DiffEq - Ch 3 - Extra Practice

3.1

#1b. The population of bacteria in a culture grows at a rate proportional to the number of bacteria present at time *t*. After 3 hours it is observed that 400 bacteria are present. After 10 hours 2000 bacteria are present.
What was the initial number of bacteria?

Deriving Solution ... $\frac{dP}{dt} = kP$

$\int \frac{1}{P}\,dP = \int k\,dt, \quad \ln P = kt + C_1$

$e^{\ln P} = e^{kt + C_1}, \quad P = e^{kt} e^{C_1} = Ce^{kt}$

if $P(0) = P_0$, $\quad P_0 = Ce^{k(0)}$, $\quad C = P_0$

so $\underline{P = P_0 e^{kt}}$

Can just start here

t	P
3	400
10	2000

2 unknowns: k, P_0

2 data points → 2 equations:

$$\begin{cases} 400 = P_0 e^{k(3)} \\ 2000 = P_0 e^{k(10)} \end{cases}$$

Solve the system...

$P_0 = \frac{400}{e^{3k}} \rightarrow 2000 = P_0 e^{10k}$

$2000 = \left(\frac{400}{e^{3k}}\right) e^{10k} = 400 e^{(10k - 3k)}$

$2000 = 400 e^{7k}$

$5 = e^{7k}$

$\ln(5) = 7k, \quad k = \frac{\ln(5)}{7}$

$P_0 = \frac{400}{e^{3k}} = \frac{400}{e^{3\left(\frac{\ln(5)}{7}\right)}}$

$P_0 = 200.679$

$\boxed{P_0 \approx 201 \text{ bacteria}}$

#2b. The population of a town grows at a rate proportional to the population present at time *t*. The initial population of 500 increases by 15% in 10 years.
(a) What will the population be in 30 years?
(b) How fast is the population growing at $t = 30$ years?

solution form: $P = P_0 e^{kt}$

t	P
0	500
10	$500(1.15) = 575$

$P = 500e^{kt}$

$575 = 500e^{k(10)}$

$e^{10k} = \frac{575}{500}$

$10k = \ln\left(\frac{575}{500}\right), \quad k = \frac{\ln\left(\frac{575}{500}\right)}{10}$

$= 0.01397619942$

$P = 500e^{0.01397619942t}$

(a) $P(30) = 500e^{0.01397619942(30)}$

$= 760.43$

$\approx \boxed{760 \text{ people}}$

(b) $\frac{dP}{dt} = kP$

$= 0.01397619942(760.43)$

$= \boxed{10.63 \text{ people/yr}}$

#3b. Initially 5 grams of a radioactive substance was present. After 8 hours the mass had decreased by 4%. If the rate of decay is proportional to the amount of the substance present at time t, find the amount remaining after 10 hours.

t (hrs)	Q (g)
0	5
8	5(.96) = 4.8 g

Solution form: $Q = Q_0 e^{kt}$

$4.8 = 5e^{k(8)}$

$e^{8k} = \frac{4.8}{5}$, $8k = \ln\left(\frac{4.8}{5}\right)$

$k = \frac{\ln\left(\frac{4.8}{5}\right)}{8} = -0.005102749\,3$

$Q = 5e^{-0.0051027493t}$

$Q(10) = 5e^{-0.0051027493(10)}$

$= \boxed{4.751 \text{ g}}$

#4b. Determine the half-life of the radioactive substance described in Problem #3b

t	Q
0	5
t_{HL}	2.5 g

$2.5 = 5e^{-0.0051027493t}$

$-0.0051027493t = \frac{2.5}{5}$

$t = \frac{\ln\left(\frac{2.5}{5}\right)}{-0.0051027493} = \boxed{135.84 \text{ hrs}}$

#5b. When interest is compounded continuously, the amount of money increases at a rate proportional to the amount A present at time t, that is $\frac{dA}{dt} = rA$, where r is the annual rate of interest.

(a) Find the amount of money accrued at the end of 10 years when \$10,000 is deposited in an investment account drawing 8% annual interest compounded continuously.

(b) In how many years will the initial sum deposited have doubled?

