DiffEq - Ch 1-2 - Extra Practice

1.1

State the order of the given ordinary differential equation. Determine whether the equation is linear or nonlinear.

#1b. $x\dfrac{d^3y}{dx^3}-\left(\dfrac{dy}{dx}\right)^4+y=0$

$\boxed{\text{3rd order, nonlinear}}$

$(\ )^4$

#2b. $\dfrac{d^2u}{dr^2}+\dfrac{du}{dr}+u=\cos(r+u)$

$\boxed{\text{2nd order, nonlinear}}$

(cos)

#3b. $\dfrac{d^2R}{dt^2}=-\dfrac{k}{R^2}$

$\boxed{\text{2nd order, nonlinear}}$

(R^2)

#4b. $\ddot{x}-\left(1-\dfrac{\dot{x}^2}{3}\right)\dot{x}+x=0$

$\boxed{\text{2nd order, nonlinear}}$

$(\dot{x}^2)$

#5b. Determine whether

$udv+\left(v+uv-ue^u\right)du=0$

(i) is linear in v?

$\dfrac{udv}{du}+\dfrac{(v+uv-ue^u)du}{du}=\dfrac{0}{du}$

$(u)\dfrac{dv}{du}+v+uv-ue^u=0$

$(u)\dfrac{dv}{du}+(1+u)v=ue^u$

$\boxed{\text{linear}}$

(ii) is linear is u?

$\dfrac{udv}{dv}+\dfrac{(v+uv-ue^u)du}{dv}=\dfrac{0}{dv}$

$u+(v+uv-ue^u)\dfrac{du}{dv}=0$

$(v+uv-ue^u)\dfrac{du}{dv}=-u$

$\boxed{\text{nonlinear}}$

#6b. Verify that $y=e^{3x}\cos(2x)$ is an explicit solution of $y''-6y'+13y=0$ (don't worry about interval of definition).

$y=e^{3x}\cos(2x)$

$y'=(e^{3x})(-2\sin(2x))+(3e^{3x})(\cos 2x)$

$=-2e^{3x}\sin(2x)+3e^{3x}\cos(2x)$

$y''=(-2e^{3x})(2\cos(2x))+(\sin(2x))(-6e^{3x})$

$+(3e^{3x})(-2\sin(2x))+(\cos(2x))(9e^{3x})$

$y''-6y'+13y\overset{?}{=}0$

$[-4e^{3x}\cos 2x-6e^{3x}\sin 2x-6e^{3x}\sin 2x+9e^{3x}\cos 2x]$

$-6[-2e^{3x}\sin 2x+3e^{3x}\cos 2x]+13[e^{3x}\cos 2x]\overset{?}{=}0$

$(-4+9-18+13)e^{3x}\cos 2x+(-6-6+12)e^{3x}\sin 2x\overset{?}{=}0$

$0+0=0$

✓

#7b. Verify that $y = \dfrac{1}{4-x^2}$ is an explicit solution of $y' = 2xy^2$

Then give at least one interval I of definition for this solutions.

$y = (4-x^2)^{-1}$

$y' = -(4-x^2)^{-2}(-2x) = \dfrac{2x}{(4-x^2)^2}$

$y' \stackrel{?}{=} 2xy^2$

$\dfrac{2x}{(4-x^2)^2} \stackrel{?}{=} 2x\left(\dfrac{1}{4-x^2}\right)^2$

$\dfrac{2x}{(4-x^2)^2} = \dfrac{2x}{(4-x^2)^2}$ ✓

avoid divide by 0: $4-x^2=0$, $x^2=4$, $x=\pm 2$

y domain $(-\infty,-2)\cup(-2,2)\cup(2,\infty)$

y' has same $\dfrac{1}{(4-x^2)^2}$ so for diffeq too, interval is

$\boxed{(-\infty,-2)\cup(-2,2)\cup(2,\infty)}$

#8b. Verify that $-2x^2y + y^2 = 1$ is an implicit solution of $2xy\,dx + \left(x^2 - y\right)dy = 0$

Then find at least one explicit solution $y = \phi(x)$, and graph this solutions. Finally, give an interval I of definition for your solution.

implicit differentiation of solution...

