e^{-t}$

$$\boxed{\begin{aligned} \vec{X} &= C_1\left(\begin{bmatrix} 4 \\ 1 \end{bmatrix}\cos 2t - \begin{bmatrix} 0 \\ -1 \end{bmatrix}\sin 2t\right)e^{-t} + C_2\left(\begin{bmatrix} 0 \\ -1 \end{bmatrix}\cos 2t + \begin{bmatrix} 4 \\ 1 \end{bmatrix}\sin 2t\right)e^{-t} \\ &\text{—or—} \\ \vec{X} &= C_1\begin{bmatrix} 4\cos 2t \\ \cos 2t - \sin 2t \end{bmatrix}e^{-t} + C_2\begin{bmatrix} 4\sin 2t \\ -\cos 2t + \sin 2t \end{bmatrix}e^{-t} \end{aligned}}$$

—or—

$K_1 + -(2+2i)K_2 = 0$

$K_1 = (2+2i)K_2$

$((2+2i)K_2, K_2) \quad (2+2i, 1)$

$\begin{bmatrix} 2+2i \\ 1 \end{bmatrix} \quad \vec{B}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \vec{B}_2 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

$\vec{X}_1 = \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\cos 2t - \begin{bmatrix} 2 \\ 0 \end{bmatrix}\sin 2t\right)e^{-t}$

$\vec{X}_2 = \left(\begin{bmatrix} 2 \\ 0 \end{bmatrix}\cos 2t + \begin{bmatrix} 2 \\ 1 \end{bmatrix}\sin 2t\right)e^{-t}$

$$\boxed{\begin{aligned} \vec{X} &= C_1\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\cos 2t - \begin{bmatrix} 2 \\ 0 \end{bmatrix}\sin 2t\right)e^{-t} + C_2\left(\begin{bmatrix} 2 \\ 0 \end{bmatrix}\cos 2t + \begin{bmatrix} 2 \\ 1 \end{bmatrix}\sin 2t\right)e^{-t} \\ &\text{—or—} \\ \vec{X} &= C_1\begin{bmatrix} 2\cos 2t - 2\sin 2t \\ \cos 2t \end{bmatrix}e^{-t} + C_2\begin{bmatrix} 2\cos 2t + 2\sin 2t \\ \sin 2t \end{bmatrix}e^{-t} \end{aligned}}$$

#3b. Find the general solution of the system:

$$\frac{dx}{dt} = 2x + y + 2z$$

$$\frac{dy}{dt} = 3x + 6z$$

$$\frac{dz}{dt} = -4x - 3z$$

$$\begin{vmatrix} 2-\lambda & 1 & 2 \\ 3 & 0-\lambda & 6 \\ -4 & 0 & -3-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)[-\lambda(-3-\lambda)-0]-1[3(-3-\lambda)+24]+2[0-4\lambda]=0$$
$$-\lambda(2-\lambda)(-3-\lambda)-3(-3-\lambda)-24-8\lambda=0$$
$$-\lambda(\lambda^2+\lambda-6)+9+3\lambda-24-8\lambda=0$$
$$-\lambda^3-\lambda^2+6\lambda+9+3\lambda-24-8\lambda=0$$
$$-\lambda^3-\lambda^2+\lambda-15=0 \quad \text{try } -3$$
$$\lambda^3+\lambda^2-\lambda+15=0$$

$$\begin{array}{r|rrrr} -3 & 1 & 1 & -1 & 15 \\ & & -3 & 6 & -15 \\ \hline & 1 & -2 & 5 & 0 \end{array}$$

$(\lambda+3)(\lambda^2-2\lambda+5)$

$\lambda=-3 \quad \lambda=\frac{2\pm\sqrt{4-4(5)}}{2}$

$\lambda=\frac{2\pm 4i}{2}=1\pm 2i$

$\lambda=-3$

$$\left[\begin{array}{ccc|c} 2+3 & 1 & 2 & 0 \\ 3 & 0+3 & 6 & 0 \\ -4 & 0 & -3+3 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 5 & 1 & 2 & 0 \\ 3 & 3 & 6 & 0 \\ -4 & 0 & 0 & 0 \end{array}\right]$$

rref

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$k_1=0$

$k_2+2k_3=0 \rightarrow k_2=-2k_3$

$(0,-2k_3,k_3)$

$(0,-2,1) \quad \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$ for $\lambda=-3$

$$\vec{X}_1=\begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}e^{-3t}$$

$\lambda=1+2i$

$$\left[\begin{array}{ccc|c} 2-(1+2i) & 1 & 2 & 0 \\ 3 & 0-(1+2i) & 6 & 0 \\ -4 & 0 & -3-(1+2i) & 0 \end{array}\right]$$

$$\begin{array}{c} ① \\ \\ ③ \end{array}\left[\begin{array}{ccc|c} 1-2i & 1 & 2 & 0 \\ 3 & -1-2i & 6 & 0 \\ -4 & 0 & -4-2i & 0 \end{array}\right] \leftarrow -4k_1+(-4-2i)k_3=0$$

$k_1=\frac{(-4-2i)}{4}k_3=(-1-\frac{1}{2}i)k_3$

① $(1-2i)k_1+k_2+2k_3=0$

$(1-2i)(-1-\frac{1}{2}i)k_3+k_2+2k_3=0$

$(-2+\frac{3}{2}i)k_3+2k_3+k_2=0$

$\frac{3}{2}ik_3+k_2=0 \rightarrow k_2=-\frac{3}{2}ik_3$

$((-1-\frac{1}{2}i)k_3, -\frac{3}{2}ik_3, k_3)$

$(-2-i, -3i, 2) \quad \begin{bmatrix} -2-i \\ -3i \\ 2 \end{bmatrix} \quad \vec{B}_1=\begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix} \quad \vec{B}_2=\begin{bmatrix} -1 \\ -3 \\ 0 \end{bmatrix}$

$$\vec{X}_2=\left(\begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix}\cos 2t-\begin{bmatrix} -1 \\ -3 \\ 0 \end{bmatrix}\sin 2t\right)e^{t}$$

$$\vec{X}_3=\left(\begin{bmatrix} -1 \\ -3 \\ 0 \end{bmatrix}\cos 2t+\begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix}\sin 2t\right)e^{t}$$

$$\vec{X}=c_1\begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}e^{-3t}+c_2\left(\begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix}\cos 2t-\begin{bmatrix} -1 \\ -3 \\ 0 \end{bmatrix}\sin 2t\right)e^{t}+c_3\left(\begin{bmatrix} -1 \\ -3 \\ 0 \end{bmatrix}\cos 2t+\begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix}\sin 2t\right)e^{t}$$

— or —

$$\vec{X}=c_1\begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}e^{-3t}+c_2\begin{bmatrix} -2\cos 2t+\sin 2t \\ 3\sin 2t \\ 2\cos 2t \end{bmatrix}e^{t}+c_3\begin{bmatrix} -\cos 2t-2\sin 2t \\ -3\cos 2t \\ 2\sin 2t \end{bmatrix}e^{t}$$

#4b. Find the general solution of the system:

$$\vec{X}' = \begin{bmatrix} 2 & 4 & 4 \\ -1 & -2 & 0 \\ -1 & 0 & -2 \end{bmatrix} \vec{X}$$

$$\begin{array}{ccc} + & - & + \end{array}$$

$$\begin{vmatrix} 2-\lambda & 4 & 4 \\ -1 & -2-\lambda & 0 \\ -1 & 0 & -2-\lambda \end{vmatrix}$$

$$(2-\lambda)\left[(-2-\lambda)(-2-\lambda)-0\right]-4\left[-(-2-\lambda)-0\right]+4\left[0+(-2-\lambda)\right]$$

$$(2-\lambda)(-2-\lambda)^2+4(-2-\lambda)+4(-2-\lambda)=0$$

$$(-2-\lambda)\left[(2-\lambda)(-2-\lambda)+8\right]=0$$

$$(-2-\lambda)\left[\lambda^2+4\right]=0$$

$$\underline{\lambda=-2} \quad \underline{\lambda=0\pm 2i}$$

$\underline{\lambda=-2}$

$$\left[\begin{array}{ccc|c} 2+2 & 4 & 4 & 0 \\ -1 & -2+2 & 0 & 0 \\ -1 & 0 & -2+2 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 4 & 4 & 4 & 0 \\ -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{array}\right]$$

rref

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$k_1=0$

$k_2+k_3=0 \rightarrow k_2=-k_3$

$(0,-k_3,k_3)$ $\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

$(0,-1,1)$

for $\underline{\lambda=-2}$

$\underline{\lambda=0+2i}$

$$\left[\begin{array}{ccc|c} 2-(2i) & 4 & 4 & 0 \\ -1 & -2-(2i) & 0 & 0 \\ -1 & 0 & -2-(2i) & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 2-2i & 4 & 4 & 0 \\ -1 & -2-2i & 0 & 0 \\ -1 & 0 & -2-2i & 0 \end{array}\right]$$

$$(2-2i)k_1+4k_2+4k_3=0$$
$$-k_1+(-2-2i)k_2 \quad =0$$
$$-k_1 \quad +(-2-2i)k_3=0$$

$k_1=(-2-2i)k_2$

$k_1=(-2-2i)k_3$

$(-2-2i)k_2=(-2-2i)k_3$

$k_2=k_3$

$(2-2i)k_1+4k_2+4k_2=0 \leftarrow$

$(2-2i)k_1+8k_2=0$ $\quad k_1=\dfrac{-8}{2-2i}k_2$

$\left(\dfrac{-8}{2-2i}k_2,\ k_2,\ k_2\right)$ $\quad \left(\dfrac{-8}{2-2i},\ 1,\ 1\right)$

$\dfrac{-8}{(2-2i)}\dfrac{(2+2i)}{(2+2i)} = \dfrac{-8(2+2i)}{4+4} = -2-2i$

$\begin{bmatrix} -2-2i \\ 1 \\ 1 \end{bmatrix}$ $\vec{B}_1=\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$ $\vec{B}_2=\begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix}$

$$\vec{X}_1=\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}e^{-2t}$$

$$\vec{X}_2=\left(\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}\cos 2t-\begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix}\sin 2t\right)\cancel{e^{0t}}$$

$$\vec{X}_3=\left(\begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix}\cos 2t+\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}\sin 2t\right)\cancel{e^{0t}}$$

$$\boxed{\vec{X}=C_1\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}e^{-2t}+C_2\left(\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}\cos 2t-\begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix}\sin 2t\right)+C_3\left(\begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix}\cos 2t+\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}\sin 2t\right)}$$

#5b. Solve the initial-value problem:

$$\vec{X}' = \begin{bmatrix} 1 & -12 & -14 \\ 1 & 2 & -3 \\ 1 & 1 & -2 \end{bmatrix}\vec{X}, \qquad \vec{X}(0) = \begin{bmatrix} 4 \\ 6 \\ -7 \end{bmatrix}$$

$$\begin{array}{c} + \\ - \\ + \end{array}\begin{vmatrix} 1-\lambda & -12 & -14 \\ 1 & 2-\lambda & -3 \\ 1 & 1 & -2-\lambda \end{vmatrix} = 0$$

