DiffEq - Ch 4 - Extra Practice

4.1

#1b. The given family of functions is the general solution of the differential equation on the indicated interval. Find a member of the family that is a solution of the initial-value problem.

$y = C_1 e^{4x} + C_2 e^{-x}, \ (-\infty, \infty);$

$y'' - 3y' - 4y = 0, \quad y(0) = 1, \ y'(0) = 2$

$y = C_1 e^{4x} + C_2 e^{-x} \qquad y' = 4C_1 e^{4x} - C_2 e^{-x}$

$1 = C_1 e^0 + C_2 e^0 \qquad 2 = 4C_2 e^0 - C_2 e^0$

$\qquad\qquad\qquad\qquad 4C_1 - C_2 = 2$

$$\begin{cases} C_1 + C_2 = 1 \\ 4C_1 - C_2 = 2 \end{cases}$$

$5C_1 = 3$

$C_1 = 3/5, \quad C_2 = 1 - \frac{3}{5} = \frac{2}{5}$

$$\boxed{y = \frac{3}{5} e^{4x} + \frac{2}{5} e^{-x}}$$

#2b. The given family of functions is the general solution of the differential equation on the indicated interval. Find a member of the family that is a solution of the initial-value problem.

$y = C_1 + C_2 \cos x + C_3 \sin x, \ (-\infty, \infty);$

$y''' + y' = 0, \quad y(\pi) = 0, \quad y'(\pi) = 2, \ y''(\pi) = -1$

$y = C_1 + C_2 \cos x + C_3 \sin x$

$0 = C_1 + C_2 \cos \pi + C_3 \sin x \rightarrow C_1 - C_2 = 0$

$y' = -C_2 \sin x + C_3 \cos x$

$2 = -C_2 \sin \pi + C_3 \cos \pi \rightarrow -C_3 = 2$

$y'' = -C_2 \cos x - C_3 \sin x$

$-1 = -C_2 \cos \pi - C_3 \sin \pi \rightarrow C_2 = -1$

System $\begin{cases} C_1 - C_2 = 0 \\ -C_3 = 2 \\ C_2 = -1 \end{cases}$ $\quad C_2 = -1, \ C_3 = -2$

$\qquad C_1 = C_2 = -1$

$$\boxed{y = -1 - \cos x - 2\sin x}$$

#3b. Determine whether the given set of functions is linearly independent on the interval $(-\infty, \infty)$.

$f_1(x) = 5, \ f_2(x) = \cos^2 x, \ f_3(x) = \sin^2 x$

$$W = \begin{vmatrix} 5 & (\cos x)^2 & (\sin x)^2 \\ 0 & -2\cos x \sin x & 2\sin x \cos x \\ 0 & (-2\cos^2 x + \sin^2 x(2)) & (2\sin x(-\sin x) + \cos x(2c \end{vmatrix}$$

$$= \begin{vmatrix} 5 & \cos^2 x & \sin^2 x \\ 0 & -2\cos x \sin x & 2\sin x \cos x \\ 0 & 2(\cos^2 x - \sin^2 x) & 2(\cos^2 x - \sin^2 x) \end{vmatrix}$$

Use this column

$= 5[-4\cos x \sin x(\cos^2 x - \sin^2 x) - 4\cos x \sin x(\cos^2 x - \sin^2 x)$

$= 0$ <u>not</u> linearly independent

#4b. Determine whether the given set of functions is linearly independent on the interval $(-\infty, \infty)$.

$f_1(x) = x, \ f_2(x) = x - 1, \ f_3(x) = x + 3$

$$W = \begin{vmatrix} x & x-1 & x+3 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{vmatrix}$$

$W = x(0-0) - (x-1)(0-0) + (x+3)(0-0)$

$= 0$

<u>not</u> linearly independent

#5b. Determine whether the given set of functions is linearly independent on the interval $(-\infty,\infty)$.

$f_1(x)=e^{-3x},\ f_2(x)=e^{4x}$

$$W=\begin{vmatrix} e^{-3x} & e^{4x} \\ -3e^{-3x} & 4e^{4x} \end{vmatrix}$$

$$= 4e^{4x}e^{-3x}-(-3e^{-3x}e^{4x})$$

$$= 4e^{(4x-3x)}+3e^{(4x-3x)}$$

$$=4e^{x}+3e^{x}$$

$$=7e^{x}\neq 0$$

so these solutions are linearly independent

#6b. Verify that the given functions form a fundamental set of solutions of the differential equation on the indicated interval. Then form the general solution.

$4y''-4y'+y=0;\quad e^{\frac{1}{2}x},\ xe^{\frac{1}{2}x},\ (-\infty,\infty)$

verify these are solutions of the DE:

$y=e^{1/2x}$ $\quad 4y''-4y'+y=0$

$y'=\frac{1}{2}e^{1/2x}$ $\quad 4(\frac{1}{4}e^{1/2x})-4(\frac{1}{2}e^{1/2x})+e^{1/2x}=0$

$y''=\frac{1}{4}e^{1/2x}$ $\quad e^{1/2x}-2e^{1/2x}+e^{1/2x}=0$

$0=0$ verified

$y=xe^{1/2x}$

$y'=x\frac{1}{2}e^{1/2x}+e^{1/2x}(1)=\frac{1}{2}xe^{1/2x}+e^{1/2x}$

$y''=\frac{1}{2}x\frac{1}{2}e^{1/2x}+e^{1/2x}(\frac{1}{2})+\frac{1}{2}e^{1/2x}$

$=\frac{1}{4}xe^{1/2x}+e^{1/2x}$

$4y''-4y'+y=0$

$4(\frac{1}{4}xe^{1/2x}+e^{1/2x})-4(\frac{1}{2}xe^{1/2x}+e^{1/2x})+xe^{1/2x}=0$

$xe^{1/2x}+4e^{1/2x}-2xe^{1/2x}-4e^{1/2x}+xe^{1/2x}=0$

$0=0$ verified

verify solutions are independent

$$W=\begin{vmatrix} e^{1/2x} & xe^{1/2x} \\ \frac{1}{2}e^{1/2x} & x\frac{1}{2}e^{1/2x}+e^{1/2x}(1) \\ & (\frac{1}{2}xe^{1/2x}+e^{1/2x}) \end{vmatrix}$$

$$=e^{1/2x}(\frac{1}{2}xe^{1/2x}+e^{1/2x})-\frac{1}{2}e^{1/2x}xe^{1/2x}$$

$$=\frac{1}{2}xe^{1/2x}e^{1/2x}+e^{1/2x}e^{1/2x}-\frac{1}{2}xe^{1/2}e^{1/2x}$$

$$=\frac{1}{2}xe^{x}+e^{x}-\frac{1}{2}xe^{x}$$

$=e^{x}\neq 0$ so are linearly independent

general solution is...

$$\boxed{y=c_1e^{1/2x}+c_2xe^{1/2x}}$$

#7b. Verify that the given functions form a fundamental set of solutions of the differential equation on the indicated interval. Then form the general solution.

$x^2y''+xy'+y=0;\quad \cos(\ln x), \sin(\ln x),\ (0,\infty)$

verify solutions to DE:

$y=\cos(\ln x)$

$y'=\underline{-\sin(\ln x)}\,\underline{\frac{1}{x}}$

$y''=-\sin(\ln x)(-\frac{1}{x^2})+\frac{1}{x}(-\cos(\ln(x))\frac{1}{x})$

$=\frac{\sin(\ln x)}{x^2}-\frac{\cos(\ln x)}{x^2}$

$x^2y''+xy'+y=0$

$x^2\left(\frac{\sin(\ln x)}{x^2}-\frac{\cos(\ln x)}{x^2}\right)+x\left(-\frac{\sin(\ln x)}{x}\right)+\cos(\ln x)\overset{?}{=}0$

$\sin(\ln x)-\cos(\ln x)-\sin(\ln x)+\cos(\ln x)=0$

$0=0$ ✓

$y=\sin(\ln x)$

$y'=\underline{\cos(\ln x)\frac{1}{x}}$

$y''=\cos(\ln x)(-\frac{1}{x^2})+\frac{1}{x}(-\sin(\ln x)\frac{1}{x})$

$x^2y''+xy'+y=0$

$x^2\left(-\frac{\cos(\ln x)}{x^2}-\frac{\sin(\ln x)}{x^2}\right)+x\left(\frac{\cos(\ln x)}{x}\right)+\sin(\ln x)\overset{?}{=}0$

$-\cos(\ln x)-\sin(\ln x)+\cos(\ln x)+\sin(\ln x)\overset{?}{=}0$

$0=0$ ✓

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verify linearly independent...

$$W=\begin{vmatrix}\cos(\ln x) & \sin(\ln x)\\ -\frac{\sin(\ln x)}{x} & \frac{\cos(\ln x)}{x}\end{vmatrix}$$

$=\frac{1}{x}\cos^2(\ln x)-(-\frac{1}{x}\sin^2(\ln x))$

$=\frac{1}{x}(\cos^2(\ln x)+\sin^2(\ln x))$

$=\frac{1}{x}\neq 0$ over $(0,\infty)$ ✓

solutions are linearly independent

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general solution is...

$$\boxed{y=c_1\cos(\ln x)+c_2\sin(\ln x)}$$

#8b. Verify that the given functions form a fundamental set of solutions of the differential equation on the indicated interval. Then form the general solution.

$x^3y'''+6x^2y''+4xy'-4y=0;\quad x, x^{-2}, x^{-2}\ln x,\ (0,\infty)$

verify solutions to DE...

$\underline{y=x}\quad x^3y'''+6x^2y''+4xy'-4y=0$

$y'=1\quad x^3(0)+6x^2(0)+4x(1)-4(x)=0$

$y''=0\quad 4x-4x=0$

$y'''=0\quad 0=0$ ✓

$\underline{y=x^{-2}}\quad x^3y''+6x^2y''+4xy'-4y=0$

$y'=-2x^{-3}\quad x^3(-24x^{-5})+6x^2(6x^{-4})+4x(-2x^{-3})$

$y''=6x^{-4}$

$y'''=-24x^{-5}\quad -24x^{-2}+36x^{-2}-8x^{-2}-4x^{-2}=0$

$0=0$ ✓

$y=\underline{x^{-2}\ln x}$

$y'=x^{-2}\frac{1}{x}+\ln x(-2x^{-3})=x^{-3}+(-2x^{-3})(\ln x)$

$y''=-3x^{-4}+(-2x^{-3})(\frac{1}{x})+\ln x(6x^{-4})$

$=-3x^{-4}-2x^{-4}+6x^{-4}\ln x=-5x^{-4}+\underline{(6x^{-4})\ln}$

$y'''=20x^{-5}+(6x^{-4})\frac{1}{x}+\ln x(-24x^{-5})$

$=20x^{-5}+6x^{-5}+(-24x^{-5})\ln x=26x^{-5}-24x^{-5}\ln$

DE: $x^3y'''+6x^2y''+4xy'-4y=0$

$x^3(26x^{-5}-24x^{-5}\ln x)+6x^2(-5x^{-4}+6x^{-4}\ln x)$

$+4x(x^{-3}-2x^{-3}\ln x)-4(x^{-2}\ln x)=0$

$\underline{26x^{-2}}-24x^{-2}\ln x\underline{-30x^{-2}}+36x^{-2}\ln x$

$\underline{+4x^{-2}}-8x^{-2}\ln x-4x^{-2}\ln x\overset{?}{=}0$

$0=0$ ✓

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verify linearly independent

$$W=\begin{vmatrix} x & x^{-2} & \underline{x^{-2}\ln x}\\ 1 & -2x^{-3} & x^{-2}\frac{1}{x}+\ln x(-2x^{-3})\\ & & (x^{-3}+(-2x^{-3})\ln x)\\ 0 & 6x^{-4} & -3x^{-4}+(-2x^{-3}\frac{1}{x})+(\ln x)(6x \end{vmatrix}$$

$$=\begin{vmatrix} +x & x^{-2} & x^{-2}\ln x\\ -1 & -2x^{-3} & x^{-3}-2x^{-3}\ln x\\ +0 & 6x^{-4} & -5x^{-4}+6x^{-4}\ln x\end{vmatrix}$$

$=$

...continued...