Solution form: $A = A_0 e^{rt}$

(a) $A = 10000e^{0.08t}$

$A(10) = 10000e^{0.08(10)}$

$= \boxed{\$22255.41}$

(b) $20000 = 10000e^{0.08t}$

$e^{0.08t} = \frac{20000}{10000} = 2$

$0.08t = \ln(2)$

$t = \frac{\ln(2)}{0.08}$

$= \boxed{8.66 \text{ years}}$

#6b. A thermometer is taken from an inside room to the outside, where the air temperature is 5 °F. After 1 minute the thermometer reads 55 °F, and after 5 minutes it reads 30 °F.
What was the initial temperature of the inside room?

Newton's Law of Cooling: $\frac{dT}{dt} = k(T-T_m)$

deriving solution form... $\int \frac{1}{T-T_m}\,dT = \int k\,dt$

$u = T-T_m,\ \frac{du}{dT} = 1,\ du = dT$

$\int \frac{1}{u}\,du = \int k\,dt,\quad \ln u = kt + C_1$

$\ln(T-T_m) = kt + C_1$

$T - T_m = e^{(kt+C_1)} = e^{kt}e^{C_1} = Ce^{kt}$

$T = T_m + Ce^{kt}$ ← can start here

here, $T_m = 5\,°F$,

min t	°F T
0	T_{room}
1	55
5	30

need 2 constants, C, k

have 2 datapts,

2 eqns;

$$\begin{cases} 55 = 5 + Ce^{k(1)} \\ 30 = 5 + Ce^{k(5)} \end{cases}$$

solve system: $Ce^{k(1)} = 55 - 5 = 50$

$C = \frac{50}{e^k}$

into $30 = 5 + Ce^{5k}$, $30 = 5 + \left(\frac{50}{e^k}\right)e^{5k}$

$25 = 50\frac{e^{5k}}{e^k} = 50e^{4k}$

$e^{4k} = \frac{25}{50},\quad 4k = \ln\left(\frac{25}{50}\right)$

$k = \frac{\ln\left(\frac{25}{50}\right)}{4} = -0.1732867951$

and $C = \frac{50}{e^k} = \frac{50}{e^{-0.1732867951}} = 59.4604$

so $T(t) = 5 + 59.4604\,e^{-0.1732867951t}$

$T(0) = T_{room} = 5 + 59.4604\,e^0$

$= \boxed{64.46\ °F}$

#7b. A thermometer reading 70 °F, is placed in an oven preheated to a constant temperature. Through a glass window in the oven door, an observer records that the thermometer reads 110 °F after 30 seconds and 145 °F after 1 minute.
How hot is the oven?

$T = T_m + Ce^{kt}$

min t	°F T
0	70
0.5	110
1	145

3 constants to find, k

3 datapts = 3 equations;

$$\begin{cases} 70 = T_m + Ce^0 \\ 110 = T_m + Ce^{k(0.5)} \\ 145 = T_m + Ce^{k(1)} \end{cases}$$

$\rightarrow C = 70 - T_m$

$$\begin{cases} 110 = T_m + (70-T_m)e^{0.5k} \\ 145 = T_m + (70-T_m)e^{k} \end{cases}$$ ← solve this for k

$(70-T_m)e^k = 145 - T_m$

$e^k = \frac{145-T_m}{70-T_m}$

$k = \ln\left(\frac{145-T_m}{70-T_m}\right)$

into $110 = T_m + (70-T_m)e^{0.5k}$

$e^{0.5k} = e^{0.5\left(\ln\left(\frac{145-T_m}{70-T_m}\right)\right)}$

$= e^{\ln\left(\left(\frac{145-T_m}{70-T_m}\right)^{0.5}\right)} = \left(\frac{145-T_m}{70-T_m}\right)^{0.5}$

$= \sqrt{\frac{145-T_m}{70-T_m}}$

So $110 = T_m + (70-T_m)\sqrt{\frac{145-T_m}{70-T_m}}$

$110 - T_m = (70-T_m)\sqrt{\frac{145-T_m}{70-T_m}}$

$\left(\frac{110-T_m}{70-T_m}\right)^2 = \left(\sqrt{\frac{145-T_m}{70-T_m}}\right)^2 = \frac{145-T_m}{70-T_m}$

$\frac{(110-T_m)^2}{\cancel{(70-T_m)^2}} = \frac{(145-T_m)(70-T_m)}{\cancel{(70-T_m)(70-T_m)}}$

$12100 - 220T_m + \cancel{T_m^2} = 10150 - 215T_m + \cancel{T_m^2}$

$5T_m = 1950$

$T_m = \frac{1950}{5} = \boxed{390\,°F}$

#8b. A 40-Volt electromotive force is applied to an LR series circuit in which the inductance is 0.3 Henry and the resistance is 150 Ohms.