$(1-\lambda)[(2-\lambda)(-2-\lambda)+3]-1[-12(-2-\lambda)+14]+1[36+14(2-\lambda)]=0$

$(1-\lambda)[\lambda^2-1]-24-12\lambda-14+36+28-14\lambda=0$

$(1-\lambda)[\lambda^2-1]-24-12\lambda-14+36+28-14\lambda=0 \quad -\lambda^3+\lambda^2-25\lambda+25=0$

$\lambda^2-1-\lambda^3+\lambda-24-12\lambda-14+36+28-14\lambda \quad \lambda^3-\lambda^2+25\lambda-25=0$

$$\begin{array}{c|cccc} 1 & 1 & -1 & 25 & -25 \\ & & 1 & 0 & 25 \\ \hline & 1 & 0 & 25 & 0 \end{array}$$

$(\lambda-1)(\lambda^2+25)$

$\underline{\lambda=1}$

$$\left[\begin{array}{ccc|c} 1-1 & -12 & -14 & 0 \\ 1 & 2-1 & -3 & 0 \\ 1 & 1 & -2-1 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 0 & -12 & -14 & 0 \\ 1 & 1 & -3 & 0 \\ 1 & 1 & -3 & 0 \end{array}\right]$$

rref

$$\left[\begin{array}{ccc|c} 1 & 0 & -25/6 & 0 \\ 0 & 1 & 7/6 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$k_1 - \frac{25}{6}k_3 = 0 \to k_1 = \frac{25}{6}k_3$

$k_2 + 7/6\, k_3 = 0 \to k_2 = \frac{-7}{6}k_3$

$(\frac{25}{6}k_3, -\frac{7}{6}k_3, k_3) \quad \begin{bmatrix} 25 \\ -7 \\ 6 \end{bmatrix}$

$(25, -7, 6)$

$\underline{\lambda = 0+5i}$

$$\left[\begin{array}{ccc|c} 1-5i & -12 & -14 & 0 \\ 1 & 2-5i & -3 & 0 \\ 1 & 1 & -2-5i & 0 \end{array}\right]$$

$(1-5i)k_1 - 12k_2 - 14k_3 = 0$

$k_1 + (2-5i)k_2 - 3k_3 = 0$

$-(k_1 + 1k_2 + (-2-5i)k_3 = 0)$

$(1-5i)k_2 + (-1+5i)k_3 = 0 \to k_2 = \frac{-(-1+5i)}{1-5i}k_3$

$(1-5i)k_1 - 12k_2 - 14k_2 = 0 \quad \leftarrow \quad k_2 = k_3$

$(1-5i)k_1 - 26k_2 = 0$

$k_1 = \frac{26}{(1-5i)}k_2 \frac{(1+5i)}{(1+5i)} = \frac{26(1+5i)}{26}k_2 = (1+5i)k_2$

$((1+5i)k_2, k_2, k_2) \quad \begin{bmatrix} 1+5i \\ 1 \\ 1 \end{bmatrix} \quad \vec{B}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \vec{B}_2 = \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}$

$(1+5i, 1, 1)$

$\vec{X} = c_1\begin{bmatrix} 25 \\ -7 \\ 6 \end{bmatrix}e^t + c_2\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\cos 5t - \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}\sin 5t\right)\cancel{e^{0t}} + c_3\left(\begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}\cos 5t + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\sin 5t\right)\cancel{e^{0t}}$

$x = 25c_1e^t + c_2(\cos 5t - 5\sin 5t) + c_3(5\cos 5t + \sin 5t)$

$4 = 25c_1e^0 + c_2(1-5(0)) + c_3(5(1)+0)$

$4 = 25c_1 + c_2 + 5c_3$

$y = -7c_1e^t + c_2(\cos 5t) + c_3(\sin 5t)$

$6 = -7c_1e^0 + c_2(1) + c_3(0)$

$6 = -7c_1 + c_2$

$z = 6c_1e^t + c_2(\cos 5t) + c_3(\sin 5t)$

$-7 = 6c_1e^0 + c_2(1) + c_3(0)$

$-7 = 6c_1 + c_2$

$$\begin{cases} 25c_1 + c_2 + 5c_3 = 4 \\ -7c_1 + c_2 = 6 \\ 6c_1 + c_2 = -7 \end{cases}$$

$$\left[\begin{array}{ccc|c} 25 & 1 & 5 & 4 \\ -7 & 1 & 0 & 6 \\ 6 & 1 & 0 & -7 \end{array}\right] \text{rref} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 6 \end{array}\right] \begin{array}{l} = c_1 \\ = c_2 \\ = c_3 \end{array}$$

$$\boxed{\vec{X} = -1\begin{bmatrix} 25 \\ -7 \\ 6 \end{bmatrix}e^t - 1\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\cos 5t - \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}\sin 5t\right) + 6\left(\begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}\cos 5t + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\sin 5t\right)}$$

8.3 day 1

#1b. Use the method of undermined coefficients to solve the system:

$$\frac{dx}{dt} = 5x + 9y + 2$$

$$\frac{dy}{dt} = -x + 11y + 6$$

$\vec{X}' = \begin{bmatrix} 5 & 9 \\ -1 & 11 \end{bmatrix}\vec{X} + \begin{bmatrix} 2 \\ 6 \end{bmatrix}$ $\quad \begin{vmatrix} 5-\lambda & 9 \\ -1 & 11-\lambda \end{vmatrix} = 0$

$(5-\lambda)(11-\lambda) + 9 = 0$

$\lambda^2 - 16\lambda + 64 = 0$

$(\lambda - 8)(\lambda - 8) = 0$

$\lambda = 8$ repeated

$\underline{\lambda = 8}$

$\left[\begin{array}{cc|c} 5-8 & 9 & 0 \\ -1 & 11-8 & 0 \end{array}\right]$

$\left[\begin{array}{cc|c} -3 & 9 & 0 \\ -1 & 3 & 0 \end{array}\right]$

rref

$\left[\begin{array}{cc|c} 1 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$k_1 - 3k_2 = 0 \rightarrow k_1 = 3k_2$

$(3k_2, k_2)$ $(3,1)$ $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$\underline{\text{repeat } \lambda = 8}$

$\left[\begin{array}{cc|c} -3 & 9 & 3 \\ -1 & 3 & 1 \end{array}\right]$

rref

$\left[\begin{array}{cc|c} 1 & -3 & -1 \\ 0 & 0 & 0 \end{array}\right]$

$k_1 - 3k_2 = -1$

$k_1 = 3k_2 - 1$

$(3k_2 - 1, k_2)$

$(2, 1)$ $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\vec{X}_c = c_1\begin{bmatrix} 3 \\ 1 \end{bmatrix}e^{8t} + c_2\left(\begin{bmatrix} 3 \\ 1 \end{bmatrix}te^{8t} + \begin{bmatrix} 2 \\ 1 \end{bmatrix}e^{8t}\right)$

for $\begin{bmatrix} 2 \\ 6 \end{bmatrix}$, $\vec{X}_p = \begin{bmatrix} A \\ B \end{bmatrix}$ $\quad \vec{X}' = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 & 9 \\ -1 & 11 \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} + \begin{bmatrix} 2 \\ 6 \end{bmatrix}$

$5A + 9B + 2 = 0$

$-A + 11B + 6 = 0$

$\begin{cases} 5A + 9B = -2 \\ -A + 11B = -6 \end{cases}$ $\left[\begin{array}{cc|c} 5 & 9 & -2 \\ -1 & 11 & -6 \end{array}\right]$

rref

$\left[\begin{array}{cc|c} 1 & 0 & 1/2 \\ 0 & 1 & -1/2 \end{array}\right]$ $\begin{matrix} = A \\ = B \end{matrix}$

$\vec{X}_p = \begin{bmatrix} 1/2 \\ -1/2 \end{bmatrix}$

$$\boxed{\vec{X} = c_1\begin{bmatrix} 3 \\ 1 \end{bmatrix}e^{8t} + c_2\left(\begin{bmatrix} 3 \\ 1 \end{bmatrix}te^{t} + \begin{bmatrix} 2 \\ 1 \end{bmatrix}e^{8t}\right) + \begin{bmatrix} 1/2 \\ -1/2 \end{bmatrix}}$$

#2b. Use the method of undermined coefficients to solve the system:

$$\vec{X}' = \begin{bmatrix}1 & -4\\ 4 & 1\end{bmatrix}\vec{X} + \begin{bmatrix}4t+9e^{6t}\\ -t+e^{6t}\end{bmatrix}$$

$$\begin{vmatrix}1-\lambda & -4\\ 4 & 1-\lambda\end{vmatrix} = 0 \quad (1-\lambda)(1-\lambda)+16=0 \qquad \lambda^2-2\lambda+17=0$$

$$\lambda = \frac{2\pm\sqrt{4-4(17)}}{2} = \frac{2\pm 8i}{2} = 1\pm 4i$$

$\underline{\lambda = 1+4i}$

$$\left[\begin{array}{cc|c}1-(1+4i) & -4 & 0\\ 4 & 1-(1+4i) & 0\end{array}\right]$$

$$\left[\begin{array}{cc|c}-4i & -4 & 0\\ 4 & -4i & 0\end{array}\right]$$

$4K_1 - 4iK_2 = 0$

$4K_1 = 4iK_2$

$K_1 = iK_2$ $\quad (iK_2, K_2)$ $\quad (i,1)$

$\begin{bmatrix}i\\1\end{bmatrix}$ $\vec{B}_1 = \begin{bmatrix}0\\1\end{bmatrix}$ $\vec{B}_2 = \begin{bmatrix}1\\0\end{bmatrix}$

$$\vec{X}_c = c_1\left(\begin{bmatrix}0\\1\end{bmatrix}\cos 4t - \begin{bmatrix}1\\0\end{bmatrix}\sin 4t\right)e^t + c_2\left(\begin{bmatrix}1\\0\end{bmatrix}\cos 4t + \begin{bmatrix}0\\1\end{bmatrix}\sin 4t\right)e^t$$

for $\begin{bmatrix}4t+9e^{6t}\\ -t+e^{6t}\end{bmatrix}$ $\quad \vec{X}_p = \begin{bmatrix}At+B+Ce^{6t}\\ Dt+E+Fe^{6t}\end{bmatrix}$ $\quad \vec{X}' = \begin{bmatrix}A+6Ce^{6t}\\ D+6Fe^{6t}\end{bmatrix}$

into DE:

$$\begin{bmatrix}A+6Ce^{6t}\\ D+6Fe^{6t}\end{bmatrix} = \begin{bmatrix}1 & -4\\ 4 & 1\end{bmatrix}\begin{bmatrix}At+B+Ce^{6t}\\ Dt+E+Fe^{6t}\end{bmatrix} + \begin{bmatrix}4t+9e^{6t}\\ -t+e^{6t}\end{bmatrix}$$