8b continued...

$$\begin{vmatrix} +\,x & x^{-2} & x^{-2}\ln x \\ -\,1 & -2x^{-3} & x^{-3}-2x^{-3}\ln x \\ +\,0 & 6x^{-4} & -5x^{-4}+6x^{-4}\ln x \end{vmatrix}$$

$$= x\left[(-2x^{-3})(-5x^{-4}+6x^{-4}\ln x)-(6x^{-4})(x^{-3}-2x^{-3}\ln x)\right]$$

$$-1\left[(x^{-2})(-5x^{-4}+6x^{-4}\ln x)-(6x^{-4})(x^{-2}\ln x)\right]$$

$$+0(\sim)$$

$$= x\left[10x^{-7}-12x^{-7}\ln x-6x^{-7}+12x^{-7}\ln x\right]-\left[-5x^{-6}+6x^{-6}\ln x-6x^{-6}\ln x\right]$$

$$= \underline{10x^{-6}}\ \underline{-12x^{-6}\ln x}\ \underline{-6x^{-6}}\ \underline{+12x^{-6}\ln x}\ \underline{+5x^{-6}}\ \underline{-6x^{-6}\ln x}\ \underline{+6x^{-6}\ln x}$$

$$= 9x^{-6} = \frac{9}{x^6} \neq 0 \quad \text{over } (0,\infty)$$

So these 3 solutions are linearly independent

general solution:

$$y = C_1 x + C_2 x^{-2} + C_3 x^{-2}\ln x$$

or

$$\boxed{y = C_1 x + \frac{C_2}{x^2} + C_3\frac{\ln x}{x^2}}$$

4.2

#1b. The indicated function is a solution of the given differential equation (no need to verify). Use reduction of order to find a second solution.

$y'' + 2y' + y = 0; \quad y_1 = xe^{-x}$

$y_2 = \underline{u}(\underline{xe^{-x}})$

$y' = u(x(-e^{-x}) + e^{-x}(1)) + (xe^{-x})u'$

$= \underline{-xe^{-x}}\,\underline{u} + \underline{e^{-x}}\,\underline{u} + \underline{xe^{-x}}\,\underline{u'}$

$y'' = (-xe^{-x})u' + u((-x)(-e^{-x}) + e^{-x}(-1))$

$+ (e^{-x})u' + u(-e^{-x})$

$+ (xe^{-x})u'' + u'(x(-e^{-x}) + e^{-x}(1))$

$= (xe^{-x})u'' + (-xe^{-x} + e^{-x} - xe^{-x} + e^{-x})u'$

$+ (xe^{-x} - e^{-x} - e^{-x})u$

$= xe^{-x}u'' + (2e^{-x} - 2xe^{-x})u' + (xe^{-x} - 2e^{-x})u$

$y'' + 2y' + y = 0$

$[xe^{-x}u'' + 2e^{-x}u' - 2xe^{-x}u' + xe^{-x}u - 2e^{-x}u]$

$+2[-xe^{-x}u + e^{-x}u + xe^{-x}u'] + xe^{-x}u = 0$

$(xe^{-x})u'' + (2e^{-x} - 2xe^{-x} + 2xe^{-x})u'$

$+ (\cancel{xe^{-x}} \cancel{-2e^{-x}} \cancel{-2xe^{-x}} \cancel{+2e^{-x}} \cancel{+xe^{-x}})u = 0$

let $w = u'$:

$\frac{xe^{-x}}{xe^{-x}}w' + \frac{2e^{-x}}{xe^{-x}}w = \frac{0}{xe^{-x}}$

$w' + \frac{2}{x}w = 0$ linear, w/ $P(x) = \frac{2}{x}$

$I.F. = e^{\int \frac{2}{x}dx} = e^{2\ln x} = e^{\ln(x^2)} = x^2$

$x^2 w = \int 0\,dx, \quad x^2 w = c_1$

$w = u' = x^{-2}c_1$

$u = \int c_1 x^{-2}dx = c_1(-x^{-1}) + c_2$

$= -\frac{c_1}{x} + c_2$ simplest if $c_1 = -1$, $c_2 = 0$

$u = \frac{1}{x}$

so $y_2 = u\,xe^{-x} = \frac{1}{x}\,xe^{-x} \boxed{= e^{-x}}$

#2b. The indicated function is a solution of the given differential equation (no need to verify). Use reduction of order to find a second solution.

$y'' + 9y = 0; \quad y_1 = \sin(3x)$

$y_2 = \underline{u}\,\underline{\sin(3x)}$

$y' = \underline{u}\,(\underline{3\cos(3x)}) + \sin\underline{(3x)u'}$

$y'' = u(-9\sin(3x)) + 3\cos(3x)u' + \sin(3x)u'' + u'(3\cos(3x$

$= -9\sin(3x)u + 6\cos(3x)u' + \sin(3x)u''$

$y'' + 9y = 0$

$[-9\sin(3x)u + 6\cos(3x)u' + \sin(3x)u'']$

$+9[u\sin(3x)] = 0$

$(\sin(3x))u'' + (6\cos(3x))u' = 0$

let $w = u'$

$\frac{\sin(3x)}{\sin(3x)}w' + \frac{6\cos(3x)}{\sin(3x)}w = \frac{0}{\sin(3x)}$

$w' + 6\cot(3x)w = 0$ linear, w/ $P = 6\cot(3x)$

$I.F. = e^{\int 6\cot(3x)dx}$ $\quad 6\int \frac{\cos(3x)}{\sin(3x)}dx$ $\quad n = 3x$, $\frac{dn}{dx} = 3$

$= 6\left(\frac{1}{3}\right)\int \frac{\cos n}{\sin n}dn$ $\quad dx = \frac{1}{3}dn$

now, let $m = \sin n$, $\frac{dm}{dn} = \cos n$, $dm = \cos n\,dn$

$= 2\int \frac{1}{m}dm = 2\ln|m|$

$= 2\ln|\sin n|$

$= \ln|\sin^2 n| = \ln(\sin^2(3$

so $I.F. = e^{\ln|\sin^2(3x)|}$

$= \sin^2(3x)$

$\sin^2(3x))w = \int 0\,dx = c_1$

$w = \frac{c_1}{\sin^2(3x)} = c_1\csc^2(3x) = u'$

$t = 3x, \frac{dt}{dx} = 3, dx = \frac{1}{3}dt$

so $u = \int c_1\csc^2(3x)dx$

$u = \frac{1}{3}c_1(-\cot(3x)) + c_2$ simplest form: $-\frac{1}{3}c_1 = 1, c_2 = 0$

$u = \cot(3x)$

so $y_2 = u\sin(3x) = \cot(3x)\sin 3x = \frac{\cos 3x}{\sin 3x}\sin 3x$

$\boxed{y_2 = \cos(3x)}$

#3b. The indicated function is a solution of the given differential equation (no need to verify). Use reduction of order to find a second solution.

$6y'' + y' - y = 0; \quad y_1 = e^{\left(\frac{1}{3}x\right)}$

$y_2 = ue^{1/3x}$

$y' = u(\frac{1}{3}e^{1/3x}) + (e^{1/3x})u'$

$y'' = u(\frac{1}{9}e^{1/3x}) + (\frac{1}{3}e^{1/3x})u' + (e^{1/3x})u'' + u'(\frac{1}{3}e^{1/3x})$

into DE...

$6y'' + y' - y = 0$

$6[\frac{1}{9}e^{1/3x}u + \frac{2}{3}e^{1/3x}u' + e^{1/3x}u'']$

$+ [\frac{1}{3}e^{1/3x}u + e^{1/3x}u'] - e^{1/3x}u = 0$

$(6e^{1/3x})u'' + (4e^{1/3x} + e^{1/3x})u' + (\underbrace{\frac{2}{3}e^{1/3x} + \frac{1}{3}e^{1/3x} - e^{1/3x}}_{=0}) = 0$

$\frac{6e^{1/3x}u''}{6e^{1/3x}} + \frac{5e^{1/3x}u'}{6e^{1/3x}} = \frac{0}{6e^{1/3x}}$

$u'' + \frac{5}{6}u' = 0$ define $w = u'$

$w' + \frac{5}{6}w = 0$ linear, $P(x) = 5$

$I.F. = e^{\int \frac{5}{6}dx} = e^{\frac{5}{6}x}$

$e^{5/6x}w = \int 0 e^{5/6x}dx = \int 0 dx = C_1$

$w = C_1 e^{-5/6x} = u'$

so $u = \int C_1 e^{-5/6x}dx$

$u = \frac{C_1}{(-\frac{5}{6})}e^{-5/6x} + C_2$

simplest if $\frac{C_1}{(-5/6)} = 1, C_2 = 0$

$u = e^{-5/6x}$

so $y_2 = ue^{1/3x} = e^{-5/6x}e^{1/3x}$

$= e^{(-5/6 + 1/3)x}$

$\boxed{y_2 = e^{-1/2x}}$

#4b. The indicated function is a solution of the given differential equation (no need to verify). Use reduction of order to find a second solution.

$x^2y'' + 2xy' - 6y = 0; \quad y_1 = x^2$

$y_2 = ux^2$

$y' = u(2x) + x^2u'$

$y'' = u(2) + 2xu' + x^2u'' + u'(2x)$

$= 2u + 4xu' + x^2u''$

into DE...

$x^2y'' + 2xy' - 6y = 0$

$x^2[2u + 4xu' + x^2u''] + 2x[2xu + x^2u']$

$-6[ux^2] = 0$

$x^4u'' + (4x^3 + 2x^3)u' + (\underbrace{2x^2 + 4x^2 - 6x^2}_{0})u = 0$

$x^4u'' + 6x^3u' = 0$ Let $w = u'$

$x^4w' + 6x^3w = 0$

$w' + \frac{6}{x}w = 0$ linear, w/ $P(x) = \frac{6}{x}$

$I.F. = e^{\int \frac{6}{x}dx} = e^{6\ln x} = e^{\ln x^6} = x$

$x^6w = \int 0x^6 dx = C_1$

$w = \frac{C_1}{x^6} = u' = C_1x^{-6}$

so $u = C_1\int x^{-6}dx = \frac{C_1}{-5}x^{-5} + C_2$

simplest: $\frac{C_1}{-5} = 1, \; C_2 = 0$

$u = x^{-5}$

then $y_2 = ux^2$

$= x^{-5}x^2$

$\boxed{y_2 = x^{-3} = \frac{1}{x^3}}$

#5b. The indicated function is a solution of the given differential equation (no need to verify). Use reduction of order to find a second solution.