(a) Find the equation for current as a function of time $i(t)$ if $i(0)=0$.

(b) Determine the current at $t \to \infty$.

(a) $L\frac{di}{dt} + Ri = E$

$0.3\frac{di}{dt} + 150i = 40$

linear w/ $P(t)=500$

$\frac{di}{dt} + 500i = \frac{400}{3}$

$I.F. = e^{\int 500\,dt} = e^{500t}$

$e^{500t} i = \int \frac{400}{3} e^{500t}\,dt$

$e^{500t} i = \left(\frac{400}{3(500)}\right) e^{500t} + C$

$\frac{e^{500t} i}{e^{500t}} = \frac{\frac{4}{15} e^{500t}}{e^{500t}} + \frac{C}{e^{500t}}$

$i(t) = \frac{4}{15} + Ce^{-500t}$

now, $i(0)=0$

$0 = \frac{4}{15} + Ce^{0}$, $C = -\frac{4}{15}$

so $\boxed{i(t) = \frac{4}{15} - \frac{4}{15}e^{-500t}}$

(b) $i(t) = \frac{4}{15} - \underbrace{\frac{4}{15}e^{-500t}}_{\text{transient term} \to 0}$

as $t \to \infty$

so $\boxed{i \to \frac{4}{15}\text{ Amp}}$

#9b. A 200-Volt electromotive force is applied to an RC series circuit in which the resistance is 1000 Ohms and the capacitance is $5 \bullet 10^{-6}$ Farad.

(a) Find the charge on the capacitor as a function of time $q(t)$ if $i(0)=0.4$ *Amps*.

(b) Determine the charge and current at $t = 0.005$ s.

(c) Determine the current at $t \to \infty$.

(a) $R\frac{dq}{dt} + \frac{1}{C}q = E$

$1000\frac{dq}{dt} + \frac{1}{5 \cdot 10^{-6}}q = 200$

linear, w/ $P(t)=200$

$\frac{dq}{dt} + 200q = \frac{1}{5}$ $\quad I.F. = e^{\int 200\,dt} = e^{200t}$

$e^{200t} q = \int \frac{1}{5} e^{200t}\,dt$

$\frac{e^{200t} q}{e^{200t}} = \frac{\frac{1}{1000} e^{200t}}{e^{200t}} + \frac{C}{e^{200t}}$

$q(t) = \frac{1}{1000} + Ce^{-200t}$

now, $i(0)=0.4$,

$i = \frac{dq}{dt}$

so $i = Ce^{-200t}(-200)$

$0.4 = -200Ce^{-200(0)} = -200C$

so $C = \frac{-0.4}{200} = -\frac{1}{500}$

so $\boxed{q(t) = \frac{1}{1000} - \frac{1}{500}e^{-200t}}$

and $i(t) = \frac{2}{5}e^{-200t}$

(b) $q(0.005) = \frac{1}{1000} - \frac{1}{500}e^{-200(.005)} = \boxed{2.64 \cdot 10^{-4}\text{ Coulombs}}$

$i(0.005) = \frac{2}{5}e^{-200(.005)} = \boxed{0.147\text{ Amps}}$

(c) $i(t) = \underbrace{\frac{2}{5}e^{-200t}}_{\text{transient term} \to 0}$

as $t \to \infty$

so $\boxed{i \to 0}$

#10b. Suppose a small cannonball weighing 22 pounds is shot vertically upward with an initial velocity $v_0 = 400\ ft/s$. The answer to the question "How high does the cannonball go?" depends upon whether we take air resistance into account or not.

(a) Suppose air resistance is ignored. If the positive direction is upward, then a model for the state of the cannonball is given by:

$$\frac{d^2 s}{dt^2} = -g$$

Since $\frac{ds}{dt} = v(t)$ the last differential equation is the same as:

$$\frac{dv}{dt} = -g$$

where we take $g = 32\ ft/s^2$.

Find the velocity of the cannonball at time t.

$\frac{dv}{dt} = -g$ (if v + upward)

$\frac{dv}{dt} = -32$ separable

$\int dv = \int -32\,dt$

$v = -32t + C$

$v(0) = 400$

$400 = -32(0) + C, \quad C = 400$

$\boxed{v(t) = -32t + 400}$

(b) Use the result obtained in part (a) to determine the height $s(t)$ of the cannonball measured from ground level.