$$\begin{cases} At+B+Ce^{6t}-4Dt-4E-4Fe^{6t}+4t+9e^{6t} = A+6Ce^{6t}\\ 4At+4B+4Ce^{6t}+Dt+E+Fe^{6t}-t+e^{6t} = D+6Fe^{6t}\end{cases}$$

$$\begin{cases}(A-4D+4)t+(B-4E)+(C-4F+9)e^{6t} = (0)t+(A)+(6C)e^{6t}\\ (4A+D-1)t+(4B+E)+(4C+F+1)e^{6t} = (0)t+(D)+(6F)e^{6t}\end{cases}$$

$$\begin{cases}A-4D+4=0\\ B-4E=A\\ C-4F+9=6C\\ 4A+D-1=0\\ 4B+E=D\\ 4C+F+1=6F\end{cases}
\quad
\begin{bmatrix}1A+0B+0C-4D+0E+0F=-4\\ 1A+0B+0C+0D-4E+0F=0\\ -1A+1B+0C+0D-4E+0F=-9\\ 0A+0B-5C+0D+0E-4F=1\\ 4A+0B+0C+1D+0E+0F=0\\ 0A+4B+0C-1D+1E+0F=0\\ 0A+0B+4C+0D+0E-5F=-1\end{bmatrix}
\xrightarrow{\text{rref}}
\begin{array}{l}A=0\\ B=4/17\\ C=1\\ D=1\\ E=1/17\\ F=1\end{array}$$

$$\vec{X}_p = \begin{bmatrix}0t+4/17+1e^{6t}\\ 1t+1/17+1e^{6t}\end{bmatrix}$$

$$\boxed{\vec{X} = c_1\left(\begin{bmatrix}0\\1\end{bmatrix}\cos 4t - \begin{bmatrix}1\\0\end{bmatrix}\sin 4t\right)e^t + c_2\left(\begin{bmatrix}1\\0\end{bmatrix}\cos 4t + \begin{bmatrix}0\\1\end{bmatrix}\sin 4t\right)e^t + \begin{bmatrix}4/17+e^{6t}\\ t+\frac{1}{17}+e^{6t}\end{bmatrix}}$$

or can be written as

$$\boxed{\vec{X} = c_1\left(\begin{bmatrix}0\\1\end{bmatrix}\cos 4t - \begin{bmatrix}1\\0\end{bmatrix}\sin 4t\right)e^t + c_2\left(\begin{bmatrix}1\\0\end{bmatrix}\cos 4t + \begin{bmatrix}0\\1\end{bmatrix}\sin 4t\right)e^t + \begin{bmatrix}0\\1\end{bmatrix}t + \begin{bmatrix}4/17\\1/17\end{bmatrix} + \begin{bmatrix}1\\1\end{bmatrix}e^{6t}}$$

#3b. Use the method of undermined coefficients to solve the system:

$$\vec{X}' = \begin{bmatrix}-1 & 5\\ -1 & 1\end{bmatrix}\vec{X} + \begin{bmatrix}\sin t\\ -2\cos t\end{bmatrix}$$

$$\begin{vmatrix}-1-\lambda & 5\\ -1 & 1-\lambda\end{vmatrix} \Rightarrow (-1-\lambda)(1-\lambda)+5=0$$

$$\lambda^2+4 \Rightarrow \quad \underline{\lambda = 0 \pm 2i}$$

$\underline{\lambda = 0+2i}$

$$\left[\begin{array}{cc|c}-1-(0+2i) & 5 & 0\\ -1 & 1-(0+2i) & 0\end{array}\right]$$

$$\left[\begin{array}{cc|c}-1-2i & 5 & 0\\ -1 & 1-2i & 0\end{array}\right]$$

$-K_1+(1-2i)K_2=0$

$K_1=((1-2i)K_2$

$((1-2i)K_2, K_2)$

$(1-2i, 1) \quad \begin{bmatrix}1-2i\\ 1\end{bmatrix} \quad \vec{B_1}=\begin{bmatrix}1\\1\end{bmatrix} \quad \vec{B_2}=\begin{bmatrix}-2\\0\end{bmatrix}$

$$\vec{X_c} = c_1\left(\begin{bmatrix}1\\1\end{bmatrix}\cos 2t - \begin{bmatrix}-2\\0\end{bmatrix}\sin 2t\right)\cancel{e^{0t}}$$
$$+c_2\left(\begin{bmatrix}-2\\0\end{bmatrix}\cos 2t + \begin{bmatrix}1\\1\end{bmatrix}\sin 2t\right)\cancel{e^{0t}}$$

for $\begin{bmatrix}\sin t\\ -2\cos t\end{bmatrix}$ $\quad \vec{X_p} = \begin{bmatrix}A\cos t + B\sin t\\ C\cos t + D\sin t\end{bmatrix} \quad x' = \begin{bmatrix}-A\sin t + B\cos t\\ -C\sin t + D\cos t\end{bmatrix}$

$$\begin{bmatrix}-A\sin t + B\cos t\\ -C\sin t + D\cos t\end{bmatrix} = \begin{bmatrix}-1 & 5\\ -1 & 1\end{bmatrix}\begin{bmatrix}A\cos t + B\sin t\\ C\cos t + D\sin t\end{bmatrix} + \begin{bmatrix}\sin t\\ -2\cos t\end{bmatrix}$$

$$\begin{cases}-A\cos t - B\sin t + 5C\cos t + 5D\sin t + \sin t = -A\sin t + B\cos t\\ -A\cos t - B\sin t + C\cos t + D\sin t - 2\cos t = -C\sin t + D\cos t\end{cases}$$

$$\begin{cases}(-A+5C)\cos t + (-B+5D+1)\sin t = (B)\cos t + (-A)\sin t\\ (-A+C-2)\cos t + (-B+D)\sin t = (D)\cos t + (-C)\sin t\end{cases}$$

$$\begin{cases}-A+5C=B\\ -B+5D+1=-A\\ -A+C-2=D\\ -B+D=-C\end{cases}$$

$$\begin{bmatrix}-1A-1B+5C+0D=0\\ 1A-1B+0C+5D=-1\\ -1A+0B+1C-1D=2\\ 0A-1B+1C+1D=0\end{bmatrix} \xrightarrow{\text{rref}} \quad A=-3,\ B=-1/3,\ C=-2/3,\ D=1/3$$

$$\vec{X_p} = \begin{bmatrix}-3\cos t - 1/3\sin t\\ -2/3\cos t + 1/3\sin t\end{bmatrix}$$

$$\boxed{\vec{X} = c_1\left(\begin{bmatrix}1\\1\end{bmatrix}\cos 2t - \begin{bmatrix}-2\\0\end{bmatrix}\sin 2t\right) + c_2\left(\begin{bmatrix}-2\\0\end{bmatrix}\cos 2t + \begin{bmatrix}1\\1\end{bmatrix}\sin 2t\right) + \begin{bmatrix}-3\\-2/3\end{bmatrix}\cos t + \begin{bmatrix}-1/3\\1/3\end{bmatrix}\sin t}$$

#4b. Use the method of undermined coefficients to solve the system:

$$\vec{X}' = \begin{bmatrix} 0 & 0 & 5 \\ 0 & 5 & 0 \\ 5 & 0 & 0 \end{bmatrix}\vec{X} + \begin{bmatrix} 5 \\ -10 \\ 40 \end{bmatrix}$$

$$\begin{vmatrix} 0-\lambda & 0 & 5 \\ 0 & 5-\lambda & 0 \\ 5 & 0 & 0-\lambda \end{vmatrix} = 0$$

$(-\lambda)[(5-\lambda)(-\lambda)-0]-0(\sim)+5[0-5(5-\lambda)]$

$\lambda^2(5-\lambda)-25(5-\lambda)=0$

$(5-\lambda)(\lambda^2-25)=0$

$(5-\lambda)(\lambda-5)(\lambda+5)=0$

$\lambda=5$ repeated, $\lambda=-5$

$\lambda=-5$

$$\left[\begin{array}{ccc|c} 0+5 & 0 & 5 & 0 \\ 0 & 5+5 & 0 & 0 \\ 5 & 0 & 0+5 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 5 & 0 & 5 & 0 \\ 0 & 10 & 0 & 0 \\ 5 & 0 & 5 & 0 \end{array}\right]$$

maf

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$K_1+K_3=0 \Rightarrow K_1=-K_3$

$K_2=0$

$(-K_3, 0, K_3)$ $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

$(1,0,-1)$

$\lambda=5$

$$\left[\begin{array}{ccc|c} 0-5 & 0 & 5 & 0 \\ 0 & 5-5 & 0 & 0 \\ 5 & 0 & 0-5 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} -5 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 \\ 5 & 0 & -5 & 0 \end{array}\right]$$

mef

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$K_1-K_3=0 \rightarrow K_1=K_3$

(K_1, K_2, K_1) $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$(1,1,1)$

$(1,0,1)$

$$\vec{X}_c = C_1\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}e^{-5t} + C_2\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}e^{5t} + C_2\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}e^{5t}$$

for $\begin{bmatrix} 5 \\ -10 \\ 40 \end{bmatrix}$ $\vec{X}_p = \begin{pmatrix} A \\ B \\ C \end{pmatrix}$ $\vec{X}' = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 5 \\ 0 & 5 & 0 \\ 5 & 0 & 0 \end{bmatrix}\begin{bmatrix} A \\ B \\ C \end{bmatrix} + \begin{bmatrix} 5 \\ -10 \\ 40 \end{bmatrix}$$

$$\begin{cases} 5C+5=0 \\ 5B-10=0 \\ 5A+40=0 \end{cases}$$

$C=-1$

$B=2$

$A=-8$

$$\vec{X}_p = \begin{bmatrix} -8 \\ 2 \\ -1 \end{bmatrix}$$

$$\boxed{\vec{X} = C_1\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}e^{-5t} + C_2\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}e^{5t} + C_3\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}e^{5t} + \begin{bmatrix} -8 \\ 2 \\ -1 \end{bmatrix}}$$

#5b. Use the method of undermined coefficients to solve the initial-value problem:

$$\vec{X}' = \begin{bmatrix} -1 & 5 \\ -1 & 1 \end{bmatrix}\vec{X} + \begin{bmatrix} \sin t \\ -2\cos t \end{bmatrix}, \qquad \vec{X}(0) = \begin{bmatrix} 7 \\ -14/3 \end{bmatrix}$$

$\begin{vmatrix} -1-\lambda & 5 \\ -1 & 1-\lambda \end{vmatrix} = 0 \quad (-1-\lambda)(1-\lambda)+5=0$

$\lambda^2+4=0, \quad \lambda = 0 \pm 2i$

$\underline{\lambda = 0+2i}$

$\left[\begin{array}{cc|c} -1-(0+2i) & 5 & 0 \\ -1 & 1-(0+2i) & 0 \end{array}\right]$

$\left[\begin{array}{cc|c} -1-2i & 5 & 0 \\ -1 & 1-2i & 0 \end{array}\right]$

$-k_1 + (1-2i)k_2 = 0$

$k_1 = (1-2i)k_2$

$((1-2i)k_2, k_2) \quad \begin{bmatrix} 1-2i \\ 1 \end{bmatrix} \quad \vec{B_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \vec{B_2} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$