$(1-2x-x^2)y''+2(1+x)y'-2y=0; \quad y_1=x+1$

$y_2 = u(x+1) = \underline{u}\underline{x} + u$

$y' = u(1) + \underline{x}\underline{u'} + u'$

$y'' = u' + xu'' + u'(1) + u'' = 2u' + xu'' + u''$

into DE; $(1-2x-x^2)y''+2(1+x)y'-2y=0$

$(1-2x-x^2)[2u'+xu''+u'']+2(1+x)[u+xu'+u']-2\overset{(x+1)u}{[xu+u]}=0$

$((1-2x-x^2)x+(1-2x-x^2))u''+(2(1-2x-x^2)+2(1+x)x+2(1+x))u'+\underbrace{(2(1+x)-2(x+1))}_{=0}u=0$

$(x-2x^2-x^3+1-2x-x^2)$

$(-x^3-3x^2-x+1)u''+(2-4x-2x^2+2x+2x^2+2+2x)u'=0$

$(-x^3-3x^2-x+1)u''+4u'=0$ Let $w=u'$

$w'+\frac{4}{-x^3-3x^2-x+1}w=0$ Linear, $w/ P(x)=\frac{4}{-x^3-3x^2-x+1}$

not easy to integrate for I.F. so switch to the formula;

$y_2 = y_1\int \frac{e^{-\int P(x)dx}}{(y_1(x))^2}dx$

$(1-2x-x^2)y''+2(1+x)y'-2y=0$

$y''+\underset{P(x)}{\frac{2(1+x)}{1-2x-x^2}}y'-\underset{Q(x)}{\frac{2}{1-2x-x^2}}y=0$

here, $P(x)=\frac{2(1+x)}{-x^2-2x+1}$; So $-\int P(x)dx$

$=-\int\frac{2(1+x)}{-x^2-2x+1}dx$

u-sub (use symbol m)

$m=-x^2-2x+1$

$\frac{dm}{dx}=-2x-2$
$=-2(x+1)$

$dm=-2(x+1)dx$

$=-\int\frac{1}{m}(-dm)$

$=\ln|m|$

$=\ln|-x^2-2x+1|$

into formula

$y_2=(x+1)\int\frac{e^{\ln|-x^2-2x+1|}}{(x+1)^2}dx$

$=(x+1)\int\frac{|-x^2-2x+1|}{(x+1)^2}dx$ ← textbook ignores | | here, so we will too ☺

simplify $\frac{-x^2-2x+1}{(x+1)^2}=\frac{-x^2-2x+1}{x^2+2x+1}$ now polynomial division

$$\begin{array}{r} -1 \\ x^2+2x+1\,\overline{)\,-x^2-2x+1} \\ \underline{-(-x^2-2x-1)} \\ 2 \end{array}$$

$\frac{-x^2-2x+1}{(x+1)^2}=-1+\frac{2}{x^2+2x+1}=-1+\frac{2}{(x+1)^2}$

$y_2=(x+1)\int\left(-1+\frac{2}{(x+1)^2}\right)dx$

$=(x+1)\left[\int -1dx+2\int\frac{1}{(x+1)^2}dx\right]$

u-sub: $m=x+1$, $dm=dx$

$2\int m^{-2}dm$

$y_2=(x+1)\left[-x+\frac{2}{x+1}\right]$

so $y_2=-x(x+1)+\frac{2(x+1)}{(x+1)}=\boxed{-x^2-x+2}$

whew! (required problem 5 is a little easier but does require the formula ☺)

4.3

#1b. Find the general solution of the differential equation.
$y'' - 36y' = 0$

$m^2 - 36m = 0$

$m(m-36) = 0$

$m = 0 \quad m = 36$

$y = C_1 e^{0x} + C_2 e^{36x}$

$$\boxed{y = C_1 + C_2 e^{36x}}$$

#2b. Find the general solution of the differential equation.
$y'' - 10y' + 25y = 0$

$m^2 - 10m + 25 = 0$

$(m-5)(m-5) = 0$

$m = 5$ (repeated)

$$\boxed{y = C_1 e^{5x} + C_2 x e^{5x}}$$

#3b. Find the general solution of the differential equation.
$y'' + 4y' - y = 0$

$m^2 + 4m - 1 = 0$

no integer factorizations, try quadratic formula

$m = \frac{-4 \pm \sqrt{16 - 4(1)(-1)}}{2(1)}$

$= \frac{-4 \pm \sqrt{20}}{2} = \frac{-4 \pm \sqrt{4}\sqrt{5}}{2} = \frac{-4 \pm 2\sqrt{5}}{2}$

$= -2 \pm \sqrt{5}$ (two distinct, real, solutions)

$$\boxed{y = C_1 e^{(-2+\sqrt{5})x} + C_2 e^{(-2-\sqrt{5})x}}$$

#4b. Find the general solution of the differential equation.
$3y'' + y = 0$

$3m^2 + 1 = 0$

$m^2 = \frac{-1}{3}$

$m = \pm\sqrt{\frac{-1}{3}} = \pm\sqrt{\frac{1}{3}}\sqrt{-1} = \pm\frac{\sqrt{1}}{\sqrt{3}} i = \pm\frac{1}{\sqrt{3}} i$

$m = \underset{\alpha}{0} \pm \underset{\beta}{\frac{1}{\sqrt{3}}} i$

$y = C_1 e^{0x} \cos(\frac{1}{\sqrt{3}}x) + C_2 e^{0x} \sin(\frac{1}{\sqrt{3}}x)$

$$\boxed{y = C_1 \cos(\frac{1}{\sqrt{3}}x) + C_2 \sin(\frac{1}{\sqrt{3}}x)}$$

#5b. Find the general solution of the differential equation.

$2y'' + 2y' + y = 0$

$2m^2 + 2m + 1 = 0$

$m = \frac{-2 \pm \sqrt{4 - 4(2)(1)}}{2(2)} = \frac{-2 \pm \sqrt{-4}}{4}$

$= \frac{-2 \pm \sqrt{4}\sqrt{-1}}{4} = \frac{-2 \pm 2i}{4} = -\frac{1}{2} \pm \frac{1}{2}i$

$$\boxed{y = C_1 e^{\left(-\frac{1}{2}x\right)} \cos\left(\tfrac{1}{2}x\right) + C_2 e^{\left(-\frac{1}{2}x\right)} \sin\left(\tfrac{1}{2}x\right)}$$

#6b. Find the general solution of the differential equation.

$2y'' - 3y' + 4y = 0$

$2m^2 - 3m + 4 = 0$

$m = \frac{3 \pm \sqrt{9 - 4(2)(4)}}{2(2)} = \frac{3 \pm \sqrt{-23}}{4}$

$= \frac{3 \pm \sqrt{23}\sqrt{-1}}{4} = \frac{3 \pm \sqrt{23}\,i}{4}$

$= \underset{\alpha}{\frac{3}{4}} \pm \underset{\beta}{\frac{\sqrt{23}}{4}}\, i$

$$\boxed{y = C_1 e^{\left(\frac{3}{4}x\right)} \cos\left(\tfrac{\sqrt{23}}{4}x\right) + C_2 e^{\left(\frac{3}{4}x\right)} \sin\left(\tfrac{\sqrt{23}}{4}x\right)}$$

#7b. Find the general solution of the differential equation.

$y''' + 3y'' - 4y' - 12y = 0$

$m^3 + 3m^2 - 4m - 12 = 0$

guess a root and test w/synthetic division:

try 1:
$$\begin{array}{r|rrrr} 1 & 1 & 3 & -4 & -12 \\ & & 1 & 4 & 0 \\ \hline & 1 & 4 & 0 & \boxed{-12} \neq 0 \end{array}$$
not 1

try 2:
$$\begin{array}{r|rrrr} 2 & 1 & 3 & -4 & -12 \\ & & 2 & 10 & 12 \\ \hline & 1 & 5 & 6 & \boxed{0} \end{array}$$
✓ yes!

$(m^2 + 5m + 6)$

so $m^3 + 3m^2 - 4m - 12 = 0$

is $(m-2)\underbrace{(m^2 + 5m + 6)}_{\text{factor further or quadratic eqn}} = 0$

$(m-2)(m+2)(m+3) = 0$

$m = 2,\ m = -2,\ m = -3$

3 distinct, real, roots

so

$$\boxed{y = C_1 e^{2x} + C_2 e^{-2x} + C_3 e^{-3x}}$$

#8b. Find the general solution of the differential equation.

$y''' - 6y'' + 12y' - 8y = 0$

$m^3 - 6m^2 + 12m - 8 = 0$

try 1 $\begin{array}{r|rrrr} & 1 & -6 & 12 & -8 \\ & & 1 & -5 & 7 \\ \hline & 1 & -5 & 7 & \boxed{-1 \neq 0} \end{array}$

try −1 $\begin{array}{r|rrrr} & 1 & -6 & 12 & -8 \\ & & -1 & 7 & -19 \\ \hline & 1 & -7 & 19 & \boxed{-27 \neq 0} \end{array}$

try 2 $\begin{array}{r|rrrr} & 1 & -6 & 12 & -8 \\ & & 2 & -8 & 8 \\ \hline & 1 & -4 & 4 & \boxed{0} \end{array}$ ✓

$(m-2)(m^2-4m+4) = 0$

$(m-2)(m-2)(m-2) = 0$

$m = 2$ multiplicity 3

so...

$$\boxed{y = C_1e^{2x} + C_2xe^{2x} + C_3x^2e^{2x}}$$

#9b. Solve the initial-value problem.

$4\frac{d^2y}{dt^2} - 4\frac{dy}{dt} - 3y = 0, \quad y(0) = 1, \; y'(0) = 5$

M	A
−12	−4
(2)(−6)	−4 ✓
(−2)(6)	4

$4m^2 - 4m - 3 = 0$

$\frac{(4m+2)}{2}\frac{(4m-6)}{2}$

$(2m+1)(2m-3) = 0$

$m = -1/2, \; m = 3/2$

$y = C_1e^{-1/2x} + C_2e^{3/2x}$ $\quad y' = -\frac{1}{2}C_1e^{-1/2x} + \frac{3}{2}C_2e^{3/2x}$

$y(0) = 1$ $\quad y'(0) = 5$

$1 = C_1e^0 + C_2e^0$ $\quad 5 = -\frac{1}{2}C_1e^0 + \frac{3}{2}C_2e^0$

system: $\begin{cases} C_1 + C_2 = 1 \\ -\frac{1}{2}C_1 + \frac{3}{2}C_2 = 5 \end{cases}$

$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ -1/2 & 3/2 & 5 \end{array}\right] \text{rref} \left[\begin{array}{cc|c} 1 & 0 & -7/4 \\ 0 & 1 & 11/4 \end{array}\right]$

$C_1 = -7/4, \; C_2 = 11/4$

so $\boxed{y = \left(-\frac{7}{4}\right)e^{-1/2x} + \left(\frac{11}{4}\right)e^{3/2x}}$

#10b. Solve the initial-value problem.

$y''-4y'+5y=0, \quad y(0)=2, \quad y'(0)=1$

$m^2-4m+5=0$

$$m=\frac{4\pm\sqrt{16-4(1)(5)}}{2(1)}=\frac{4\pm\sqrt{-4}}{2}=\frac{4\pm\sqrt{4}\sqrt{-1}}{2}=\frac{4\pm 2i}{2}=2\pm i$$

$y=C_1e^{2x}\cos(x)+C_2e^{2x}\sin(x)$

$y(0)=2$

$2=C_1e^{0}\cos(0)+C_2e^{0}\sin(0)$

$2=C_1+0 \rightarrow C_1=2$

$y=2e^{2x}\cos(x)+C_2e^{2x}\sin(x)$

$y'=(2e^{2x})(-\sin x)+\cos x(4e^{2x})+(C_2e^{2x})(\cos x)+\sin x(2C_2e^{2x})$

$y'(0)=1$

$1=2e^{0}(-1)\sin 0+\cos 0(4)e^{0}+C_2e^{0}\cos 0+\sin 0(2)C_2e^{0}$

$1=0+4+C_2+0 \rightarrow C_2=-3$

so $\boxed{y=2e^{2x}\cos(x)-3e^{2x}\sin(x)}$

#11b. Solve the initial-value problem.

$y''' + 2y'' - 5y' - 6y = 0, \quad y(0) = y'(0) = 0, \quad y''(0) = 1$

$m^3 + 2m^2 - 5m - 6 = 0$

try -1
$$\begin{array}{r|rrrr} -1 & 1 & 2 & -5 & -6 \\ & & -1 & -1 & 6 \\ \hline & 1 & 1 & -6 & \boxed{0} \ ✓ \end{array}$$

$(m+1)(m^2+m-6) = 0$

$(m+1)(m+3)(m-2) = 0$

$m = -1, \ m = -3, \ m = 2$

so $y = C_1 e^{-x} + C_2 e^{-3x} + C_3 e^{2x}$

since $y(0) = 0$

$0 = C_1 e^0 + C_2 e^0 + C_3 e^0 \rightarrow \underline{C_1 + C_2 + C_3 = 0}$

take derivatives...