(c) Find the maximum height attained by the cannonball.

(b) $v = \frac{ds}{dt}$, so $s = \int v\,dt$

$s = \int (-32t + 400)\,dt$

$s = -16t^2 + 400t + C$

assuming cannonball starts from ground:

$s(0) = 0$

$0 = -16(0)^2 + 400(0) + C, \quad C = 0$

so $\boxed{s(t) = -16t^2 + 400t}$

(c) max height when $v = 0$

$v = -32t + 400$

$0 = -32t + 400$

$32t = 400, \quad t = \frac{400}{32} = 12.5$ sec

height $= s(t) = -16t^2 + 400t$

$s(12.5) = -16(12.5)^2 + 400(12.5)$

$\boxed{S_{max} = 2500\ ft}$

#11b. Repeat problem #10b, but this time, assume that air resistance is proportional to instantaneous velocity. Use the following differential equation:

$$m\frac{dv}{dt} = mg - kv$$

…and assume $k = 0.005$.

(a) Find the velocity of the cannonball at time t.
(b) Use the result obtained in part (a) to determine the height $s(t)$ of the cannonball measured from ground level.
(c) Find the maximum height attained by the cannonball.

(a) $m\frac{dv}{dt} = mg - kv$ assumes v+ downward

to change to v+ upward make mg term −
(kv term stays negative—it still opposes motion)

$m\frac{dv}{dt} = -mg - kv$ mass: $F = mg$

$22 = m\,32$

$\frac{dv}{dt} = -g - \frac{k}{m}v$ $m = \frac{22}{32} = \frac{11}{16}$ slug

$\frac{dv}{dt} = -32 - \frac{.005}{(11/16)}v$

linear, w/ $P(t) = \frac{2}{275}$

$\frac{dv}{dt} + \frac{2}{275}v = -32$ I.F. $= e^{\int \frac{2}{275}dt} = e^{\frac{2}{275}t}$

$e^{2/275 t}v = \int -32e^{\frac{2}{275}t}$

$\frac{e^{\frac{2}{275}t}v}{e^{2/275t}} = \frac{-32\left(\frac{1}{\left(\frac{2}{275}\right)}\right)e^{2/275t}}{e^{2/275t}} + \frac{C}{e^{\frac{2}{275}t}}$

$v = -440 + Ce^{-\frac{2}{275}t}$

now, $v(0) = 400$

$400 = -440 + Ce^0$, $C = 840$

so $\boxed{v(t) = -440 + 840e^{-\left(\frac{2}{275}\right)t}}$

(b) $\frac{ds}{dt} = v$, so $s = \int v\,dt$

$s = \int\left(-440 + 840e^{-\left(\frac{2}{275}\right)t}\right)dt$

$s = -440t + 840\left(-\frac{275}{2}\right)e^{-\frac{2}{275}t} + C$

assume start on ground: $s(0) = 0$

$0 = -440(0) - 115500e^0 + C$

$C = 115500$

$\boxed{s(t) = -440t - 115500e^{-\left(\frac{2}{275}\right)t} + 115500}$

(c) max height when $v = 0$

$v = -440 + 840e^{-\left(\frac{2}{275}\right)t}$

$0 = -440 + 840e^{-\left(\frac{2}{275}\right)t}$

$e^{-\left(\frac{2}{275}\right)t} = \frac{440}{840} = \frac{44}{84}$

$-\frac{2}{275}t = \ln\left(\frac{44}{84}\right)$

$t = \frac{\ln\left(\frac{44}{84}\right)}{-\left(\frac{2}{275}\right)} = 88.9112$ sec

So max height $= s(88.9112)$

$= -440(88.9112) - 115500e^{-\left(\frac{2}{275}\right)(88.9112)} + 115500$

$= \boxed{15879 \text{ ft}}$

#12b. A skydiver weighs 125 pounds, and her parachute and equipment combined weight another 35 pounds. After exiting from a plane at an altitude of 15,000 ft, she waits 15 seconds and opens her parachute. Assuming that air resistance is proportional to instantaneous velocity, then velocity and time are related by this differential equation:

$$m\frac{dv}{dt} = mg - kv$$

use as is (v + downward)

(positive direction is downward). Assume $k = 0.5$ during free fall and $k = 10$ after the parachute is opened, and assume that her initial velocity on leaving the plane is zero.