$(1-2i, 1)$

$\vec{X_c} = C_1\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\cos 2t - \begin{bmatrix} -2 \\ 0 \end{bmatrix}\sin 2t\right)\cancel{e^{0t}} + C_2\left(\begin{bmatrix} -2 \\ 0 \end{bmatrix}\cos 2t + \begin{bmatrix} 1 \\ 1 \end{bmatrix}\sin 2t\right)\cancel{e^{0t}}$

for $\begin{bmatrix} \sin t \\ -2\cos t \end{bmatrix}$ $\vec{X_p} = \begin{bmatrix} A\cos t + B\sin t \\ C\cos t + D\sin t \end{bmatrix}$ $\vec{X}' = \begin{bmatrix} -A\sin t + B\cos t \\ -C\sin t + D\cos t \end{bmatrix}$

$\begin{bmatrix} -A\sin t + B\cos t \\ -C\sin t + D\cos t \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} A\cos t + B\sin t \\ C\cos t + D\sin t \end{bmatrix} + \begin{bmatrix} \sin t \\ -2\cos t \end{bmatrix}$

$\begin{cases} -A\cos t - B\sin t + 5C\cos t + 5D\sin t + \sin t = -A\sin t + B\cos t \\ -A\cos t - B\sin t - C\cos t - D\sin t - 2\cos t = -C\sin t + D\cos t \end{cases}$

$-A\cos t - B\sin t - C\cos t \quad (B)\cos t + (-A)\sin t$

$(-B+5B+1)\sin t =$

$\begin{cases} (-A+5C)\cos t + (-B+5B+1)\sin t = (B)\cos t + (-A)\sin t \\ (-A-C-2)\cos t + (-B-D)\sin t = (D)\cos t + (-C)\sin t \end{cases}$

$\begin{cases} -A+5C = B \\ -B+5B+1 = -A \\ -A-C-2 = D \\ -B-D = -C \end{cases} \quad \begin{bmatrix} -1A - 1B + 5C + 0D = 0 \\ 1A + 4B + 0C + 0D = -1 \\ -1A + 0B - 1C - 1D = 2 \\ 0A - 1B + 1C - 1D = 0 \end{bmatrix} \xrightarrow{\text{rref}} \begin{array}{l} A = -43/31 \\ B = 3/31 \\ C = -8/31 \\ D = -11/31 \end{array} \quad \vec{X_p} = \begin{bmatrix} -\frac{43}{31}\cos t + \frac{3}{31}\sin t \\ -\frac{8}{31}\cos t - \frac{11}{31}\sin t \end{bmatrix}$

$\vec{X} = C_1\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\cos 2t - \begin{bmatrix} -2 \\ 0 \end{bmatrix}\sin 2t\right) + C_2\left(\begin{bmatrix} -2 \\ 0 \end{bmatrix}\cos 2t + \begin{bmatrix} 1 \\ 1 \end{bmatrix}\sin 2t\right) + \begin{bmatrix} -\frac{43}{31}\cos t + \frac{3}{31}\sin t \\ -\frac{8}{31}\cos t - \frac{11}{31}\sin t \end{bmatrix}$

(continued ...)

8.3 day #5b continued...

$$\vec{X} = c_1\left(\begin{bmatrix}1\\1\end{bmatrix}\cos 2t - \begin{bmatrix}-2\\0\end{bmatrix}\sin 2t\right) + c_2\left(\begin{bmatrix}-2\\0\end{bmatrix}\cos 2t + \begin{bmatrix}1\\1\end{bmatrix}\sin 2t\right) + \begin{bmatrix}-43/31\cos t + 3/31\sin t\\ -8/31\cos t - 11/31\sin t\end{bmatrix}$$

now, $\vec{X}(0) = \begin{bmatrix}7\\-14/3\end{bmatrix}$

$$x = c_1(\cos 2t + 2\sin 2t) + c_2(-2\cos 2t + \sin 2t) - 43/31\cos t + 3/31\sin t$$

$$7 = c_1(1 + 2(0)) + c_2(-2(1) + 0) - 43/31(1) + 3/31(0)$$

$$c_1 - 2c_2 = \frac{260}{31}$$

$$y = c_1(\cos 2t) + c_2(\sin 2t) - 8/31\cos t - 11/31\sin t$$

$$-14/3 = c_1(1) + c_2(0) - 8/31(1) - 11/31(0)$$

$$c_1 = \frac{-410}{93}, \qquad \left(\frac{-410}{93}\right) - 2c_2 = \frac{260}{31}$$

$$2c_2 = \frac{-1190}{93}, \quad c_2 = \frac{-595}{93}$$

$$\boxed{\vec{X} = -\frac{410}{93}\left(\begin{bmatrix}1\\1\end{bmatrix}\cos 2t - \begin{bmatrix}-2\\0\end{bmatrix}\sin 2t\right) - \frac{595}{93}\left(\begin{bmatrix}-2\\0\end{bmatrix}\cos 2t + \begin{bmatrix}1\\1\end{bmatrix}\sin 2t\right) + \begin{bmatrix}-43/31\cos t + 3/31\sin t\\ -8/31\cos t - 11/31\sin t\end{bmatrix}}$$

8.3 day 2

#1b. Use variation of parameters to solve the system:

$$\vec{X}' = \begin{bmatrix} 0 & 2 \\ -1 & 3 \end{bmatrix}\vec{X} + \begin{bmatrix} 2 \\ e^{-3t} \end{bmatrix}$$

$\begin{vmatrix} 0-\lambda & 2 \\ -1 & 3-\lambda \end{vmatrix} = 0$ $\quad -\lambda(3-\lambda)+2=0$ $\quad \lambda^2-3\lambda+2=0$ $\quad (\lambda-2)(\lambda-1)=0$ $\quad \underline{\lambda=2 \;\; \lambda=1}$

$\underline{\lambda=2}$

$\left[\begin{array}{cc|c} 0-2 & 2 & 0 \\ -1 & 3-2 & 0 \end{array}\right]$

$\left[\begin{array}{cc|c} -2 & 2 & 0 \\ -1 & 1 & 0 \end{array}\right]$

rref

$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$ $\quad k_1-k_2=0$, $k_1=k_2$ $\quad \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\underline{\lambda=1}$

$\left[\begin{array}{cc|c} 0-1 & 2 & 0 \\ -1 & 3-1 & 0 \end{array}\right]$

$\left[\begin{array}{cc|c} -1 & 2 & 0 \\ -1 & 2 & 0 \end{array}\right]$

rref

$\left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$ $\quad k_1-2k_2=0$, $k_1=2k_2$ $\quad \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\vec{X_c} = c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{2t} + c_2\begin{bmatrix} 2 \\ 1 \end{bmatrix}e^{t}$

$\Phi = \begin{bmatrix} e^{2t} & 2e^{t} \\ e^{2t} & e^{t} \end{bmatrix}$ $\quad \begin{vmatrix} e^{2t} & 2e^{t} \\ e^{2t} & e^{t} \end{vmatrix} = e^{2t}e^{t} - 2e^{2t}e^{t} = e^{3t} - 2e^{3t} = -e^{3t}$

$\Phi^{-1} = \frac{1}{-e^{3t}}\begin{bmatrix} e^{t} & -2e^{t} \\ -e^{2t} & e^{2t} \end{bmatrix} = \begin{bmatrix} -e^{-2t} & 2e^{-2t} \\ e^{-t} & -e^{-t} \end{bmatrix}$

$\vec{X}_p = \Phi \int \Phi^{-1} F\,dt = \begin{bmatrix} e^{2t} & 2e^{t} \\ e^{2t} & e^{t} \end{bmatrix} \int \begin{bmatrix} -e^{-2t} & 2e^{-2t} \\ e^{-t} & -e^{-t} \end{bmatrix}\begin{bmatrix} 2 \\ e^{-3t} \end{bmatrix} dt$

$\int \begin{bmatrix} -2e^{-2t} + 2e^{-5t} \\ 2e^{-t} - e^{-4t} \end{bmatrix} dt$

$\begin{bmatrix} e^{2t} & 2e^{t} \\ e^{2t} & e^{t} \end{bmatrix}\begin{bmatrix} e^{-2t} - \frac{2}{5}e^{-5t} \\ -2e^{-t} + \frac{1}{4}e^{-4t} \end{bmatrix}$

$\begin{bmatrix} e^{2t}(e^{-2t} - \frac{2}{5}e^{-5t}) + 2e^{t}(-2e^{-t} + \frac{1}{4}e^{-4t}) \\ e^{2t}(e^{-2t} - \frac{2}{5}e^{-5t}) + e^{t}(-2e^{-t} + \frac{1}{4}e^{-4t}) \end{bmatrix}$

$\vec{X}_p = \begin{bmatrix} 1 - \frac{2}{5}e^{-3t} - 4 + \frac{1}{2}e^{-3t} \\ 1 - \frac{2}{5}e^{-3t} - 2 + \frac{1}{4}e^{-3t} \end{bmatrix} = \begin{bmatrix} -3 + \frac{1}{10}e^{-3t} \\ -1 - \frac{3}{20}e^{-3t} \end{bmatrix}$

$$\boxed{\vec{X} = c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{2t} + c_2\begin{bmatrix} 2 \\ 1 \end{bmatrix}e^{t} + \begin{bmatrix} -3 \\ -1 \end{bmatrix} + \begin{bmatrix} 1/10 \\ -3/20 \end{bmatrix}e^{-3t}}$$

#2b. Use variation of parameters to solve the system:

$\vec{X}' = \begin{bmatrix} 3 & 2 \\ -2 & -1 \end{bmatrix}\vec{X} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-t}$

$\begin{vmatrix} 3-\lambda & 2 \\ -2 & -1-\lambda \end{vmatrix} = 0$ $\quad (3-\lambda)(-1-\lambda)+4=0 \quad \lambda^2-2\lambda+1=0$ $\quad (\lambda-1)(\lambda-1)=0 \quad \underline{\lambda=1 \text{ repeated}}$

$\underline{\lambda=1}$

$\left[\begin{array}{cc|c} 3-1 & 2 & 0 \\ -2 & -1-1 & 0 \end{array}\right] \left[\begin{array}{cc|c} 2 & 2 & 0 \\ -2 & -2 & 0 \end{array}\right]$ rref $\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$ $\quad K_1+K_2=0 \quad K_1=-K_2$

$(-K_2, K_2) \quad (1,-1) \quad \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\underline{\text{repeat}}$

$\left[\begin{array}{cc|c} 2 & 2 & 1 \\ -2 & -2 & -1 \end{array}\right]$ rref $\left[\begin{array}{cc|c} 1 & 1 & 1/2 \\ 0 & 0 & 0 \end{array}\right]$ $\quad K_1+K_2=1/2 \quad K_2=1/2-K_1$

$(K_1, 1/2-K_1) \quad (0, 1/2) \quad \begin{bmatrix} 0 \\ 1/2 \end{bmatrix}$

$\vec{X}_c = C_1\begin{bmatrix} 1 \\ -1 \end{bmatrix}e^t + C_2\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}te^t + \begin{bmatrix} 0 \\ 1/2 \end{bmatrix}e^t\right)$