$y' = -C_1 e^{-x} - 3C_2 e^{-3x} + 2C_3 e^{2x}$

$y'(0) = 0$

$0 = -C_1 e^0 - 3C_2 e^0 + 2C_3 e^0$

$0 = -C_1 - 3C_2 + 2C_3 \longrightarrow \underline{-C_1 - 3C_2 + 2C_3 = 0}$

$y'' = C_1 e^{-x} + 9C_2 e^{-3x} + 4C_3 e^{2x}$

$y''(0) = 1$

$1 = C_1 e^0 + 9C_2 e^0 + 4C_3 e^0$

$1 = C_1 + 9C_2 + 4C_3 \longrightarrow \underline{C_1 + 9C_2 + 4C_3 = 1}$

System:

$$\begin{cases} C_1 + C_2 + C_3 = 0 \\ -C_1 - 3C_2 + 2C_3 = 0 \\ C_1 + 9C_2 + 4C_3 = 1 \end{cases} \quad \left[\begin{array}{rrr|r} 1 & 1 & 1 & 0 \\ -1 & -3 & 2 & 0 \\ 1 & 9 & 4 & 1 \end{array}\right] \text{rref} \left[\begin{array}{rrr|r} 1 & 0 & 0 & -1/6 \\ 0 & 1 & 0 & 1/10 \\ 0 & 0 & 1 & 1/15 \end{array}\right]$$

$C_1 = -1/6, \ C_2 = 1/10, \ C_3 = 1/15$

So $\boxed{y = \left(-\frac{1}{6}\right)e^{-x} + \left(\frac{1}{10}\right)e^{-3x} + \left(\frac{1}{15}\right)e^{2x}}$

4.4

#1b. Solve the differential equation using the method of undermined coefficients.
$y''+9y=15$

y_c: $y''+9y=0$

$m^2+9=0$

$m^2=-9$

$m=\pm\sqrt{-9}=\pm 3i=0\pm 3i$

$y_c=C_1e^{0x}\cos(3x)+C_2e^{0x}\sin(3x)$

$y_c=C_1\cos(3x)+C_2\sin(3x)$

y_p: from table...

$y_p=A$ (no absorption)

$y'=0$

$y''=0$

into full DE:

$y''+9y=15$

$[0]+9[A]=15$

$A=\frac{15}{9}=\frac{3(5)}{3(3)}=\frac{5}{3}$

$y_p=\frac{5}{3}$

general solution:

$y=y_c+y_p$

$$\boxed{y=C_1\cos(3x)+C_2\sin(3x)+\frac{5}{3}}$$

#2b. Solve the differential equation using the method of undermined coefficients.
$y''+y'-6y=2x$

y_c: $y''+y'-6y=0$

$m^2+m-6=0$

$(m+3)(m-2)=0$

$m=-3$, $m=2$

$y_c=C_1e^{-3x}+C_2e^{2x}$

y_p: from table for $2x$

$y_p=Ax+B$ (no absorption)

$y'=A$

$y''=0$

into full DE:

$y''+y'-6y=2x$

$[0]+[A]-6[Ax+B]=2x$

$(-6A)x+(A-6B)=(2)x+(0)$

System: $\begin{cases}-6A=2\\ A-6B=0\end{cases}$ $\quad A=\frac{-2}{6}=\frac{-1}{3}$ $\swarrow$

$-\frac{1}{3}-6B=0$ $\quad 6B=\frac{-1}{3}$

$B=\frac{-1}{18}$

$y_p=-\frac{1}{3}x-\frac{1}{18}$

general solution:

$y=y_c+y_p$

$$\boxed{y=C_1e^{-3x}+C_2e^{2x}-\frac{1}{3}x-\frac{1}{18}}$$

#3b. Solve the differential equation using the method of undermined coefficients.

$4y'' - 4y' - 3y = \cos 2x$

$\underline{y_c}$: $4y'' - 4y' - 3y = 0$

$4m^2 - 4m - 3 = 0$

$\left(\frac{4m+2}{2}\right)\left(\frac{4m-6}{2}\right) = 0$

$(2m+1)(2m-3) = 0$

$m = -1/2 \quad m = 3/2$

$$\begin{array}{c|c} m & A \\ -12 & -4 \\ \hline (2)(-6) & \checkmark \end{array}$$

$y_c = C_1 e^{-1/2x} + C_2 e^{3/2x}$

$\underline{y_p}$: table for $\cos(2x)$

$y_p = A\cos(2x) + B\sin(2x)$ (no absorption)

$y' = -2A\sin(2x) + 2B\cos(2x)$

$y'' = -4A\cos(2x) - 4B\sin(2x)$

into DE; $4y'' - 4y' - 3y = \cos 2x$

$4[-4A\cos(2x) - 4B\sin(2x)]$
$-4\{-2A\sin(2x) + 2B\cos(2x)]$
$-3[A\cos(2x) + B\sin(2x)] = \cos(2x)$

$(-16A - 8B - 3A)\cos(2x)$
$+(-16B + 8A - 3B)\sin(2x) = \cos(2x)$

System: $\begin{cases} -19A - 8B = 1 \\ 8A - 19B = 0 \end{cases}$

$\left[\begin{array}{cc|c} -19 & -8 & 1 \\ 8 & -19 & 0 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{cc|c} 1 & 0 & -19/425 \\ 0 & 1 & -8/425 \end{array}\right]$

$A = -19/425, \quad B = -8/425$

$y_p = \left(\frac{-19}{425}\right)\cos(2x) + \left(\frac{-8}{425}\right)\sin(2x)$

general solution:

$y = y_c + y_p$

$$\boxed{y = C_1 e^{-1/2x} + C_2 e^{3/2x} - \frac{19}{425}\cos(2x) - \frac{8}{425}\sin(2x)}$$

#4b. Solve the differential equation using the method of undermined coefficients.

$y'' - 8y' + 20y = 100x^2 - 26xe^x$

$\underline{y_c}$: $y'' - 8y' + 20y = 0$

$m^2 - 8m + 20 = 0 \qquad m = \frac{8 \pm \sqrt{64 - 4(1)(20)}}{2(1)}$

$m = \frac{8 \pm \sqrt{-16}}{2} = \frac{8 \pm 4i}{2} = 4 \pm 2i$

$y_c = C_1 e^{4x}\cos(2x) + C_2 e^{4x}\sin(2x)$

$\underline{y_p}$: from table for $100x^2 - 26xe^x$:

$y_p = Ax^2 + Bx + C + (Dx + E)e^x$ (no absorption)

$y' = 2Ax + B + (Dx + E)e^x + e^x(D)$

$y'' = 2A + (Dx + E)e^x + e^x(D) + De^x$

into DE… $y'' - 8y' + 20y = 100x^2 - 26xe^x$

$[\underline{2A} + \underline{Dxe^x} + \underline{Ee^x} + \underline{De^x} + \underline{De^x}]$
$-8[\underline{2Ax} + \underline{B} + \underline{Dxe^x} + \underline{Ee^x} + \underline{De^x}]$
$+20[\underline{Ax^2} + \underline{Bx} + \underline{C} + \underline{Dxe^x} + \underline{Ee^x}] = 100x^2 - 26xe^x$

$(D - 8D + 20D)xe^x + (E + 2D - 8E - 8D + 20E)e^x$
$+ (20A)x^2 + (-16A + 20B)x + (2A - 8B + 20C) = 100x^2$
$- 26xe^x$

System:

$\begin{cases} 13D = -26 \\ 13E - 6D = 0 \\ 20A = 100 \\ -16A + 20B = 0 \\ 2A - 8B + 20C = 0 \end{cases}$

$\left[\begin{array}{ccccc|c} 0 & 0 & 0 & 13 & 0 & -26 \\ 0 & 0 & 0 & -6 & 13 & 0 \\ 20 & 0 & 0 & 0 & 0 & 100 \\ -16 & 20 & 0 & 0 & 0 & 0 \\ 2 & -8 & 20 & 0 & 0 & 0 \end{array}\right]$

rref

$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & 0 & 11/10 \\ 0 & 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 & 1 & -12/13 \end{array}\right] \begin{array}{l} = A \\ = B \\ = C \\ = D \\ = E \end{array}$

$y_p = 5x^2 + 4x + \frac{11}{10} + \left(-2x - \frac{12}{13}\right)e^x$

general solution:

$y = y_c + y_p$

$$\boxed{\begin{array}{l} y = C_1 e^{4x}\cos(2x) + C_2 e^{4x}\sin(2x) \\ \quad + 5x^2 + 4x + \frac{11}{10} - 2xe^x - \frac{12}{13}e^x \end{array}}$$

#5b. Solve the differential equation using the method of undermined coefficients.

$y'' + 4y = 3\sin 2x$

$\underline{y_c}$: $y'' + 4y = 0$

$m^2 + 4 = 0$

$m^2 = -4$

$m = \pm\sqrt{-4} = \pm 2i = 0 \pm 2i$

$y_c = C_1 e^{0x}\cos(2x) + C_2 e^{0x}\sin(2x)$

$y_c = C_1\cos(2x) + C_2\sin(2x)$

$\underline{y_p}$; table for $3\sin 2x$

$y_p = A\cos(2x) + B\sin(2x)$ (absorbed ... match terms in y_c)

So $y_p = \underline{Ax\cos(2x)} + \underline{Bx\sin(2x)}$

$y' = Ax(-2\sin(2x)) + \cos(2x)A + Bx(2\cos(2x)) + \sin(2x)B$

$y'' = Ax(-4\cos(2x)) + (-2\sin(2x))A - 2A\sin(2x) + Bx(-4\sin(2x)) + (2\cos(2x))B + 2B\cos(2x)$

into DE... $y'' + 4y = 3\sin(2x)$

$[-4Ax\cos 2x - 4A\sin 2x - 4Bx\sin 2x + 4B\cos 2x] + 4[Ax\cos 2x + Bx\sin 2x] = 3\sin 2x$

$\underset{0}{(-4A \cancel{+4A})}x\cos 2x + \underset{0}{(-4B \cancel{+4B})}x\sin 2x + (-4A)\sin 2x + (4B)\cos 2x = (3)\sin 2x + (0)\cos 2x$

system: $\begin{cases} -4A = 3 \\ 4B = 0 \end{cases}$ $A = -\frac{3}{4}$, $B = 0$

$y_p = -\frac{3}{4}x\cos(2x) + (0)x\sin(2x)$

$y_p = -\frac{3}{4}x\cos(2x)$

general solution: $y = y_c + y_p$

$$\boxed{y = C_1\cos(2x) + C_2\sin(2x) - \frac{3}{4}x\cos(2x)}$$

#6b. Solve the differential equation using the method of undermined coefficients.

$y''-2y'+5y=e^x\cos 2x$

y_c: $y''-2y'+5y=0$ $\quad m=\frac{2\pm\sqrt{4-4(1)(5)}}{2(1)}=\frac{2\pm\sqrt{-16}}{2}=\frac{2\pm 4i}{2}=1\pm 2i$

$m^2-2m+5=0$

$y_c=C_1e^x\cos(2x)+C_2e^x\sin(2x)$

y_p: from table, for $e^x\cos 2x$: $y_p=Ae^x\cos 2x+Be^x\sin 2x$ (both terms absorbed)

so $y_p=Axe^x\cos 2x+Bxe^x\sin 2x$ [NOTE! on #6 you can fix absorption by just dropping absorbed terms

$y_p=xe^x(A\cos 2x+B\sin 2x)$ deriv. easier when factored

$y'=xe^x(-2A\sin 2x+2B\cos 2x)+(A\cos 2x+B\sin 2x)[xe^x+e^x(1)]$

$y''=xe^x(-4A\cos 2x-4B\sin 2x)+(-2A\sin 2x+2B\cos 2x)(xe^x+e^x)+(xe^x+e^x)(-2A\sin 2x+2B\cos 2x)$

$+(A\cos 2x+B\sin 2x)(xe^x+2e^x)$

Into full DE... $y''-2y'+5y=e^x\cos 2x$

$[-4Axe^x\cos 2x-4Bxe^x\sin 2x-2Axe^x\sin 2x-2Ae^x\sin 2x+2Bxe^x\cos 2x+2Be^x\cos 2x$

$+Axe^x\cos 2x+2Ae^x\cos 2x+Bxe^x\sin 2x+2Be^x\sin 2x-2Axe^x\sin 2x+2Bxe^x\cos 2x-2Ae^x\sin 2x+2Be^x\cos 2x]$

$-2[-2Axe^x\sin 2x+2Bxe^x\cos 2x+Axe^x\cos 2x+Ae^x\cos 2x+Bxe^x\sin 2x+Be^x\sin 2x]$

$+5[Axe^x\cos 2x+Bxe^x\sin 2x]=e^x\cos 2x$

$(-4A+2B+A\cancel{+2B-4B}\overset{0}{-2A}+5A)xe^x\cos 2x+(-4B-2A+B\cancel{-2A}\overset{0}{+4A}-2B+5B)xe^x\sin 2x$

$+(-2A+2B-2A-2B)e^x\sin 2x+(2B+2A+2B-2A)e^x\cos 2x=e^x\cos 2x$

system: $\begin{cases}-4A=0\\4B=1\end{cases}$ $\quad A=0,\ B=1/4$

$y_p=xe^x(0\cos 2x+\frac{1}{4}\sin 2x)=\frac{1}{4}xe^x\sin 2x$

general solution:

$y=y_c+y_p$

$$\boxed{y=C_1e^x\cos(2x)+C_2e^x\sin(2x)+\frac{1}{4}xe^x\sin(2x)}$$

#7b. Solve the initial-value problem.