What is her velocity 20 seconds after leaving the plane?

2 separate phases:

free fall	parachute open
$m\frac{dv}{dt} = mg - kv$	$m\frac{dv}{dt} = mg - kv$
w/ $k = 0.5$	w/ $k = 10$
$v(0) = 0$	
find $v(15\text{sec})$	becomes $v(0)$ for this phase (restart time at $t=0$)

mass: $F = mg$

$(125+35)\text{lbs} = m(32)$

$m = \frac{125+35}{32} = 5$ slugs

phase 1 (freefall)

$m\frac{dv}{dt} = mg - 0.5v$

$\frac{dv}{dt} = 32 - \frac{0.5}{5}v$

$\frac{dv}{dt} + \frac{1}{10}v = 32$ linear, $P(t) = \frac{1}{10}$

I.F. $= e^{\int \frac{1}{10}dt} = e^{\frac{1}{10}t}$

$e^{\frac{1}{10}t}v = \int 32e^{\frac{1}{10}t}dt$

$e^{\frac{1}{10}t}v = 32\left(\frac{1}{(\frac{1}{10})}\right)e^{\frac{1}{10}t} + C$

$\frac{e^{\frac{1}{10}t}v}{e^{\frac{1}{10}t}} = \frac{320e^{\frac{1}{10}t}}{e^{\frac{1}{10}t}} + \frac{C}{e^{\frac{1}{10}t}}$

$v = 320 + Ce^{-\frac{1}{10}t}$

initial velocity = 0

$0 = 320 + Ce^{0}$, $C = -320$

$v = 320 - 320e^{-\frac{1}{10}t}$

at end of free-fall, $t = 15$ sec

$v(15) = 320 - 320e^{-\frac{1}{10}(15)}$

$= 248.598 = 248.6$ ft/sec

becomes the initial velocity for parachute phase...

phase 2 (parachute)

$m\frac{dv}{dt} = mg - 10v$

$\frac{dv}{dt} = 32 - \frac{10}{5}v$

$\frac{dv}{dt} + 2v = 32$ linear, $P(t) = 2$

I.F. $= e^{\int 2dt} = e^{2t}$

$e^{2t}v = \int 32e^{2t}$

$\frac{e^{2t}v}{e^{2t}} = \frac{32(\frac{1}{2})e^{2t}}{e^{2t}} + \frac{C}{e^{2t}}$

$v = 16 + Ce^{-2t}$ (t starts over again in this phase)

$v(0) = 248.6$

$248.6 = 16 + Ce^{0}$, $C = 248.6 - 16 = 232.6$

$v = 16 + 232.6e^{-2t}$

20 sec after leaving plane − 15 = 5 seconds into parachute phase:

$v(5) = 16 + 232.6e^{-2(5)}$

$= \boxed{16.01 \text{ ft/sec}}$

3.2

#1b. If a constant number h of fish are harvested from a fishery per unit time, than a model for the population $P(t)$ of the fishery at time t is given by:

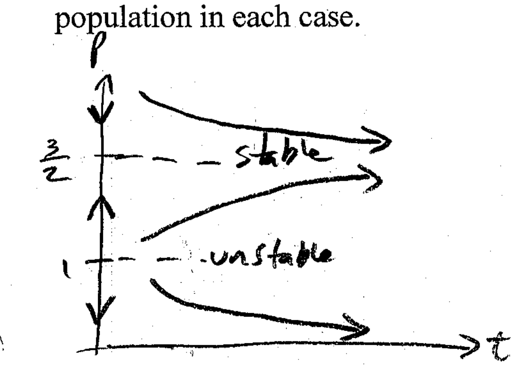
$$\frac{dP}{dt} = P(a-bP)-h, \quad P(0)=P_0$$

where a, b, h, and P_0 are positive constants. Suppose $a=5$, $b=2$, and $h=3$.

(a) Since the DE is autonomous, use the phase portrait concept to sketch representative solution curves corresponding to the cases

$$P_0 > \frac{3}{2}, \quad 1 < P_0 < \frac{3}{2}, \quad \text{and} \quad 0 < P_0 < 1 .$$

Determine the long-term behavior of the population in each case.