$\Phi = \begin{bmatrix} e^t & te^t \\ -e^t & -te^t+\frac{1}{2}e^t \end{bmatrix} \quad \begin{vmatrix} e^t & te^t \\ -e^t & -te^t+\frac{1}{2}e^t \end{vmatrix} = e^t(-te^t+\frac{1}{2}e^t)+te^{2t} = -te^{2t}+\frac{1}{2}e^{2t}+te^{2t} = \frac{1}{2}e^{2t}$

$\Phi^{-1} = \frac{1}{\frac{1}{2}e^{2t}}\begin{bmatrix} -te^t+\frac{1}{2}e^t & -te^t \\ e^t & e^t \end{bmatrix} = \begin{bmatrix} -2te^{-t}+e^{-t} & -2te^{-t} \\ 2e^{-t} & 2e^{-t} \end{bmatrix}$

$\vec{X}_p = \Phi\int\Phi^{-1}F\,dt = \begin{bmatrix} e^t & te^t \\ -e^t & -te^t+\frac{1}{2}e^t \end{bmatrix}\int\begin{bmatrix} -2te^{-t}+e^{-t} & -2te^{-t} \\ 2e^{-t} & 2e^{-t} \end{bmatrix}\begin{bmatrix} 2e^{-t} \\ e^{-t} \end{bmatrix}dt$

$\int\begin{bmatrix} -4te^{-2t}+2e^{-2t}-2te^{-2t} \\ 4e^{-2t}+2e^{-2t} \end{bmatrix}dt$

$\int te^{-2t}dt \quad u=t \quad du=dt \quad dv=e^{-2t}dt \quad v=-\frac{1}{2}e^{-2t}$

$-\frac{1}{2}te^{-2t}+\frac{1}{2}\int e^{-2t}dt$

$-\frac{1}{2}te^{-2t}-\frac{1}{4}e^{-2t}$

$\int\begin{bmatrix} -6te^{-2t}+2e^{-2t} \\ 6e^{-2t} \end{bmatrix}dt$

$\vec{X}_p = \begin{bmatrix} e^t & te^t \\ -e^t & -te^t+\frac{1}{2}e^t \end{bmatrix}\begin{bmatrix} 3te^{-2t}-e^{-2t} \\ -3e^{-2t} \end{bmatrix}$

$\vec{X}_p = \begin{bmatrix} 3te^{-t}-e^{-t}-3te^{-t} \\ -3te^{-t}+e^{-t}+3te^{-3t}-\frac{3}{2}e^{-t} \end{bmatrix} = \begin{bmatrix} -e^{-t} \\ -1/2e^{-t} \end{bmatrix}$

$$\boxed{\vec{X} = C_1\begin{bmatrix} 1 \\ -1 \end{bmatrix}e^t + C_2\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}te^t + \begin{bmatrix} 0 \\ 1/2 \end{bmatrix}e^t\right) + \begin{bmatrix} -1 \\ -1/2 \end{bmatrix}e^{-t}}$$

#3b. Use variation of parameters to solve the system:

$$\vec{X}' = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\vec{X} + \begin{bmatrix} 0 \\ \sec t \tan t \end{bmatrix}$$

$\begin{vmatrix} 0-\lambda & 1 \\ -1 & 0-\lambda \end{vmatrix} = 0 \quad (-\lambda)(-\lambda)+1=0 \quad \lambda^2+1=0 \quad \lambda = 0 \pm i$

$\lambda = 0+i$

$\left[\begin{array}{cc|c} -i & 1 & 0 \\ -1 & -i & 0 \end{array}\right]$ $\quad k_2 = ik_1 \quad (k_1, ik_1) \quad \begin{bmatrix} 1 \\ i \end{bmatrix}$

$-ik_1 + k_2 = 0$

$\vec{B}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \vec{B}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\vec{X}_c = C_1\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\cos t - \begin{bmatrix} 0 \\ 1 \end{bmatrix}\sin t\right)e^{0t} + C_2\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\cos t + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\sin t\right)e^{0t}$

$\vec{X}_c = C_1\begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} + C_2\begin{bmatrix} \sin t \\ \cos t \end{bmatrix}$

$\Phi = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \quad \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = \cos^2 t + \sin^2 t = 1$

$\Phi^{-1} = \frac{1}{1}\begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}$

$\vec{X}_p = \Phi \int \Phi^{-1} F\, dt = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \int \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}\begin{bmatrix} 0 \\ \sec t \tan t \end{bmatrix} dt$

$\int \begin{bmatrix} -\sin t \sec t \tan t \\ \cos t \sec t \tan t \end{bmatrix} dt$

$\begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}\begin{bmatrix} -\tan t + t \\ -\ln|\cos t| \end{bmatrix}$

$\vec{X}_p = \begin{bmatrix} -\cos t \tan t + t\cos t - \sin t \ln|\cos t| \\ \sin t \tan t - t\sin t - \cos t \ln|\cos t| \end{bmatrix}$

$-\int \sin t \sec t \tan t\, dt$

$-\int \sin t \frac{1}{\cos t}\frac{\sin t}{\cos t} dt$

$-\int \frac{\sin^2 t}{\cos^2 t} dt$

$-\int \frac{(1-\cos^2 t)}{\cos^2 t} dt$

$-\int \sec^2 t\, dt + \int 1\, dt$

$-\tan t + t$

$\int \cos t \sec t \tan t\, dt$

$\int \cos t \frac{1}{\cos t}\frac{\sin t}{\cos t} dt$

$\int \frac{\sin t}{\cos t} dt \quad u = \cos t \quad du = -\sin t\, dt$

$-\int \frac{1}{u} du = -\ln|\cos t|$

$\cos t \frac{\sin t}{\cos t} = \sin t$

$$\vec{X} = C_1\begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} + C_2\begin{bmatrix} \sin t \\ \cos t \end{bmatrix} + \begin{bmatrix} -\cos t \tan t + t\cos t - \sin t \ln|\cos t| \\ \sin t \tan t - t\sin t - \cos t \ln|\cos t| \end{bmatrix}$$

— or —

$$\vec{X} = C_1\begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} + C_2\begin{bmatrix} \sin t \\ \cos t \end{bmatrix} + \begin{bmatrix} -\sin t \\ \sin t \tan t \end{bmatrix} + \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} t + \begin{bmatrix} -\sin t \\ -\cos t \end{bmatrix} \ln|\cos t|$$

#4b. Use variation of parameters to solve the system:

$$\vec{X}' = \begin{bmatrix} 1 & 2 \\ -\frac{1}{2} & 1 \end{bmatrix}\vec{X} + \begin{bmatrix} \csc t \\ \sec t \end{bmatrix} e^t$$

$$\begin{vmatrix} 1-\lambda & 2 \\ -\frac{1}{2} & 1-\lambda \end{vmatrix} = 0 \quad (1-\lambda)(1-\lambda)+1=0$$

$$\lambda^2 - 2\lambda + 2 = 0$$

$$\lambda = \frac{2 \pm \sqrt{4-4(2)}}{2} = \frac{2 \pm 2i}{2} = 1 \pm i$$

$\lambda = 1+i$

$$\left[\begin{array}{cc|c} 1-(1+i) & 2 & 0 \\ -\frac{1}{2} & 1-(1+i) & 0 \end{array}\right] \quad -ik_1 + 2k_2 = 0$$

$$2k_2 = ik_1$$

$$\left[\begin{array}{cc|c} -i & 2 & 0 \\ -\frac{1}{2} & -i & 0 \end{array}\right] \quad k_2 = \frac{1}{2}ik_1$$

$$(k_1, \tfrac{1}{2}ik_1) \quad \begin{bmatrix} 2 \\ i \end{bmatrix}$$

$$\vec{B_1} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \quad \vec{B_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\vec{X_c} = c_1\left(\begin{bmatrix} 2 \\ 0 \end{bmatrix}\cos t - \begin{bmatrix} 0 \\ 1 \end{bmatrix}\sin t\right)e^t + c_2\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\cos t + \begin{bmatrix} 2 \\ 0 \end{bmatrix}\sin t\right)e^t$$

$$\vec{X_c} = c_1\begin{bmatrix} 2e^t\cos t \\ -e^t\sin t \end{bmatrix} + c_2\begin{bmatrix} 2e^t\sin t \\ e^t\cos t \end{bmatrix}$$

$$\Phi = \begin{bmatrix} 2e^t\cos t & 2e^t\sin t \\ -e^t\sin t & e^t\cos t \end{bmatrix} \quad \begin{vmatrix} 2e^t\cos t & 2e^t\sin t \\ -e^t\sin t & e^t\cos t \end{vmatrix} = 2e^{2t}\cos^2 t + 2e^{2t}\sin^2 t = 2e^{2t}(\cos^2 t + \sin^2 t) = 2e^{2t}$$

$$\Phi^{-1} = \frac{1}{2e^{2t}}\begin{bmatrix} e^t\cos t & -2e^t\sin t \\ e^t\sin t & 2e^t\cos t \end{bmatrix} = \begin{bmatrix} \frac{1}{2}e^{-t}\cos t & -e^{-t}\sin t \\ \frac{1}{2}e^{-t}\sin t & e^{-t}\cos t \end{bmatrix}$$

$$\vec{X_p} = \Phi\int\Phi^{-1}F\,dt = \begin{bmatrix} 2e^t\cos t & 2e^t\sin t \\ -e^t\sin t & e^t\cos t \end{bmatrix}\int\begin{bmatrix} \frac{1}{2}e^{-t}\cos t & -e^{-t}\sin t \\ \frac{1}{2}e^{-t}\sin t & e^{-t}\cos t \end{bmatrix}\begin{bmatrix} e^t\csc t \\ e^t\sec t \end{bmatrix}dt$$

$$\int\begin{bmatrix} \frac{1}{2}\cos t\csc t - \sin t\sec t \\ \frac{1}{2}\sin t\csc t + \cos t\sec t \end{bmatrix}dt$$

$$\int\begin{bmatrix} \frac{1}{2}\frac{\cos t}{\sin t} - \frac{\sin t}{\cos t} \\ \frac{1}{2} + 1 \end{bmatrix}dt$$

$$\frac{1}{2}\int\frac{\cos t}{\sin t}dt \quad u = \sin t \quad du = \cos t\,dt$$

$$\frac{1}{2}\int\frac{1}{u}du$$

$$\frac{1}{2}\ln|\sin t|$$

$$\int\frac{\sin t}{\cos t}dt \quad u = \cos t \quad du = -\sin t\,dt$$

$$-\int\frac{1}{u}du$$

$$-\ln|\cos t|$$

$$\vec{X_p} = \begin{bmatrix} 2e^t\cos t & 2e^t\sin t \\ -e^t\sin t & e^t\cos t \end{bmatrix}\begin{bmatrix} \frac{1}{2}\ln|\sin t| + \ln|\cos t| \\ \frac{3}{2}t \end{bmatrix}$$

$$\vec{X_p} = \begin{bmatrix} e^t\cos t\ln|\sin t| + 2e^t\cos t\ln|\cos t| + 3te^t\sin t \\ -\frac{1}{2}e^t\sin t\ln|\sin t| - e^t\sin t\ln|\cos t| + \frac{3}{2}te^t\cos t \end{bmatrix}$$

$$\boxed{\vec{X} = c_1\begin{bmatrix} 2e^t\cos t \\ -e^t\sin t \end{bmatrix} + c_2\begin{bmatrix} 2e^t\sin t \\ e^t\cos t \end{bmatrix} + \begin{bmatrix} e^t\cos t\ln|\sin t| + 2e^t\cos t\ln|\cos t| + 3te^t\sin t \\ -\frac{1}{2}e^t\sin t\ln|\sin t| - e^t\sin t\ln|\cos t| + \frac{3}{2}te^t\cos t \end{bmatrix}}$$