$y''+4y'+5y=35e^{-4x}, \quad y(0)=-3, \quad y'(0)=1$

$\underline{y_c}$: $y''+4y'+5y=0$ $\quad m=\frac{-4\pm\sqrt{16-4(1)(5)}}{2(1)}=\frac{-4\pm\sqrt{-4}}{2}=\frac{-4\pm 2i}{2}=-2\pm i$

$m^2+4m+5=0$

$y_c=C_1e^{-2x}\cos x+C_2e^{-2x}\sin x$

$\underline{y_p}$: from table for $35e^{-4x}$: $y_p=Ae^{-4x}$ (no absorption)

$y'=-4Ae^{-4x}$

$y''=16Ae^{-4x}$

into full DE... $y''+4y'+5y=35e^{-4x}$

$[16Ae^{-4x}]+4[-4Ae^{-4x}]+5[Ae^{-4x}]=35e^{-4x}$

$[16A-16A+5A)e^{-4x}=35e^{-4x}$, $5A=35$, $A=7$

so $y_p=7e^{-4x}$

general solution: $y=y_c+y_p$,, $y=C_1e^{-2x}\cos x+C_2e^{-2x}\sin x+7e^{-4x}$

particular solution: use $y(0)=-3$ and $y'(0)=1$

$y=C_1e^{-2x}\cos x+C_2e^{-2x}\sin x+7e^{-4x}$

$-3=C_1e^0\cos 0+C_2e^0\sin 0+7e^0$

$-3=C_1+0+7$

$C_1=-10$, so $y=\underline{-10e^{-2x}}\,\underline{\cos x}+\underline{C_2e^{-2x}}\,\underline{\sin x}+7e^{-4x}$

$y'=(-10e^{-2x})(-\sin x)+\cos x\,(20e^{-2x})+C_2e^{-2x}\cos x+\sin x(-2C_2e^{-2x})-28e^{-4x}$

$1=-10e^0(-1)\sin 0+\cos 0\,(20)e^0+C_2e^0\cos 0+\sin 0(-2)C_2e^0-28e^0$

$1=\quad 0+20+C_2+0-28$, $C_2=9$

$1=$

particular solution: $\boxed{y=-10e^{-2x}\cos x+9e^{-2x}\sin x+7e^{-4x}}$

4.6 (we skip 4.5)

#1b. Solve the differential equation using the method of variation of parameters.
$y'' + y = \csc x$

$\underline{y_c}$: $y''+y=0 \quad m=\pm\sqrt{-1}=0\pm i$

$m^2+1=0 \quad y_c=C_1\cos x+C_2\sin x$

$\underline{y_p}$: $y_p=u_1\cos x+u_2\sin x$

$$u_1'=\frac{\begin{vmatrix}0 & \sin x\\ \csc x & \cos x\end{vmatrix}}{\begin{vmatrix}\cos x & \sin x\\ -\sin x & \cos x\end{vmatrix}}=\frac{0-\sin x\csc x}{\cos^2x+\sin^2x}=\frac{1}{1}=1$$

$$u_1=\int 1dx=x$$

$$u_2'=\frac{\begin{vmatrix}\cos x & 0\\ -\sin x & \csc x\end{vmatrix}}{\begin{vmatrix}\cos x & \sin x\\ -\sin x & \cos x\end{vmatrix}}=\frac{\cos x\csc x}{1}=\frac{\cos x}{\sin x}$$

$$u_2=\int\frac{\cos x}{\sin x}dx \qquad \begin{matrix}u=\sin x\\ du=\cos x dx\end{matrix}$$

$$=\int\frac{1}{u}du$$

$$=\ln|u|=\ln|\sin x|$$

$$y_p=x\cos x+\ln|\sin x|\sin x$$

general solution:

$$\boxed{y=C_1\cos x+C_2\sin x+x\cos x+\ln|\sin x|\sin x}$$

#2b. Solve the differential equation using the method of variation of parameters.
$y'' + y = \sec\theta\tan\theta$

$\underline{y_c}$: $y''+y=0 \quad m=\pm\sqrt{-1}=0\pm i$

$m^2+1=0 \quad y_c=C_1\cos\theta+C_2\sin\theta$

$\underline{y_p}$: $y_p=u_1\cos\theta+u_2\sin\theta$

$$u_1'=\frac{\begin{vmatrix}0 & \sin\theta\\ \sec\theta\tan\theta & \cos\theta\end{vmatrix}}{\begin{vmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{vmatrix}}=\frac{0-\sin\theta\sec\theta\tan\theta}{\cos^2\theta+\sin^2\theta}=\frac{\left(-\frac{\sin^2\theta}{\cos^2\theta}\right)}{1}$$

$$u_1=-\int\frac{\sin^2\theta}{\cos^2\theta}d\theta=-\int\frac{1-\cos^2\theta}{\cos^2\theta}d\theta=-\int\left(\frac{1}{\cos^2\theta}-1\right)d\theta$$

$$=-\int\sec^2\theta d\theta+\int 1d\theta=-\tan\theta+\theta$$

$$u_2'=\frac{\begin{vmatrix}\cos\theta & 0\\ -\sin\theta & \sec\theta\tan\theta\end{vmatrix}}{\begin{vmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{vmatrix}}=\frac{\cos\theta\sec\theta\tan\theta}{1}=\tan\theta$$

$$u_2=\int\tan\theta d\theta=\int\frac{\sin\theta}{\cos\theta}d\theta \qquad \begin{matrix}u=\cos\theta\\ du=-\sin\theta d\theta\end{matrix}$$

$$=-\int\frac{1}{u}du=-\ln|u|$$

$$=-\ln|\cos\theta|=\ln|(\cos\theta)^{-1}|=\ln|\sec\theta$$

$$y_p=(-\tan\theta+\theta)\cos\theta+\ln|\sec\theta|\sin\theta$$

general solution:

$$\boxed{y=C_1\cos\theta+C_2\sin\theta-\tan\theta\cos\theta+\theta\cos\theta+\ln|\sec\theta|\sin\theta}$$

#3b. Solve the differential equation using the method of variation of parameters.

$$y'' - 4y = \frac{e^{2x}}{x}$$

y_c: $y'' - 4y = 0$ $\quad m = \pm\sqrt{4} = \pm 2$ $\quad y_c = C_1 e^{2x} + C_2 e^{-2x}$

$m^2 - 4 = 0$

y_p: $y_p = u_1 e^{2x} + u_2 e^{-2x}$

$$u_1' = \frac{\begin{vmatrix} 0 & e^{-2x} \\ \frac{e^{2x}}{x} & -2e^{-2x} \end{vmatrix}}{\begin{vmatrix} e^{2x} & e^{-2x} \\ 2e^{2x} & -2e^{-2x} \end{vmatrix}} = \frac{0 - \frac{e^{2x}}{x} e^{-2x}}{-2e^{2x}e^{-2x} - 2e^{2x}e^{-2x}} = \frac{0 - \frac{e^0}{x}}{-2e^0 - 2e^0} = \frac{-\frac{1}{x}}{-4} = \frac{1}{4}\frac{1}{x}$$

$$u_1 = \frac{1}{4}\int \frac{1}{x}dx = \frac{1}{4}\ln|x|$$

$$u_2' = \frac{\begin{vmatrix} e^{2x} & 0 \\ 2e^{2x} & \frac{e^{2x}}{x} \end{vmatrix}}{\begin{vmatrix} e^{2x} & e^{-2x} \\ 2e^{2x} & -2e^{-2x} \end{vmatrix}} = \frac{e^{2x}\frac{e^{2x}}{x} - 0}{-4} = -\frac{1}{4}\frac{1}{x}e^{4x}$$

there is no straight forward way to integrate this. So we use Fundamental Theorem of Calc to express answer as an integral:

$$u_2 = -\frac{1}{4}\int \frac{1}{x}e^{4x}dx$$

$$u_2 = -\frac{1}{4}\int_{x_0}^{x} \frac{1}{t}e^{4t}dt$$

$$y_p = \left(\frac{1}{4}\ln|x|\right)e^{2x} + \left(-\frac{1}{4}\int_{x_0}^{x}\frac{1}{t}e^{4t}dt\right)e^{-2x}$$

general solution:

$$\boxed{y = C_1 e^{2x} + C_2 e^{-2x} + \frac{1}{4}e^{2x}\ln|x| - \frac{1}{4}e^{-2x}\int_{x_0}^{x}\frac{1}{t}e^{4t}dt}$$

#4b. Solve the differential equation using the method of variation of parameters.

$$y''-2y'+y=\frac{e^x}{1+x^2}$$

y_c: $y''-2y'+y=0$ $\quad$ $m=1$ repeated so $y_c = C_1e^x+C_2xe^x$

$m^2-2m+1=0$

$(m-1)(m-1)$

y_p: $y_p = u_1e^x+u_2xe^x$

$$u_1' = \frac{\begin{vmatrix} 0 & xe^x \\ \frac{e^x}{1+x^2} & xe^x+e^x \end{vmatrix}}{\begin{vmatrix} e^x & xe^x \\ e^x & xe^x+e^x \end{vmatrix}} = \frac{0-\frac{xe^xe^x}{1+x^2}}{e^x(xe^x+e^x)-e^xxe^x} = \frac{\frac{-xe^{2x}}{1+x^2}}{xe^{2x}+e^{2x}-xe^{2x}} = \frac{\frac{-xe^{2x}}{1+x^2}}{e^{2x}} = \frac{-x}{1+x^2}$$

$$u_1 = -\int\frac{x}{1+x^2}dx \quad \begin{matrix} u=1+x^2 \\ du=2xdx \end{matrix} \quad u_1=-\frac{1}{2}\int\frac{1}{u}du=-\frac{1}{2}\ln|u|=-\frac{1}{2}\ln|1+x^2|$$

$$u_2' = \frac{\begin{vmatrix} e^x & 0 \\ e^x & \frac{e^x}{1+x^2} \end{vmatrix}}{\begin{vmatrix} e^x & xe^x \\ e^x & xe^x+e^x \end{vmatrix}} = \frac{\frac{e^xe^x}{1+x^2}-0}{e^{2x}} = \frac{\left(\frac{e^{2x}}{1+x^2}\right)}{e^{2x}} = \frac{1}{1+x^2}$$

$$u_2=\int\frac{1}{1+x^2}dx = \arctan(x)$$

$$y_p=\left(-\frac{1}{2}\ln|1+x^2|\right)e^x+(\arctan(x))xe^x = -\frac{1}{2}e^x\ln|1+x^2|+xe^x\arctan(x)$$

general solution:

$$\boxed{y=C_1e^x+C_2xe^x-\frac{1}{2}e^x\ln|1+x^2|+xe^x\arctan(x)}$$

#5b. Solve the initial value problem using variation of parameters.