(test $P=2$)

$$\frac{dP}{dt} = 2(5-2(2))-3 < 0$$

(test $P=1.25$)

$$\frac{dP}{dt} = 1.25(5-2(1.25))-3 > 0$$

(test $P=0.5$)

$$\frac{dP}{dt} = 0.5(5-2(0.5))-3 < 0$$

for $P_0 > 3/2$, Population decreases to $3/2$

for $1 < P_0 < 3/2$, population increases to $3/2$

for $P_0 < 1$, population decreases to 0

(b) Solve the differential equation initial-value problem. Then verify the results of your phase portrait in part (a) by graphing the solution with an initial condition taken from each of the three intervals given.

$$\frac{dP}{dt} = P(5-2P)-3$$
$$= -2P^2+5P-3$$
$$= -(2P^2-5P+3)$$

Separable:

$$\int \frac{1}{2P^2-5P+3}\,dP = \int -dt$$

for left side, factor and use partial fractions

$$2P^2-5P+3 = (2P-3)(2P-2) = (2P-3)(P-1)$$

$$\frac{1}{(2P-3)(P-1)} = \frac{A}{2P-3} + \frac{B}{P-1} = \frac{A(P-1)+B(2P-3)}{(2P-3)(P-1)}$$

$$1 = AP - A + 2BP - 3B = (A+2B)P + (-A-3B)$$

$$\begin{cases} A+2B=0 \\ -A-3B=1 \end{cases} \quad A=2, \quad B=-1$$

$$2\int \frac{1}{2P-3}\,dP - \int \frac{1}{P-1}\,dP = \int -dt$$

$u = 2P-3$

$\frac{du}{dP} = 2$, $du = 2dP$

$$2\left(\tfrac{1}{2}\right)\int \frac{1}{u}\,du - \int \frac{1}{P-1}\,dP = \int -dt$$

$$\ln(2P-3) - \ln(P-1) = -t + C_1$$

$$\ln\left(\frac{2P-3}{P-1}\right) = -t + C_1$$

continued ...

$$\frac{2P-3}{P-1} = e^{-t+C_1} = e^{-t}e^{C_1} = Ce^{-t}$$

(c) Use the information in parts (a) and (b) to determine whether the fishery population becomes extinct in finite time. If so, find that time.

any $P_0 < 1$ leads to extinction when denom $=0$:

$$2 - \left(\frac{2P_0-3}{P_0-1}\right)e^{-t} = 0$$

$$\frac{2P_0-3}{P_0-1}e^{-t} = 2, \quad e^{-t} = 2\,\frac{P_0-1}{2P_0-3}$$

$$\boxed{\text{at } t = -\ln\left(2\,\frac{P_0-1}{2P_0-3}\right)}$$

$\frac{2p-3}{p-1} = Ce^{-t}$ if $P(0)=P_0$, $\frac{2P_0-3}{P_0-1} = Ce^0$, $C = \frac{2P_0-3}{P_0-1}$ (will plug in later)

$2p-3 = (p-1)Ce^{-t} = pCe^{-t} - Ce^{-t}$

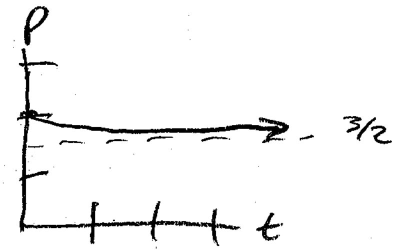
$2p - pCe^{-t} = 3 - Ce^{-t}$

$p(2-Ce^{-t}) = 3-Ce^{-t}$

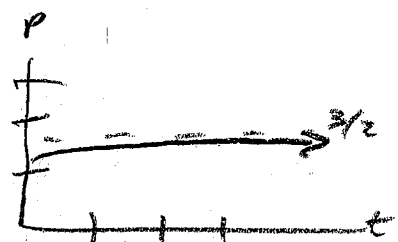
$p = \frac{3-Ce^{-t}}{2-Ce^{-t}}$ so $\boxed{P(t) = \frac{3-\left(\frac{2P_0-3}{P_0-1}\right)e^{-t}}{2-\left(\frac{2P_0-3}{P_0-1}\right)e^{-t}}}$

trying the test values from part a:

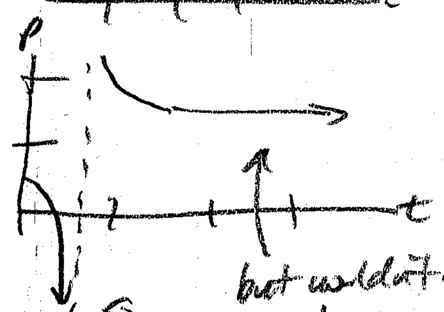
<u>$P_0=2$</u> $\quad P = \frac{3-\left(\frac{2(2)-3}{2-1}\right)e^{-t}}{2-\left(\frac{2(2)-3}{2-1}\right)e^{-t}} = \frac{3-e^{-t}}{2-e^{-t}}$



<u>$P_0=1.25$</u> $\quad P = \frac{3-\left(\frac{2(1.25)-3}{1.25-1}\right)e^{-t}}{2-\left(\frac{2(1.25)-3}{1.25-1}\right)e^{-t}} = \frac{3+2e^{-t}}{2+2e^{-t}}$



<u>$P_0=0.5$</u> $\quad P = \frac{3-\left(\frac{2(0.5)-3}{0.5-1}\right)e^{-t}}{2-\left(\frac{2(0.5)-3}{0.5-1}\right)e^{-t}} = \frac{3-4e^{-t}}{2-4e^{-t}}$



has vertical asymptote because denom can be zero:

at $2-4e^{-t}=0$

$4e^{-t}=2$

$e^{-t}=\frac{2}{4}$

$-t=\ln\left(\frac{2}{4}\right)$

$t=-\ln\left(\frac{2}{4}\right)=0.693$

Differential Equations: Formulas for Ch3 Test

Unrestricted Population Growth

$$\frac{dP}{dt} = kP \quad solution:\ P(t) = P_0 e^{kt}$$

Continuously Compounded Interest

$$\frac{dA}{dt} = kA \quad solution:\ A(t) = Pe^{rt}$$

$(P = Principal,\ r = annual\ interest\ rate)$

Radioactive Decay

$$\frac{dQ}{dt} = kQ \quad solution:\ Q(t) = Q_0 e^{kt}$$

$(halflife = time\ for\ quantity\ to\ be\ cut\ in\ half)$

$\left(\begin{array}{l} carbon\ dating.... \\ halflife\ of\ C-14 = 5600\ years \end{array}\right)$

Newton's Law of Cooling / Warming

$$\frac{dT}{dt} = k(T - T_m) \quad solution:\ T(t) = T_m + Ce^{kt}$$

Falling Masses

$(assuming\ positive\ direction\ is\ down)$

$air - resistance\ proportion\ to\ velocity...$

$$m\frac{dv}{dt} = mg - kv$$

$air - resistance\ proportion\ to\ (velocity)^2 ...$

$$m\frac{dv}{dt} = mg - kv^2$$

$m = mass\ (kg\ or\ slugs)$

$g = gravity\ constant\ (9.81\ m/s^2\ or\ 32\ ft/s^2)$

$v = velocity$

Growth limited by environment

$logistic\ model:$

$$\frac{dP}{dt} = P(a - bP)$$

$a, b\ are\ constants\ unique\ to\ each\ environment$

Electrical Circuits

LR series circuit

$$L\frac{di}{dt} + Ri = E(t) \quad solution:\ i(t) = \frac{E}{R} + Ce^{-\left(\frac{R}{L}\right)t}$$

$L = inductance\ (in\ Henrys)$

$R = resistance\ (in\ Ohms)$

$E = voltage\ (in\ Volts)$

RC series circuit

$$Ri + \frac{1}{C}q = E(t) \qquad i = \frac{dq}{dt}$$

$$R\frac{dq}{dt} + \frac{1}{C}q = E(t)$$

$$solution:\ q(t) = EC + C_1 e^{-\left(\frac{1}{RC}\right)t}$$

$$i(t) = \frac{dq}{dt} = -\left(\frac{1}{RC}\right)C_1 e^{-\left(\frac{1}{RC}\right)t}$$

$C = capacitance\ (in\ Farads)$

$R = resistance\ (in\ Ohms)$

$E = voltage\ (in\ Volts)$

$[C_1\ is\ the\ integration\ constant]$

Integrals you may need

$$\int \frac{1}{a^2+u^2}\,du = \frac{1}{a}\arctan\left(\frac{u}{a}\right)+C$$

$$\int \tan u\,du = -\ln\left|\cos u\right|+C$$