8.3 day 3

#1b. Solve the initial-value problem:

$$\vec{X}' = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \vec{X} + \begin{bmatrix} 1/t \\ 1/t \end{bmatrix}, \quad \vec{X}(1) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$\begin{vmatrix} 1-\lambda & -1 \\ 1 & -1-\lambda \end{vmatrix} = 0 \quad (1-\lambda)(-1-\lambda)+1=0$

$\lambda^2 = 0 \quad \lambda = 0$ (repeated)

$\lambda = 0$

$\left[\begin{array}{cc|c} 1-0 & -1 & 0 \\ 1 & -1-0 & 0 \end{array}\right] \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 1 & -1 & 0 \end{array}\right]$ rref $\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$ $k_1 - k_2 = 0$, $k_1 = k_2$ $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

repeat

$\left[\begin{array}{cc|c} 1 & -1 & 1 \\ 1 & -1 & 1 \end{array}\right]$ rref $\left[\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 0 & 0 \end{array}\right]$ $k_1 - k_2 = 1$, $k_1 = k_2 + 1$ $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\vec{X}_c = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{0t} + c_2 \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} te^{0t} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{0t} \right)$

$\vec{X}_c = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} t + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)$

$\Phi = \begin{bmatrix} 1 & t+1 \\ 1 & t \end{bmatrix} \quad \begin{vmatrix} 1 & t+1 \\ 1 & t \end{vmatrix} = t - (t+1) = -1$

$\Phi^{-1} = \frac{1}{-1} \begin{bmatrix} t & -(t+1) \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -t & t+1 \\ 1 & -1 \end{bmatrix}$

$\vec{X}_p = \Phi \int \Phi^{-1} F \, dt = \begin{bmatrix} 1 & t+1 \\ 1 & t \end{bmatrix} \int \begin{bmatrix} -t & t+1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1/t \\ 1/t \end{bmatrix} dt$

$\int \begin{bmatrix} -1 + 1 + \frac{1}{t} \\ \frac{1}{t} - \frac{1}{t} \end{bmatrix} dt = \int \begin{bmatrix} 1/t \\ 0 \end{bmatrix} dt$

$\vec{X}_p = \begin{bmatrix} 1 & t+1 \\ 1 & t \end{bmatrix} \begin{bmatrix} \ln|t| \\ 0 \end{bmatrix}$

$\vec{X}_p = \begin{bmatrix} \ln|t| \\ \ln|t| \end{bmatrix}$

$\vec{X} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} t + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) + \begin{bmatrix} \ln|t| \\ \ln|t| \end{bmatrix}$ now, use $\vec{X}(1) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$x = c_1 + c_2(t+1) + \ln|t|$

$2 = c_1 + c_2((1)+1) + \ln(1)$

$2 = c_1 + 2c_2 + 0 \longrightarrow c_1 + 2c_2 = 2$

$y = c_1 + c_2(t) + \ln|t|$

$-1 = c_1 + c_2(1) + \ln(1)$

$-1 = c_1 + c_2 + 0 \longrightarrow c_1 + c_2 = -1$

$\begin{bmatrix} c_1 + 2c_2 = 2 \\ c_1 + c_2 = -1 \end{bmatrix}$ rref $c_1 = -4$, $c_2 = 3$

$$\boxed{\vec{X} = -4 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 3\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} t + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) + \begin{bmatrix} \ln|t| \\ \ln|t| \end{bmatrix}}$$

$$\boxed{\vec{X} = \begin{bmatrix} -1 \\ -4 \end{bmatrix} + \begin{bmatrix} 3 \\ 3 \end{bmatrix} t + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \ln|t|}$$

8.3 day 4

#1b. Rewrite the following 3rd-order differential equation as a system of 3 1st-order differential equations (and also write the initial conditions in matrix form):

$2y''' + 4y'' - 6y' = 12 - 8\sin x$

$y(0) = 4, \quad y'(0) = 0, \quad y''(0) = -5,$

$y''' + 2y'' - 3y' = 6 - 4\sin x$

$$\begin{cases} y' = 0y + 1y' + 0y'' + 0 \\ y'' = 0y + 0y' + 1y'' + 0 \\ y''' = 0y + 3y' - 2y'' + 6 - 4\sin x \end{cases}$$

$$\boxed{\vec{X}' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 3 & -2 \end{bmatrix} \vec{X} + \begin{bmatrix} 0 \\ 0 \\ 6-4\sin x \end{bmatrix} \quad \text{where } \vec{X} = \begin{bmatrix} y \\ y' \\ y'' \end{bmatrix}}$$

$$\boxed{\vec{X}(0) = \begin{bmatrix} 4 \\ 0 \\ -5 \end{bmatrix}}$$

* There are other eigenvalue choices for each problem which would also be correct *

Ch8 Test Review

#1. Find the general solution of the system:

$$\frac{dx}{dt} = x + 2y$$

$$\frac{dy}{dt} = 4x + 3y$$

$\vec{X}' = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}\vec{X}$

$\begin{vmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{vmatrix} = 0$ $\quad (1-\lambda)(3-\lambda)-8=0$

$\lambda^2-4\lambda-5=0$

$(\lambda-5)(\lambda+1)=0$

$\underline{\lambda=5}$ $\underline{\lambda=-1}$

$\underline{\lambda=5}$

$\left[\begin{array}{cc|c} -4 & 2 & 0 \\ 4 & -2 & 0 \end{array}\right]$

rref

$\left[\begin{array}{cc|c} 1 & -1/2 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$k_1 - \frac{1}{2}k_2 = 0$

$k_1 = \frac{1}{2}k_2$

$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\underline{\lambda=-1}$

$\left[\begin{array}{cc|c} 2 & 2 & 0 \\ 4 & 4 & 0 \end{array}\right]$

rref

$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$k_1 + k_2 = 0$

$k_1 = -k_2$

$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$$\boxed{\vec{X} = C_1\begin{bmatrix} 1 \\ 2 \end{bmatrix}e^{5t} + C_2\begin{bmatrix} -1 \\ 1 \end{bmatrix}e^{-t}}$$

#2. Find the general solution of the system:

$$\vec{X}' = \begin{bmatrix} 10 & -5 \\ 8 & -12 \end{bmatrix}\vec{X}$$

$\begin{vmatrix} 10-\lambda & -5 \\ 8 & -12-\lambda \end{vmatrix}$ $\quad (10-\lambda)(-12-\lambda)+40=0$

$\lambda^2+2\lambda-80=0$

$(\lambda-8)(\lambda+10)=0$

$\underline{\lambda=8}$ $\underline{\lambda=-10}$

$\underline{\lambda=8}$

$\left[\begin{array}{cc|c} 2 & -5 & 0 \\ 8 & -20 & 0 \end{array}\right]$

rref

$\left[\begin{array}{cc|c} 1 & -5/2 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$k_1 - \frac{5}{2}k_2 = 0$

$k_1 = \frac{5}{2}k_2$

$\begin{bmatrix} 5 \\ 2 \end{bmatrix}$

$\underline{\lambda=-10}$

$\left[\begin{array}{cc|c} 20 & -5 & 0 \\ 8 & -2 & 0 \end{array}\right]$

rref

$\left[\begin{array}{cc|c} 1 & -1/4 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$k_1 - \frac{1}{4}k_2 = 0$

$k_1 = \frac{1}{4}k_2$

$\begin{bmatrix} 1 \\ 4 \end{bmatrix}$

$$\boxed{\vec{X} = C_1\begin{bmatrix} 5 \\ 2 \end{bmatrix}e^{8t} + C_2\begin{bmatrix} 1 \\ 4 \end{bmatrix}e^{-10t}}$$

#3. Find the general solution of the system:

$$\vec{X}' = \begin{bmatrix} -6 & 2 \\ -3 & 1 \end{bmatrix} \vec{X} \qquad \begin{vmatrix} -6-\lambda & 2 \\ -3 & 1-\lambda \end{vmatrix} = 0$$

$(-6-\lambda)(1-\lambda)+6=0$

$\lambda^2+5\lambda=0$

$\lambda(\lambda+5)=0$

$\lambda=0 \quad \lambda=-5$

$\lambda=0$

$\left[\begin{array}{cc|c} -6 & 2 & 0 \\ -3 & 1 & 0 \end{array}\right]$

rref

$\left[\begin{array}{cc|c} 1 & -1/3 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$k_1 - 1/3 k_2 = 0$

$k_1 = 1/3 k_2$

$\begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$\lambda=-5$

$\left[\begin{array}{cc|c} -1 & 2 & 0 \\ -3 & 6 & 0 \end{array}\right]$

rref

$\left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$k_1 - 2k_2 = 0$

$k_1 = 2k_2$

$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$$\boxed{\vec{X} = c_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-5t}}$$

#4. Find the general solution of the system:

$$\frac{dx}{dt} = 3x - y$$

$$\frac{dy}{dt} = 9x - 3y$$

$\vec{X}' = \begin{bmatrix} 3 & -1 \\ 9 & -3 \end{bmatrix} \vec{X}$

$\begin{vmatrix} 3-\lambda & -1 \\ 9 & -3-\lambda \end{vmatrix} = 0$

$(3-\lambda)(-3-\lambda)+9=0$

$\lambda^2 = 0 \quad \lambda=0$ repeated

$\lambda=0$

$\left[\begin{array}{cc|c} 3 & -1 & 0 \\ 9 & -3 & 0 \end{array}\right]$

rref

$\left[\begin{array}{cc|c} 1 & -1/3 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$k_1 - 1/3 k_2 = 0$

$k_1 = 1/3 k_2$

$\begin{bmatrix} 1 \\ 3 \end{bmatrix}$

repeat $\lambda=0$

$\left[\begin{array}{cc|c} 3 & -1 & 1 \\ 9 & -3 & 3 \end{array}\right]$

rref

$\left[\begin{array}{cc|c} 1 & -1/3 & 1/3 \\ 0 & 0 & 0 \end{array}\right]$

$k_1 - 1/3 k_2 = 1/3$

$k_1 = 1/3 k_2 + 1/3$

$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\vec{X} = c_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{0t} + c_2 \left(\begin{bmatrix} 1 \\ 3 \end{bmatrix} t e^{0t} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{0t} \right)$

$$\boxed{\vec{X} = c_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2 \left(\begin{bmatrix} 1 \\ 3 \end{bmatrix} t + \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right)}$$