$2y'' + y' - y = x + 1 \longrightarrow y'' + \frac{1}{2}y' - \frac{1}{2}y = \frac{x+1}{2} \Big\}$ RHS for y_p part

$\underline{y_c}$: $2y'' + y' - y = 0$

$$2m^2 + m - 1 = 0 \qquad \begin{array}{c|c} m & A \\ \hline -2 & 1 \\ (2)(-1) & \end{array} \qquad m = 1/2,\ m = -1$$

$$(\underset{1}{\underline{2m-1}})(\underset{2}{\underline{2m+2}}) \qquad y_c = C_1 e^{1/2x} + C_2 e^{-x}$$

$$(2m-1)(m+1)$$

$\underline{y_p}$: $y_p = u_1 e^{1/2x} + u_2 e^{-x}$

$$u_1' = \frac{\begin{vmatrix} 0 & e^{-x} \\ \frac{x+1}{2} & -e^{-x} \end{vmatrix}}{\begin{vmatrix} e^{1/2x} & e^{-x} \\ \frac{1}{2}e^{1/2x} & -e^{-x} \end{vmatrix}} = \frac{0 - e^{-x}\left(\frac{x+1}{2}\right)}{-e^{1/2x}e^{-x} - \frac{1}{2}e^{1/2x}e^{-x}} = \frac{-\frac{1}{2}e^{-x}(x+1)}{-\frac{3}{2}e^{-1/2x}} = \frac{1}{3}e^{-1/2x}(x+1) = \frac{1}{3}xe^{-1/2x} + \frac{1}{3}e^{-1/2x}$$

$$u_1 = \frac{1}{3}\underbrace{\int xe^{-1/2x}dx}_{\text{by parts}} + \frac{1}{3}\int e^{-1/2x}dx$$

$u = x \quad dv = e^{-1/2x}dx \qquad + \frac{1}{3}(-2e^{-1/2x})$

$\frac{du}{dx} = 1 \quad \int dv = \int e^{-1/2x}dx$

$du = dx \quad v = -2e^{-1/2x}$

$\frac{1}{3}\left[uv - \int v\,du\right]$

$\frac{1}{3}\left[-2xe^{-1/2x} - \int(-2e^{-1/2x}dx\right]$

$\frac{1}{3}\left[-2xe^{-1/2x} - 4e^{-1/2x}\right]$

$$u_1 = -\frac{2}{3}xe^{-1/2x} - \frac{4}{3}e^{-1/2x} - \frac{2}{3}e^{-1/2x} = -\frac{2}{3}xe^{-1/2x} - 2e^{-1/2x}$$

$$u_2' = \frac{\begin{vmatrix} e^{1/2x} & 0 \\ \frac{1}{2}e^{1/2x} & \frac{x+1}{2} \end{vmatrix}}{\begin{vmatrix} e^{1/2x} & e^{-x} \\ \frac{1}{2}e^{1/2x} & -e^{-x} \end{vmatrix}} = \frac{e^{1/2x}\left(\frac{x+1}{2}\right) - 0}{-\frac{3}{2}e^{-1/2x}} = \frac{\frac{1}{2}e^{1/2x}(x+1)}{-\frac{3}{2}e^{-1/2x}} = -\frac{1}{3}e^{x}(x+1) = -\frac{1}{3}xe^{x} - \frac{1}{3}e^{x}$$

Continued...

4.6 #5b continued...

$$u_2 = -\frac{1}{3}\underbrace{\int xe^x\,dx}_{\text{by parts}} - \frac{1}{3}\int e^x\,dx$$

$$u = x \quad dv = e^x dx \qquad -\frac{1}{3}e^x$$

$$\frac{du}{dx} = 1 \quad \int dv = \int e^x dx$$

$$du = dx \quad v = e^x$$

$$-\frac{1}{3}\left[uv - \int v\,du\right]$$

$$-\frac{1}{3}\left[xe^x - \int e^x dx\right]$$

$$-\frac{1}{3}\left[xe^x - e^x\right]$$

$$u_2 = -\frac{1}{3}xe^x + \frac{1}{3}e^x - \frac{1}{3}e^x = -\frac{1}{3}xe^x$$

$$y_p = \left(-\frac{2}{3}xe^{-1/2x} - 2e^{-1/2x}\right)e^{1/2x} + \left(-\frac{1}{3}xe^x\right)e^{-x}$$

$$= -\frac{2}{3}x - 2 - \frac{1}{3}x = -x - 2$$

general solution:

$$\boxed{y = c_1e^{1/2x} + c_2e^{-x} - x - 2}$$

4.7

#1b. Solve the differential equation.

$x(xy''-3y')=(0)x$

$x^2y''-3xy'=0$ now, Cauchy-Euler form

$am^2+(b-a)m+c=0$

$1m^2+(-3-1)m+0=0$

$m^2-4m=0$

$m(m-4)=0$

$m=0,\ m=4$

$y=C_1x^0+C_2x^4$

$$\boxed{y=C_1+C_2x^4}$$

#2b. Solve the differential equation.

$4x^2y''+4xy'-y=0$

$am^2+(b-a)m+c=0$

$4m^2+(4-4)m+(-1)=0$

$4m^2-1=0$

$m^2=\frac{1}{4}$

$m=\pm\sqrt{\frac{1}{4}}=\pm\frac{1}{2}$

$$\boxed{y=C_1x^{1/2}+C_2x^{-1/2}}$$

#3b. Solve the differential equation.

$x^2y''+3xy'-4y=0$

$am^2+(b-a)m+c=0$

$1m^2+(3-1)m+(-4)=0$

$m^2+2m-4=0$

$m=\frac{-2\pm\sqrt{4+4(1)(4)}}{2(1)}=\frac{-2\pm\sqrt{20}}{2}$

$m=\frac{-2\pm\sqrt{4}\sqrt{5}}{2}=\frac{-2\pm2\sqrt{5}}{2}=-1\pm\sqrt{5}$

$$\boxed{y=C_1x^{(-1+\sqrt{5})}+C_2x^{(-1-\sqrt{5})}}$$

#4b. Solve the differential equation by variation of parameters.

$2x^2y'' + 5xy' + y = x^2 - x$

$$\begin{array}{c|c} M & A \\ 2 & 3 \end{array}$$

$$\frac{(2m+1)}{1}\frac{(2m+2)}{2} \quad (1,\tfrac{1}{2})$$

$\underline{y_c}$: $2x^2y''+5xy'+y=0$

$am^2+(b-a)m+c=0 \quad (2m+1)(m+1)$

$2m^2+(5-2)m+1=0 \quad m=-1/2 \quad m=-1$

$2m^2+3m+1=0 \quad y_c=c_1x^{-1/2}+c_2x^{-1}$

$\underline{y_p}$: RHS: $\frac{x^2-x}{2x^2} \quad y_p=u_1x^{-1/2}+u_2x^{-1}$

$$u_1'=\frac{\begin{vmatrix} 0 & x^{-1} \\ \frac{x^2-x}{2x^2} & -x^{-2} \end{vmatrix}}{\begin{vmatrix} x^{-1/2} & x^{-1} \\ -\frac{1}{2}x^{-3/2} & -x^{-2} \end{vmatrix}}=\frac{0-x^{-1}\frac{x^2-x}{2x^2}}{-x^{-1/2}x^{-2}+\frac{1}{2}x^{-3/2}x^{-1}}$$

$$=\frac{\left(\frac{-x^2+x}{2x^3}\right)}{-x^{-5/2}+\frac{1}{2}x^{-5/2}}$$

$$u_1'=\frac{\left(\frac{x-x^2}{2x^3}\right)}{-\frac{1}{2}x^{-5/2}}=\frac{-2x^{5/2}(-x^2+x)}{2x^3}$$

$$=x^{3/2}-x^{1/2}$$

$$u_1=\int(x^{3/2}-x^{1/2})dx=\frac{2}{5}x^{5/2}-\frac{2}{3}x^{3/2}$$

$$u_2'=\frac{\begin{vmatrix} x^{-1/2} & 0 \\ -\frac{1}{2}x^{-1/2} & \frac{x^2-x}{2x^2} \end{vmatrix}}{\begin{vmatrix} x^{-1/2} & x^{-1} \\ -\frac{1}{2}x^{-1/2} & -x^{-2} \end{vmatrix}}=\frac{x^{-1/2}\frac{x^2-x}{2x^2}-0}{-\frac{1}{2}x^{-5/2}}$$

$$=-\frac{2x^{5/2}x^{-1/2}(x^2-x)}{2\ x^2}=-x^2+x$$

$$u_2=\int(-x^2+x)dx=-\frac{1}{3}x^3+\frac{1}{2}x^2$$

$$y_p=\left(\frac{2}{5}x^{5/2}-\frac{2}{3}x^{3/2}\right)x^{-1/2}+\left(-\frac{1}{3}x^3+\frac{1}{2}x^2\right)x^{-1}$$

$$=\frac{2}{5}x^2-\frac{2}{3}x-\frac{1}{3}x^2+\frac{1}{2}x$$

$$=\frac{1}{15}x^2-\frac{1}{6}x$$

general solution:

$$\boxed{y=c_1x^{-1/2}+c_2x^{-1}+\frac{1}{15}x^2-\frac{1}{6}x}$$

#5b. Use the substitution $x = e^t$ to transform the given Cauchy-Euler equation to a differential equation with constant coefficients, then solve.

$x^2y'' - 9xy' + 25y = 0$

$x=e^t, \quad t=\ln x$

$$y'=\frac{1}{x}\frac{dy}{dt}, \quad y''=\frac{1}{x^2}\left(\frac{d^2y}{dt^2}-\frac{dy}{dt}\right)$$

into DE...

$$x^2\left[\frac{1}{x^2}\left(\frac{d^2y}{dt^2}-\frac{dy}{dt}\right)\right]-9x\left[\frac{1}{x}\frac{dy}{dt}\right]+25y=0$$

$$\frac{d^2y}{dt^2}-\frac{dy}{dt}-9\frac{dy}{dt}+25y=0$$

$$\frac{d^2y}{dt^2}-10\frac{dy}{dt}+25y=0$$

$m^2-10m+25=0$

$(m-5)(m-5)=0$

$m=5$ repeated

$y=C_1e^{5t}+C_2te^{5t}$

resubstitute back to x:

$y=C_1e^{5\ln x}+C_2\ln x\, e^{5\ln x}$

$y=C_1e^{\ln(x^5)}+C_2\ln x\, e^{\ln(x^5)}$

$y=C_1x^5+C_2\ln x\, x^5$

$$\boxed{y=C_1x^5+C_2x^5\ln x}$$

#6b. Use the substitution $x = e^t$ to transform the given Cauchy-Euler equation to a differential equation with constant coefficients, then solve.

$x^2y'' - 4xy' + 6y = \ln\left(x^2\right)$

$\underline{y_c}$: $x^2y''-4xy'+6y=0$ substitute: $x=e^t,\ t=\ln x,\ y'=\frac{1}{x}\frac{dy}{dt},\ y''=\frac{1}{x^2}\left(\frac{d^2y}{dt}-\frac{dy}{dt}\right)$

$$x^2\left[\frac{1}{x^2}\left(\frac{d^2y}{dt^2}-\frac{dy}{dt}\right)\right]-4x\left[\frac{1}{x}\frac{dy}{dt}\right]+6y=0$$

$$\frac{d^2y}{dt^2}-\frac{dy}{dt}-4\frac{dy}{dt}+6y=0$$

$$\frac{d^2y}{dt^2}-5\frac{dy}{dt}+6y=0$$

$m^2-5m+6=0$

$(m-2)(m-3)=0,\ m=2, m=3$

$y_c = C_1e^{2t}+C_2e^{3t}$

resubstituting...

$y_c = C_1e^{2\ln x}+C_2e^{3\ln x} = C_1e^{\ln(x^2)}+C_2e^{\ln(x^3)}$

$y_c = C_1x^2+C_2x^3$

$\underline{y_p}$: $\ln(x^2)$ isn't in table so we must use the 'Wronskian method':

$y_p = u_1x^2+u_2x^3$ RHS: $=\frac{\ln(x^2)}{x^2}$

$$u_1' = \frac{\begin{vmatrix} 0 & x^3 \\ \frac{\ln(x^2)}{x^2} & 3x^2 \end{vmatrix}}{\begin{vmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{vmatrix}} = \frac{0-\frac{\ln(x^2)}{x^2}x^3}{3x^4-2x^4} = \frac{-x\ln(x^2)}{x^4} = \frac{-\ln(x^2)}{x^3}$$

$u_1 = -\int\frac{\ln(x^2)}{x^3}$ yikes! OKAY, plan B... let's go back to the DE but keep the substitution... stay in 't':

$$x^2y''-4xy'+6y=\ln(x^2)$$

$$x^2\left[\frac{1}{x^2}\left(\frac{d^2y}{dt^2}-\frac{dy}{dt}\right)\right]-4x\left[\frac{1}{x}\frac{dy}{dt}\right]+6y=\ln((e^t)^2)$$

$$\frac{d^2y}{dt^2}'-\frac{dy}{dt}-4\frac{dy}{dt}+6y=\ln(e^{2t})=2t$$

$$\frac{d^2y}{dt^2}-5\frac{dy}{dt}+6y=2t$$

Continued...