#5. Find the general solution of the system:

$\vec{X}' = \begin{bmatrix} -1 & 3 \\ -3 & 5 \end{bmatrix} \vec{X}$ $\quad \begin{vmatrix} -1-\lambda & 3 \\ -3 & 5-\lambda \end{vmatrix} = 0 \quad (-1-\lambda)(5-\lambda)+9=0 \quad (\lambda-2)(\lambda-2)=0$

$\lambda^2 - 4\lambda + 4 = 0 \qquad \underline{\lambda = 2 \text{ (repeated)}}$

$\underline{\lambda=2}$

$\left[\begin{array}{cc|c} -3 & 3 & 0 \\ -3 & 3 & 0 \end{array}\right]$

rref

$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$K_1 - K_2 = 0 \quad \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$K_1 = K_2$

repeat $\lambda = 2$

$\left[\begin{array}{cc|c} -3 & 3 & 1 \\ -3 & 3 & 1 \end{array}\right]$

rref

$\left[\begin{array}{cc|c} 1 & -1 & 1/3 \\ 0 & 0 & 0 \end{array}\right]$

$K_1 - K_2 = 1/3 \quad \begin{bmatrix} 1/3 \\ 0 \end{bmatrix}$

$K_1 = K_2 + 1/3$

$$\boxed{\vec{X} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t} + c_2 \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} t e^{2t} + \begin{bmatrix} 1/3 \\ 0 \end{bmatrix} e^{2t} \right)}$$

#6. Solve the initial-value problem:

$\vec{X}' = \begin{bmatrix} 2 & 4 \\ -1 & 6 \end{bmatrix} \vec{X}, \qquad \vec{X}(0) = \begin{bmatrix} -1 \\ 6 \end{bmatrix}$ $\quad \begin{vmatrix} 2-\lambda & 4 \\ -1 & 6-\lambda \end{vmatrix} = 0 \quad (2-\lambda)(6-\lambda)+4=0 \quad (\lambda-4)(\lambda-4)=0$

$\lambda^2 - 8\lambda + 16 = 0 \qquad \underline{\lambda = 4 \text{ (repeated)}}$

$\underline{\lambda=4}$

$\left[\begin{array}{cc|c} -2 & 4 & 0 \\ -1 & 2 & 0 \end{array}\right]$

rref

$\left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$K_1 - 2K_2 = 0 \quad \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$K_1 = 2K_2$

repeat $\lambda = 4$

$\left[\begin{array}{cc|c} -2 & 4 & 2 \\ -1 & 2 & 1 \end{array}\right]$

rref

$\left[\begin{array}{cc|c} 1 & -2 & -1 \\ 0 & 0 & 0 \end{array}\right]$

$K_1 - 2K_2 = -1 \quad \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$K_1 = 2K_2 - 1$

$\vec{X} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{4t} + c_2 \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} t e^{4t} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t} \right)$

$x = 2c_1 e^{4t} + 2c_2 t e^{4t} + c_2 e^{4t}$

$-1 = 2c_1 e^0 + 2c_2(0)e^0 + c_2 e^0$

$-1 = 2c_1 + c_2$

$y = c_1 e^{4t} + c_2 t e^{4t} + c_2 e^{4t}$

$6 = c_1 e^0 + c_2(0)e^0 + c_2 e^0$

$6 = c_1 + c_2$

$\begin{bmatrix} c_1 + c_2 = 6 \\ 2c_1 + c_2 = -1 \end{bmatrix}$ rref $\left[\begin{array}{cc|c} 1 & 0 & -7 \\ 0 & 1 & 13 \end{array}\right] \begin{matrix} = c_1 \\ = c_2 \end{matrix}$

$$\boxed{\vec{X} = -7 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{4t} + 13 \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} t e^{4t} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t} \right)}$$

#7. Find the general solution of the system:

$$\frac{dx}{dt} = x + y$$

$$\frac{dy}{dt} = -2x - y$$

$\vec{X}' = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} \vec{X}$ $\quad \begin{vmatrix} 1-\lambda & 1 \\ -2 & -1-\lambda \end{vmatrix} = 0 \quad (1-\lambda)(-1-\lambda)+2=0$

$\lambda^2 + 1 = 0 \qquad \lambda = 0 \pm i$

$\lambda = 0 + i$

$\left[\begin{array}{cc|c} 1-i & 1 & 0 \\ -2 & -1-i & 0 \end{array}\right] \quad (1-i)k_1 + k_2 = 0 \quad k_2 = -(1-i)k_1 \quad \begin{bmatrix} 1 \\ -1+i \end{bmatrix} \quad \vec{B}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \vec{B}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$\vec{X} = c_1\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\cos t - \begin{bmatrix} 0 \\ 1 \end{bmatrix}\sin t\right)e^{0t} + c_2\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\cos t + \begin{bmatrix} 1 \\ -1 \end{bmatrix}\sin t\right)e^{0t}$$

$$\boxed{\vec{X} = c_1\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\cos t - \begin{bmatrix} 0 \\ 1 \end{bmatrix}\sin t\right) + c_2\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\cos t + \begin{bmatrix} 1 \\ -1 \end{bmatrix}\sin t\right)}$$

#8. Find the general solution of the system:

$$\vec{X}' = \begin{bmatrix} 4 & -5 \\ 5 & -4 \end{bmatrix} \vec{X}$$

$\begin{vmatrix} 4-\lambda & -5 \\ 5 & -4-\lambda \end{vmatrix} = 0 \quad (4-\lambda)(-4-\lambda)+25=0$

$\lambda^2 + 9 = 0 \qquad \lambda = 0 \pm 3i$

$\lambda = 0 + 3i$

$\left[\begin{array}{cc|c} 4-3i & -5 & 0 \\ 5 & -4-3i & 0 \end{array}\right] \quad (4-3i)k_1 - 5k_2 = 0 \quad k_2 = \frac{(4-3i)}{5}k_1 \quad \begin{bmatrix} 5 \\ 4-3i \end{bmatrix} \quad \vec{B}_1 = \begin{bmatrix} 5 \\ 4 \end{bmatrix} \quad \vec{B}_2 = \begin{bmatrix} 0 \\ -3 \end{bmatrix}$

$$\boxed{\vec{X} = c_1\left(\begin{bmatrix} 5 \\ 4 \end{bmatrix}\cos 3t - \begin{bmatrix} 0 \\ -3 \end{bmatrix}\sin 3t\right) + c_2\left(\begin{bmatrix} 0 \\ -3 \end{bmatrix}\cos t + \begin{bmatrix} 5 \\ 4 \end{bmatrix}\sin t\right)}$$

#9. Solve the initial-value problem:

$\vec{X}' = \begin{bmatrix} 6 & -1 \\ 5 & 4 \end{bmatrix}\vec{X}, \qquad \vec{X}(0) = \begin{bmatrix} -2 \\ 8 \end{bmatrix}$

$\begin{vmatrix} 6-\lambda & -1 \\ 5 & 4-\lambda \end{vmatrix} = 0 \qquad (6-\lambda)(4-\lambda)+5=0$

$\lambda^2 - 10\lambda + 29 = 0$

$\lambda = \frac{10 \pm \sqrt{100-4(29)}}{2} = \frac{10 \pm 4i}{2} = 5 \pm 2i$

$\underline{\lambda = 5+2i}$

$\left[\begin{array}{cc|c} 6-(5+2i) & -1 & 0 \\ 5 & 4-(5+2i) & 0 \end{array}\right]$

$\left[\begin{array}{cc|c} -1-2i & -1 & 0 \\ 5 & -1-2i & 0 \end{array}\right] \qquad \begin{aligned} (-1-2i)K_1 - K_2 &= 0 \\ K_2 &= (-1-2i)K_1 \end{aligned} \qquad \begin{bmatrix} 1 \\ 1-2i \end{bmatrix} \qquad \vec{B}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \vec{B}_2 = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$

$\vec{X} = c_1\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\cos 2t - \begin{bmatrix} 0 \\ -2 \end{bmatrix}\sin 2t\right)e^{5t} + c_2\left(\begin{bmatrix} 0 \\ -2 \end{bmatrix}\cos 2t + \begin{bmatrix} 1 \\ 1 \end{bmatrix}\sin 2t\right)e^{5t}$

$x = c_1(\cos 2t)e^{5t} + c_2(\sin 2t)e^{5t}$

$-2 = c_1(\cos 0)e^0 + c_2(\sin 0)e^0$

$-2 = c_1(1)(1) + c_2(0)(1)$

$\underline{c_1 = -2}$

$y = c_1(\cos 2t + 2\sin 2t)e^{5t} + c_2(-2\cos 2t + \sin 2t)e^{5t}$

$8 = c_1(\cos 0 + 2\sin 0)e^0 + c_2(-2(\cos 0) + \sin 0)e^0$

$8 = c_1(1+0)(1) + c_2(-2(1)+0)(1)$

$8 = c_1 - 2c_2$

$8 = (-2) - 2c_2 \qquad c_2 = \frac{10}{-2} = -5$

$$\boxed{\vec{X} = -2\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\cos 2t - \begin{bmatrix} 0 \\ -2 \end{bmatrix}\sin 2t\right)e^{5t} - 5\left(\begin{bmatrix} 0 \\ -2 \end{bmatrix}\cos 2t + \begin{bmatrix} 1 \\ 1 \end{bmatrix}\sin 2t\right)e^{5t}}$$

#10. Solve both ways (method of undermined coefficients and variation of parameters):

$$\vec{X}' = \begin{bmatrix} 4 & 1/3 \\ 9 & 6 \end{bmatrix}\vec{X} + \begin{bmatrix} -3 \\ 10 \end{bmatrix} e^t$$

$$\begin{vmatrix} 4-\lambda & 1/3 \\ 9 & 6-\lambda \end{vmatrix} = 0 \quad (4-\lambda)(6-\lambda) - 3 = 0 \quad (\lambda-7)(\lambda-3) = 0$$

$$\lambda^2 - 10\lambda + 21 = 0 \qquad \lambda = 7,\ \lambda = 3$$

$\lambda = 7$

$$\left[\begin{array}{cc|c} -3 & 1/3 & 0 \\ 9 & -1 & 0 \end{array}\right]$$

rref

$$\left[\begin{array}{cc|c} 1 & -1/9 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$K_1 - 1/9 K_2 = 0$

$K_1 = 1/9 K_2$ $\quad \begin{bmatrix} 1 \\ 9 \end{bmatrix}$

$\lambda = 3$

$$\left[\begin{array}{cc|c} 1 & 1/3 & 0 \\ 9 & 3 & 0 \end{array}\right]$$

rref

$$\left(\begin{array}{cc|c} 1 & 1/3 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$K_1 + 1/3 K_2 = 0$

$K_1 = -1/3 K_2$ $\quad \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

$$\vec{X}_c = C_1\begin{bmatrix} 1 \\ 9 \end{bmatrix} e^{7t} + C_2\begin{bmatrix} -1 \\ 3 \end{bmatrix} e^{3t}$$

undetermined coefficients

for $\begin{bmatrix} -3e^t \\ 10e^t \end{bmatrix}$ $\quad \vec{X}_p = \begin{bmatrix} Ae^t \\ Be^t \end{bmatrix}$

(no absorption)

$$\vec{X}' = \begin{bmatrix} Ae^t \\ Be^t \end{bmatrix}$$

into DE system

$$\begin{bmatrix} Ae^t \\ Be^t \end{bmatrix} = \begin{bmatrix} 4 & 1/3 \\ 9 & 6 \end{bmatrix}\begin{bmatrix} Ae^t \\ Be^t \end{bmatrix} + \begin{bmatrix} -3e^t \\ 10e^t \end{bmatrix}$$

$$\begin{cases} 4Ae^t + \frac{1}{3}Be^t - 3e^t = Ae^t \\ 9Ae^t + 6Be^t + 10e^t = Be^t \end{cases}$$

$$\begin{cases} 4A + \frac{1}{3}B - 3 = A \\ 9A + 6B + 10 = B \end{cases}$$

$$\begin{cases} 5A + \frac{1}{3}B = 3 \\ 9A + 5B = -10 \end{cases}$$

$$\left[\begin{array}{cc|c} 5 & 1/3 & 3 \\ 9 & 5 & -10 \end{array}\right]$$

rref

$$\left[\begin{array}{cc|c} 1 & 0 & 55/36 \\ 0 & 1 & -19/4 \end{array}\right]$$

$$\vec{X}_p = \begin{bmatrix} 55/36 \\ -19/4 \end{bmatrix} e^t$$

$$\boxed{\vec{X} = C_1\begin{bmatrix} 1 \\ 9 \end{bmatrix} e^{7t} + C_2\begin{bmatrix} -1 \\ 3 \end{bmatrix} e^{3t} + \begin{bmatrix} 55/36 \\ -19/4 \end{bmatrix} e^t}$$

variation of parameters

$$\Phi = \begin{bmatrix} e^{7t} & -e^{3t} \\ 9e^{7t} & 3e^{3t} \end{bmatrix} \quad \det\Phi = 3e^{10t} + 9e^{10t} = 12e^{10t}$$

$$\Phi^{-1} = \frac{1}{12e^{10t}}\begin{bmatrix} 3e^{3t} & e^{3t} \\ -9e^{7t} & e^{7t} \end{bmatrix} = \begin{bmatrix} 1/4 e^{-7t} & 1/12 e^{-7t} \\ -3/4 e^{-3t} & 1/12 e^{-3t} \end{bmatrix}$$

$$\vec{X}_p = \Phi \int \Phi^{-1} F\,dt$$

$$= \begin{bmatrix} e^{7t} & -e^{3t} \\ 9e^{7t} & 3e^{3t} \end{bmatrix} \int \begin{bmatrix} 1/4 e^{-7t} & 1/12 e^{-7t} \\ -3/4 e^{-3t} & 1/12 e^{-3t} \end{bmatrix}\begin{bmatrix} -3e^t \\ 10e^t \end{bmatrix} dt$$

$$\int \begin{bmatrix} -3/4 e^{-6t} + 5/6 e^{-6t} \\ 9/4 e^{-2t} + 5/6 e^{-2t} \end{bmatrix} dt$$

$$\int \begin{bmatrix} 1/12 e^{-6t} \\ 37/12 e^{-2t} \end{bmatrix} dt$$

$$\begin{bmatrix} e^{7t} & -e^{3t} \\ 9e^{7t} & 3e^{3t} \end{bmatrix}\begin{bmatrix} -\frac{1}{72} e^{-6t} \\ -37/24 e^{-2t} \end{bmatrix}$$

$$\begin{bmatrix} -\frac{1}{72} e^t + \frac{37}{24} e^t \\ -\frac{9}{72} e^t - \frac{111}{24} e^t \end{bmatrix} = \begin{bmatrix} 55/36 \\ -19/4 \end{bmatrix} e^t$$

$$\boxed{\vec{X} = C_1\begin{pmatrix} 1 \\ 9 \end{pmatrix} e^{7t} + C_2\begin{bmatrix} -1 \\ 3 \end{bmatrix} e^{3t} + \begin{bmatrix} 55/36 \\ -19/4 \end{bmatrix} e^t}$$

#11. Solve both ways (method of undermined coefficients and variation of parameters):

$$\vec{X}' = \begin{bmatrix} 3 & 2 \\ -2 & -1 \end{bmatrix}\vec{X} + \begin{bmatrix} 2e^{-t} \\ e^{-t} \end{bmatrix}$$

$\begin{vmatrix} 3-\lambda & 2 \\ -2 & -1-\lambda \end{vmatrix} = 0$ $(3-\lambda)(-1-\lambda)+4=0$ $\lambda^2-2\lambda+1=0$ $(\lambda-1)(\lambda-1)=0$ $\lambda=1$ repeated

$\lambda=1$

$\left[\begin{array}{cc|c} 2 & 2 & 0 \\ -2 & -2 & 0 \end{array}\right]$ rref $\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$K_1+K_2=0$
$K_1=-K_2$ $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

repeat $\lambda=1$

$\left[\begin{array}{cc|c} 2 & 2 & 1 \\ -2 & -2 & -1 \end{array}\right]$ rref $\left[\begin{array}{cc|c} 1 & 1 & 1/2 \\ 0 & 0 & 0 \end{array}\right]$

$K_1+K_2=1/2$
$K_2=1/2-K_1$ $\begin{bmatrix} 0 \\ 1/2 \end{bmatrix}$

$$\vec{X}_c = C_1\begin{bmatrix} 1 \\ -1 \end{bmatrix}e^t + C_2\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}te^t + \begin{bmatrix} 0 \\ 1/2 \end{bmatrix}e^t\right)$$

undetermined coefficients

for $\begin{bmatrix} 2e^{-t} \\ e^{-t} \end{bmatrix}$ table:

$\vec{X}_p = \begin{bmatrix} Ae^{-t} \\ Be^{-t} \end{bmatrix}$ $\vec{X}' = \begin{bmatrix} -Ae^{-t} \\ -Be^{-t} \end{bmatrix}$

(no absorption)

into DE system:

$$\begin{bmatrix} -Ae^{-t} \\ -Be^{-t} \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} Ae^{-t} \\ Be^{-t} \end{bmatrix} + \begin{bmatrix} 2e^{-t} \\ e^{-t} \end{bmatrix}$$

$$\begin{cases} 3Ae^{-t}+2Be^{-t}+2e^{-t}=-Ae^{-t} \\ -2Ae^{-t}-Be^{-t}+e^{-t}=-Be^{-t} \end{cases}$$

$$\begin{cases} 3A+2B+2=-A \\ -2A-B+1=-B \end{cases}$$

$$\begin{cases} 4A+2B=-2 \\ -2A+0B=-1 \end{cases}$$

$\left[\begin{array}{cc|c} 4 & 2 & -2 \\ -2 & 0 & -1 \end{array}\right]$ rref $\left[\begin{array}{cc|c} 1 & 0 & 1/2 \\ 0 & 1 & -2 \end{array}\right] \begin{matrix} =A \\ =B \end{matrix}$

$\vec{X}_p = \begin{bmatrix} 1/2e^{-t} \\ -2e^{-t} \end{bmatrix}$

$$\boxed{\vec{X} = C_1\begin{bmatrix} 1 \\ -1 \end{bmatrix}e^t + C_2\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}te^t + \begin{bmatrix} 0 \\ 1/2 \end{bmatrix}e^t\right) + \begin{bmatrix} 1/2 \\ -2 \end{bmatrix}e^{-t}}$$

variation of parameters

$\Phi = \begin{bmatrix} e^t & te^t \\ -e^t & -te^t+\frac{1}{2}e^t \end{bmatrix}$ $\det\Phi = -te^{2t}+\frac{1}{2}e^{2t}+te^{2t} = \frac{1}{2}e^{2t}$

$$\Phi^{-1} = \frac{1}{\frac{1}{2}e^{2t}}\begin{bmatrix} -te^t+\frac{1}{2}e^t & -te^t \\ e^t & e^t \end{bmatrix} = \begin{bmatrix} -2te^{-t}+e^{-t} & -2te^{-t} \\ 2e^{-t} & 2e^{-t} \end{bmatrix}$$

$$\vec{X}_p = \Phi\int\Phi^{-1}F\,dt$$

$$= \begin{bmatrix} e^t & te^t \\ -e^t & -te^t+\frac{1}{2}e^t \end{bmatrix}\int\begin{bmatrix} -2te^{-t}+e^{-t} & -2te^{-t} \\ 2e^{-t} & 2e^{-t} \end{bmatrix}\begin{bmatrix} 2e^{-t} \\ e^{-t} \end{bmatrix}dt$$

$$\int\begin{bmatrix} -4te^{-2t}+2e^{-2t}-2te^{-2t} \\ 4e^{-2t}+2e^{-2t} \end{bmatrix}dt$$

$$\int\begin{bmatrix} -6te^{-2t}+2e^{-2t} \\ 6e^{-2t} \end{bmatrix}dt$$

$$\begin{bmatrix} 3te^{-2t}+\frac{3}{2}e^{-2t}-e^{-2t} \\ -3e^{-2t} \end{bmatrix}$$

$$\begin{bmatrix} e^t & te^t \\ -e^t & -te^t+\frac{1}{2}e^t \end{bmatrix}\begin{bmatrix} 3te^{-2t}+\frac{1}{2}e^{-2t} \\ -3e^{-2t} \end{bmatrix}$$

$$\vec{X}_p = \begin{bmatrix} 3te^{-t}+\frac{1}{2}e^{-t}-3te^{-t} \\ -3te^{-t}-\frac{1}{2}e^{-t}+3te^{-t}-\frac{3}{2}e^{-t} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2}e^{-t} \\ -2e^{-t} \end{bmatrix}$$

$$\boxed{\vec{X} = C_1\begin{bmatrix} 1 \\ -1 \end{bmatrix}e^t + C_2\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}te^t + \begin{bmatrix} 0 \\ 1/2 \end{bmatrix}e^t\right) + \begin{bmatrix} 1/2 \\ -2 \end{bmatrix}e^{-t}}$$

#12. Rewrite the following 3rd-order differential equation as a system of 3 1st-order differential equations (and also write the initial conditions in matrix form):

$$y''' - y'' + 2y' - 3y = t^3 - 4t$$

$$y(0) = 3, \quad y'(0) = 7, \quad y''(0) = 5$$

(do not solve the system)

$$\begin{cases} y' = 0y + y' + 0y'' + 0 \\ y'' = 0y + 0y' + y'' + 0 \\ y''' = 3y - 2y' + y'' + t^3 - 4t \end{cases}$$

$$\boxed{\vec{x}' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 3 & -2 & 1 \end{bmatrix} \vec{x} + \begin{bmatrix} 0 \\ 0 \\ t^3 - 4t \end{bmatrix} \qquad \vec{x}(0) = \begin{bmatrix} 3 \\ 7 \\ 5 \end{bmatrix} \qquad \text{where } \vec{x} = \begin{bmatrix} y \\ y' \\ y'' \end{bmatrix}}$$