4.7 #6b continued...

$$\frac{d^2y}{dt^2} - 5\frac{dy}{dt} + 6y = 2t$$ now we could use "table method";

$y_p = At+B$

$y' = A$

$y'' = 0$

into full DE... $[0] - 5[A] + 6[At+B] = 2t$

$(6A)t + (-5A+6B) = (2)t + (0)$

system; $\begin{cases} 6A = 2 \\ -5A+6B = 0 \end{cases}$ $A = \frac{1}{3}$, $-5(\frac{1}{3}) + 6B = 0$, $6B = \frac{5}{3}$, $B = \frac{5}{18}$

so $y_p = \frac{1}{3}t + \frac{5}{18}$

now resubstitute; $y_p = \frac{1}{3}\ln x + \frac{5}{18}$

general solution: $y = y_c + y_p$

$$\boxed{y = C_1 x^2 + C_2 x^3 + \frac{1}{3}\ln x + \frac{5}{18}}$$

DiffEq Ch4 Test Review

#1. Given one solution to the differential equation (no need to verify), use reduction of order to find a second solution.

$y'' - 25y = 0; \quad y_1 = e^{5x}$

$y_2 = \underline{u}\,\underline{e}^{5x}$

$y' = \underline{u}\,\underline{(5e^{5x})} + \underline{e}^{5x}\underline{u'}$

$y'' = u(25e^{5x}) + 5e^{5x}u' + e^{5x}u'' + u'(5e^{5x})$

$[25e^{5x}u + 10e^{5x}u' + e^{5x}u''] - 25[e^{5x}u] = 0$

$(e^{5x})u'' + (10e^{5x})u' + (\underset{0}{25e^{5x} \cancel{- 25e^{5x}}})u = 0$

let $w = u'$

$\frac{e^{5x}w'}{e^{5x}} + \frac{10e^{5x}w}{e^{5x}} = \frac{0}{e^{5x}}$

$w' + 10w = 0$ linear, w/ $P(x) = 10$

I.F. $= e^{\int 10dx} = e^{10x}$

$\frac{e^{10x}w}{e^{10x}} = \int 0e^{10x}dx = \int 0dx = \frac{c_1}{e^{10x}}$

$w = c_1e^{-10x}$

$u' = c_1e^{-10x}$

$u = c_1\int e^{-10x}dx = c_1\left(-\frac{1}{10}e^{-10x}\right) + c_2$

simplest form if $-\frac{c_1}{10} = 1$, $c_2 = 0$

$u = e^{-10x}$

then $y_2 = ue^{5x} = e^{-10x}e^{5x}$

$\boxed{y_2 = e^{-5x}}$

#2. Given one solution to the differential equation (no need to verify), use reduction of order to find a second solution.

$9y'' - 12y' + 4y = 0; \quad y_1 = e^{\left(\frac{2}{3}x\right)}$

$y_2 = \underline{u}\,\underline{e}^{2/3x}$

$y' = \underline{u\left(\frac{2}{3}e^{2/3x}\right)} + \underline{e}^{2/3x}\underline{u'}$

$y'' = u\left(\frac{4}{9}e^{2/3x}\right) + \frac{2}{3}e^{2/3x}u' + e^{2/3x}u'' + u'\left(\frac{2}{3}e^{2/3x}\right.$

$9\left[\frac{4}{9}e^{2/3x}u + \frac{4}{3}e^{2/3x}u' + e^{2/3x}u''\right]$

$-12\left[\frac{2}{3}e^{2/3x}u + e^{2/3x}u'\right] + 4\left[e^{2/3x}u\right] = 0$

$(9e^{2/3x})u'' + (\underset{0}{12e^{2/3x} \cancel{- 12e^{2/3x}}})u' + (\underset{0}{4e^{2/3x} - 8e^{2/3x} + 4e^{2/3x}})u$

Let $w = u'$

$\frac{9e^{2/3x}w'}{9e^{2/3x}} = \frac{0}{9e^{2/3x}}$

$w' = 0$

$w = \int 0dx = c_1$

$u' = c_1$

$u = \int c_1dx = c_1x + c_2$

simplest is $c_1 = 1$, $c_2 = 0$

$u = x$

so $y_2 = ue^{2/3x}$

$\boxed{y_2 = xe^{2/3x}}$

#3. Use a Wronskian to determine if the following set of functions is linearly independent on the interval $(0,\infty)$

$f_1(x)=e^{4x}, \quad f_2(x)=e^{2x}$

$$\begin{vmatrix} e^{4x} & e^{2x} \\ 4e^{4x} & 2e^{2x} \end{vmatrix}$$

$$2e^{2x}e^{4x}-4e^{4x}e^{2x}$$

$$-2e^{6x}\neq 0$$

So these solutions are linearly independent.

#4. Use a Wronskian to determine if the following set of functions is linearly independent on the interval $(0,\infty)$

$f_1(x)=x, \quad f_2(x)=x-1, \quad f_3(x)=x+3$

$$\begin{vmatrix} \overset{+}{x} & \overset{-}{x-1} & \overset{+}{x+3} \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{vmatrix}$$

$$x(0-0)-(x-1)(0-0)+(x+3)(0-0)$$

$$0-0-0$$

$$=0$$

So these solutions are not linearly independent.

#5. Solve the differential equation.

$y''-2y'-2y=0$

$$m^2-2m-2=0$$

$$m=\frac{2\pm\sqrt{4+4(1)2}}{2(1)}=\frac{2\pm\sqrt{12}}{2}=\frac{2\pm\sqrt{4}\sqrt{3}}{2}$$

$$m=\frac{2\pm2\sqrt{3}}{2}=1\pm\sqrt{3}$$

$$y=c_1e^{(1+\sqrt{3})x}+c_2e^{(1-\sqrt{3})x}$$

#6. Solve the differential equation.

$2y''+2y'+3y=0$

$$2m^2+2m+3=0$$

$$m=\frac{-2\pm\sqrt{4-4(2)(3)}}{2(2)}=\frac{-2\pm\sqrt{-20}}{4}$$

$$m=\frac{-2\pm\sqrt{4}\sqrt{5}\sqrt{-1}}{4}=\frac{-2\pm2\sqrt{5}\,i}{4}$$

$$m=-\frac{1}{2}\pm\frac{\sqrt{5}}{2}i$$

$$\boxed{y=c_1e^{-\frac{1}{2}x}\cos\left(\frac{\sqrt{5}}{2}x\right)+c_2e^{-\frac{1}{2}x}\sin\left(\frac{\sqrt{5}}{2}x\right)}$$

#7. Solve the differential equation.
$y''' - 5y'' + 3y' + 9y = 0$

$m^3 - 5m^2 + 3m + 9 = 0$

$$\text{try } -1 \left| \begin{array}{cccc} 1 & -5 & 3 & 9 \\ & -1 & 6 & -9 \\ \hline 1 & -6 & 9 & \underline{/0} \end{array} \right.$$

$(m+1)(m^2 - 6m + 9) = 0$

$(m+1)(m-3)(m-3) = 0$

$m = -1$, $m = 3$ repeated

$$\boxed{y = C_1 e^{-x} + C_2 e^{3x} + C_3 x e^{3x}}$$

#8. Solve the differential equation by method of undetermined coefficients (table method).
$y'' + 4y = 3\sin(2x)$

$\underline{y_c}$: $y'' + 4y = 0$ $\quad m^2 = -4$

$m^2 + 4 = 0$ $\quad m = \pm\sqrt{-4} = 0 \pm 2i$

$y_c = C_1 \cos(2x) + C_2 \sin(2x)$

$\underline{y_p}$: table for $3\sin(2x)$

$y_p = A\cos(2x) + B\sin(2x)$ (absorbed)

so $y_p = \underline{Ax}\underline{\cos(2x)} + \underline{Bx}\underline{\sin(2x)}$

$y' = \underline{Ax}(\underline{-2\sin(2x)}) + \cos(2x)A + \underline{Bx}(2\underline{\cos(2x)}) + \sin(2x)B$

$y'' = Ax(-4\cos(2x)) + (-2\sin(2x))A - 2A\sin(2x) + Bx(-4\sin(2x)) + (2\cos(2x))B + 2B\cos(2x)$

into full DE.. $y'' + 4y = 3\sin(2x)$

$[-4Ax\underline{\cos(2x)} - 4A\underline{\sin(2x)} - 4B\,\underline{x\sin(2x)} + 4B\,\underline{\cos(2x)}] + 4[A\underline{x\cos(2x)} + Bx\underline{\sin(2x)}] = 3\sin(2x)$

$(\underset{0}{-4A \cancel{+4A}})x\cos(2x) + (\underset{0}{-4B \cancel{+4B}})x\sin(2x) + (-4A)\sin(2x) + (4B)\cos(2x) = 3\sin(2x)$

System: $\begin{cases} -4A = 3 \\ 4B = 0 \end{cases}$ $\quad A = -\frac{3}{4}$, $B = 0$

$y_p = -\frac{1}{4}x\cos(2x) + (0)x\sin(2x)$

$y_p = -\frac{3}{4}x\cos(2x)$

general solution: $\boxed{y = C_1\cos(2x) + C_2\sin(2x) - \frac{3}{4}x\cos(2x)}$

#9. Solve the differential equation by method of undetermined coefficients (table method).

$y'' + 8y' + 16y = 2x^2 - 3, \quad y(0) = \frac{247}{64}, \quad y'(0) = \frac{-153}{8}$

$\underline{y_c}$: $y'' + 8y' + 16y = 0$ $\qquad m = -4$ repeated

$m^2 + 8m + 16 = 0 \qquad y_c = C_1 e^{-4x} + C_2 x e^{-4x}$

$(m+4)(m+4) = 0$

$\underline{y_p}$: table for $2x^2 - 3$: $y_p = Ax^2 + Bx + C$ (no absorption)

$y' = 2Ax + B$

$y'' = 2A$

into full DE: $y'' + 8y' + 16y = 2x^2 - 3$

$[2A] + 8[2Ax + B] + 16[Ax^2 + Bx + C] = 2x^2 - 3$

$(16A)x^2 + (16A + 16B)x + (2A + 8B + 16C) = (2)x^2 + (0)x + (-3)$

system: $\begin{cases} 16A = 2 \\ 16A + 16B = 0 \\ 2A + 8B + 16C = -3 \end{cases}$

$A = \frac{2}{16} = \frac{1}{8}$

$16(\frac{1}{8}) + 16B = 0, \quad B = -\frac{1}{8}$

$2(\frac{1}{8}) + 8(-\frac{1}{8}) + 16C = -3$

$\frac{1}{4} - 1 + 16C = -3, \quad 16C = -\frac{9}{4}, \quad C = -\frac{9}{64}$

$y_p = \frac{1}{8}x^2 - \frac{1}{8}x - \frac{9}{64}$

general solution: $y = C_1 e^{-4x} + C_2 x e^{-4x} + \frac{1}{8}x^2 - \frac{1}{8}x - \frac{9}{64}$

use $y(0) = \frac{247}{64}$ $\qquad \frac{247}{64} = C_1 e^0 + C_2(0)e^0 + \frac{1}{8}(0)^2 - \frac{1}{8}(0) - \frac{9}{64}$

$\frac{247}{64} = C_1 - \frac{9}{64}, \quad C_1 = 4$

$y = 4e^{-4x} + \underline{C_2 x}\,\underline{e}^{-4x} + \frac{1}{8}x^2 - \frac{1}{8}x - \frac{9}{64}$

$y' = -16e^{-4x} + C_2 x(-4e^{-4x}) + e^{-4x}C_2 + \frac{1}{4}x - \frac{1}{8}$

use $y'(0) = \frac{-153}{8}$ $\qquad \frac{-153}{8} = -16e^0 - 4C_2(0)e^0 + C_2 e^0 + \frac{1}{4}(0) - \frac{1}{8}$

$\frac{-153}{8} = -16 + C_2 - \frac{1}{8}, \quad C_2 = -3$

$$\boxed{y = 4e^{-4x} - 3xe^{-4x} + \frac{1}{8}x^2 - \frac{1}{8}x - \frac{9}{64}}$$

#10. Solve the differential equation by method of undetermined coefficients (table method).

$y''-16y=2e^{4x}$

$\underline{y_c}$: $y''-16y=0$ $\quad m=4,\ m=-4$

$m^2-16=0$ $\quad y_c=C_1e^{4x}+C_2e^{-4x}$

$(m-4)(m+4)=0$

$\underline{y_p}$: table for $2e^{4x}$:

$y_p=Ae^{4x}$ (absorbed)

So $y_p=\underline{Axe}^{4x}$

$y'=\underline{Ax}(\underline{4e^{4x}})+e^{4x}A$

$y''=Ax(16e^{4x})+4e^{4x}A+4Ae^{4x}$

into full DE: $y''-16y=2e^{4x}$

$[16Axe^{4x}+8Ae^{4x}]-16[Axe^{4x}]=2e^{4x}$

$\underbrace{(16A-\cancel{16A})}_{0}xe^{4x}+(8A)e^{4x}=2e^{4x}$, $8A=2,\ A=\frac{2}{8}=\frac{1}{4}$

$y_p=\frac{1}{4}xe^{4x}$

general solution: $\boxed{y=C_1e^{4x}+C_2e^{-4x}+\frac{1}{4}xe^{4x}}$

#11. Solve the differential equation by variation of parameters (Wronskian method).

$$y'' - 9y = \frac{9x}{e^{3x}}$$

$\underline{y_c}$: $y''-9y=0$ $\quad m=3,\ m=-3$

$m^2-9=0$ $\quad y_c = C_1e^{3x}+C_2e^{-3x}$

$(m-3)(m+3)=0$

$\underline{y_p}$: $y_p = u_1e^{3x}+u_2e^{-3x}$

$$u_1' = \frac{\begin{vmatrix} 0 & e^{-3x} \\ \frac{9}{e^{3x}} & -3e^{-3x} \end{vmatrix}}{\begin{vmatrix} e^{3x} & e^{-3x} \\ 3e^{3x} & -3e^{-3x} \end{vmatrix}} = \frac{0-\frac{9x}{e^{3x}}e^{-3x}}{-3e^{3x}e^{-3x}-3e^{3x}e^{-3x}} = \frac{-9xe^{-3x}e^{-3x}}{-3e^0-3e^0} = \frac{-9xe^{-6x}}{-6} = \frac{3}{2}xe^{-6x}$$

$u_1 = \frac{3}{2}\int xe^{-6x}dx$

by parts;

$u=x \quad dv=e^{-6x}dx$

$\frac{du}{dx}=1 \quad \int dv=\int e^{-6x}dx$

$du=dx \quad v=-\frac{1}{6}e^{-6x}$

$uv-\int vdu$

$-\frac{1}{6}xe^{-6x}+\int(-\frac{1}{6}e^{-6x})dx$

$-\frac{1}{6}xe^{-6x}+\frac{1}{6}\int e^{-6x}dx = -\frac{1}{6}xe^{-6x}-\frac{1}{36}e^{-6x}$

$u_1 = \frac{3}{2}[-\frac{1}{6}xe^{-6x}-\frac{1}{36}e^{-6x}] = -\frac{1}{4}xe^{-6x}-\frac{1}{24}e^{-6x}$

$$u_2' = \frac{\begin{vmatrix} e^{3x} & 0 \\ 3e^{3x} & \frac{9}{e^{3x}} \end{vmatrix}}{\begin{vmatrix} e^{3x} & e^{-3x} \\ 3e^{3x} & -3e^{-3x} \end{vmatrix}} = \frac{e^{3x}\frac{9x}{e^{3x}}}{-6} = -\frac{3}{2}x$$

$u_2 = -\frac{3}{2}\int xdx = -\frac{3}{2}(\frac{1}{2}x^2) = -\frac{3}{4}x^2$

$y_p = (-\frac{1}{4}xe^{-6x}-\frac{1}{24}e^{-6x})e^{3x}+(-\frac{3}{4}x^2)e^{-3x} = -\frac{1}{4}xe^{-6x}e^{3x}-\frac{1}{24}e^{-6x}e^{3x}-\frac{3}{4}x^2e^{-3x}$

$y_p = -\frac{1}{4}xe^{-3x}-\frac{1}{24}e^{-3x}-\frac{3}{4}x^2e^{-3x}$

general Solution:

$y = C_1e^{3x}+\underbrace{C_2e^{-3x}}_{}-\frac{1}{4}xe^{-3x}\underbrace{-\frac{1}{24}e^{-3x}}_{}-\frac{3}{4}x^2e^{-3x}$

$$\boxed{y = C_1e^{3x}+C_3e^{-3x}-\frac{1}{4}xe^{-3x}-\frac{3}{4}x^2e^{-3x}}$$

#12. Solve the differential equation by variation of parameters (Wronskian method).

$y'' + y = \sec^3 x$

y_c: $y''+y=0$ $\quad m = \pm\sqrt{-1} = 0 \pm i$ $\quad y_c = C_1\cos x + C_2 \sin x$

$m^2+1=0$

y_p: $y_p = u_1 \cos x + u_2 \sin x$

$$u_1' = \frac{\begin{vmatrix} 0 & \sin x \\ \sec^3 x & \cos x \end{vmatrix}}{\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}} = \frac{0-\sin x \sec^3 x}{\cos^2 x+\sin^2 x} = \frac{-\frac{\sin x}{\cos^3 x}}{1} = -\frac{\sin x}{\cos^3 x}$$

$u_1 = \int \frac{-\sin x}{\cos^3 x}dx$ $\quad u=\cos x$, $\frac{du}{dx} = -\sin x$, $du = -\sin x\,dx$ $\quad u_1 = \int u^{-3}du = \frac{u^{-2}}{-2} = \frac{-1}{2\cos^2 x}$

$$u_2' = \frac{\begin{vmatrix} \cos x & 0 \\ -\sin x & \sec^3 x \end{vmatrix}}{\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}} = \frac{\cos x \sec^3 x - 0}{1} = \frac{\cos x}{\cos^3 x} = \frac{1}{\cos^2 x} = \sec^2 x.$$

$u_2 = \int \sec^2 x\,dx = \tan x$

$y_p = \left(\frac{-1}{2\cos^2 x}\right)\cos x + (\tan x)\sin x = \frac{-1}{2\cos x} + \tan x \sin x$

general solution: $\boxed{y = C_1 \cos x + C_2 \sin x - \frac{1}{2\cos x} + \tan x \sin x}$

#13. Solve this Cauchy-Euler form differential equation using the Cauchy-Euler procedure for the homogeneous solution. Then use the variation of parameters (Wronskian method) to find the particular solution. Finally, join the solutions to produce the general solution.

$2x^2y'' + 5xy' + y = x^2 - x$

$\underline{y_c}$: $2x^2y''+5xy'+y=0$ $\qquad (2m+1)(m+1)=0$ $\qquad y_c = C_1 x^{-1/2} + C_2 x^{-1}$

$am^2+(b-a)m+c=0$ $\qquad m=-1/2 \quad m=-1$

$2m^2+(5-2)m+1=0$

$2m^2+3m+1=0$

$\underline{y_p}$: $y_p = u_1 x^{-1/2} + u_2 x^{-1}$ $\qquad$ RHS: $\frac{x^2-x}{2x^2}$

$$u_1' = \frac{\begin{vmatrix} 0 & x^{-1} \\ \frac{x^2-x}{2x^2} & -x^{-2} \end{vmatrix}}{\begin{vmatrix} x^{-1/2} & x^{-1} \\ -\frac{1}{2}x^{-3/2} & -x^{-2} \end{vmatrix}} = \frac{0 - x^{-1}\frac{x^2-x}{2x^2}}{-x^{-1/2}x^{-2} + \frac{1}{2}x^{-3/2}x^{-1}} = \frac{-\left(\frac{x-1}{2x^2}\right)}{-x^{-5/2}+\frac{1}{2}x^{-5/2}} = \frac{-(x-1)}{2x^2(-\frac{1}{2}x^{-5/2})} = \frac{x-1}{x^{-1/2}} = x^{3/2} - x^{1/2}$$

$u_1 = \int (x^{3/2} - x^{1/2})dx = \frac{2}{5}x^{5/2} - \frac{2}{3}x^{3/2}$

$$u_2' = \frac{\begin{vmatrix} x^{-1/2} & 0 \\ -\frac{1}{2}x^{-3/2} & \frac{x^2-x}{2x^2} \end{vmatrix}}{\begin{vmatrix} x^{-1/2} & x^{-1} \\ -\frac{1}{2}x^{-3/2} & -x^{-2} \end{vmatrix}} = \frac{x^{-1/2}\frac{(x^2-x)}{2x^2} - 0}{-\frac{1}{2}x^{-5/2}} = -(x^2-x) = -x^2+x$$

$u_2 = \int (-x^2+x)dx = -\frac{1}{3}x^3 + \frac{1}{2}x^2$

$y_p = (\frac{2}{5}x^{5/2} - \frac{2}{3}x^{3/2})x^{-1/2} + (-\frac{1}{3}x^3 + \frac{1}{2}x^2)x^{-1} = \frac{2}{5}x^2 - \frac{2}{3}x - \frac{1}{3}x^2 + \frac{1}{2}x$

$y_p = \frac{1}{15}x^2 - \frac{1}{6}x$

so $\boxed{y = C_1 x^{-1/2} + C_2 x^{-1} + \frac{1}{15}x^2 - \frac{1}{6}x}$

#14. Solve this Cauchy-Euler form differential equation using the Cauchy-Euler procedure for the homogeneous solution. Then use the variation of parameters (Wronskian method) to find the particular solution. Finally, join the solutions to produce the general solution.

$xy'' - 4y' = x^4$

y_c: $xy''-4y'=0$ (multiply by x)

$x^2y''-4xy'=0$ (now, Cauchy-Euler)

$am^2+(b-a)m+c=0$

$1 m^2+(-4-1)m+0=0$

$m^2-5m=0 \rightarrow$

$m(m-5)=0$

$m=0,\ m=5$

$y_c=C_1x^0+C_2x^5$

$y_c=C_1+C_2x^5$

y_p: $xy''-4y'=x^4$

$y''-\frac{4}{x}y'=x^3$

$y_p=u_1+u_2x^5$

$$u_1'=\frac{\begin{vmatrix}0 & x^5\\ x^3 & 5x^4\end{vmatrix}}{\begin{vmatrix}1 & x^5\\ 0 & 5x^4\end{vmatrix}}=\frac{0-x^8}{5x^4-0}=\frac{-x^8}{5x^4}=-\frac{1}{5}x^4$$

$$u_1=-\frac{1}{5}\int x^4dx=-\frac{1}{5}\frac{1}{5}x^5=-\frac{1}{25}x^5$$

$$u_2'=\frac{\begin{vmatrix}1 & 0\\ 0 & x^3\end{vmatrix}}{\begin{vmatrix}1 & x^5\\ 0 & 5x^4\end{vmatrix}}=\frac{x^3-0}{5x^4}=\frac{1}{5}\frac{1}{x}$$

$$u_2=\frac{1}{5}\int\frac{1}{x}dx=\frac{1}{5}\ln|x|$$

$$y_p=-\frac{1}{25}x^5(1)+\left(\frac{1}{5}\ln|x|\right)x^5=-\frac{1}{25}x^5+\frac{1}{5}x^5\ln|x|$$

so $y=C_1+\underbrace{C_2x^5-\frac{1}{25}x^5}+\frac{1}{5}x^5\ln|x|$

$$\boxed{y=C_1+C_3x^5+\frac{1}{5}x^5\ln|x|}$$