$\frac{d}{dx}[-2x^2y] + \frac{d}{dx}[y^2] = \frac{d}{dx}[1]$

$(-2x^2)\frac{d}{dx}[y] + (y)\frac{d}{dx}[-2x^2] + \frac{d}{dx}[y^2] = \frac{d}{dx}[1]$

$(-2x^2)(1\frac{dy}{dx}) + y(-4x) + (2y\frac{dy}{dx}) = 0$

$-2x^2\frac{dy}{dx} - 4xy + 2y\frac{dy}{dx} = 0$

$(2y-2x^2)\frac{dy}{dx} = 4xy$, $\frac{dy}{dx} = \dfrac{4xy}{2(y-x^2)}$

$\frac{dy}{dx} = \dfrac{2xy}{y-x^2}$, $2xy\,dx = (y-x^2)dy$

$2xy\,dx + (x^2-y)dy = 0$ ✓

to find explicit solution, solve given solution for y:

$-2x^2y + y^2 = 1$ too hard, so solve for x instead:

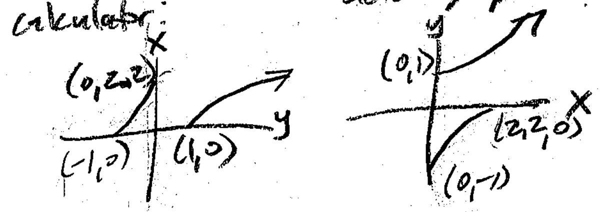
$-2x^2y = 1-y^2$, $x^2y = \dfrac{1-y^2}{-2} = \dfrac{y^2-1}{2}$

$x^2 = \dfrac{y^2-1}{2y}$, $x = \pm\sqrt{\dfrac{y^2-1}{2y}}$

$x(y) = \pm\sqrt{\dfrac{y^2-1}{2y}}$

Calculator to graph (then swap x-y)

calculator: actual graph



$\boxed{\text{Interval is either } (0,2.2) \text{ for } -y \text{ values; } (0,\infty) \text{ for } +y \text{ values}}$

#9b. Find values of m so that the function $y = x^m$ is a solution of $x^2y'' - 7xy' + 15y = 0$

$y = x^m$

$y' = mx^{m-1}$

$y'' = m(m-1)x^{m-2}$

$x^2y'' - 7xy' + 15y = 0$

$x^2[m(m-1)x^{m-2}] - 7x[mx^{m-1}] + 15[x^m] = 0$

$m(m-1)\, x^2 x^{m-2} - 7xm\, x^{m-1} + 15x^m = 0$

$(m^2 - m)x^m - (7m)x^m + 15x^m = 0$

$[(m^2-m) - (7m) + (15)]\, x^m = 0$

if this not zero, then

.. this must be zero

$m^2 - m - 7m + 15 = 0$

$m^2 - 8m + 15 = 0$

$(m-3)(m-5) = 0$

$\boxed{m = 3,\ m = 5}$

(corresponding solutions are:
$y = x^3$ and $y = x^5$)

#10b. Verify that the pair of functions…

$$x = \cos 2t + \sin 2t + \frac{1}{5}e^t$$

$$y = -\cos 2t - \sin 2t - \frac{1}{5}e^t$$

is a solution of the system of differential equations…

$$\begin{cases} ① \ \dfrac{d^2x}{dt^2} = 4y + e^t \\ ② \ \dfrac{d^2y}{dt^2} = 4x - e^t \end{cases}$$

$x = \cos 2t + \sin 2t + \frac{1}{5}e^t$ $\quad y = -\cos 2t - \sin 2t - \frac{1}{5}e^t$

$\frac{dx}{dt} = -2\sin 2t + 2\cos 2t + \frac{1}{5}e^t$ $\quad \frac{dy}{dt} = 2\sin 2t - 2\cos 2t - \frac{1}{5}e^t$

$\frac{d^2x}{dt^2} = -4\cos 2t - 4\sin 2t + \frac{1}{5}e^t$ $\quad \frac{d^2y}{dt^2} = 4\cos 2t + 4\sin 2t - \frac{1}{5}e^t$

① $\frac{d^2x}{dt^2} \stackrel{?}{=} 4y + e^t$

$-4\cos 2t - 4\sin 2t + \frac{1}{5}e^t \stackrel{?}{=} 4\left(-\cos 2t - \sin 2t - \frac{1}{5}e^t\right) + e^t$

$-4\cos 2t - 4\sin 2t + \frac{1}{5}e^t \stackrel{?}{=} -4\cos 2t - 4\sin 2t + \frac{1}{5}e^t$

✓

② $\frac{d^2y}{dt^2} \stackrel{?}{=} 4x - e^t$

$4\cos 2t + 4\sin 2t - \frac{1}{5}e^t \stackrel{?}{=} 4\left(\cos 2t + \sin 2t + \frac{1}{5}e^t\right) - e^t$

$4\cos 2t + 4\sin 2t - \frac{1}{5}e^t \stackrel{?}{=} 4\cos 2t + 4\sin 2t - \frac{1}{5}e^t$

✓

1.2

#1b. $y = \dfrac{1}{1+C_1e^{-x}}$ is a one-parameter family of solutions of the first-order DE $y' = y - y^2$. Find a solution of the first-order Initial Value Problem given initial condition $y(-1) = 2$.

$$y = \frac{1}{1+c_1e^{-x}}$$

$$(2) = \frac{1}{1+c_1e^{-(-1)}}$$

$$2 = \frac{1}{1+c_1e^1} = \frac{1}{1+c_1e}$$

$$2(1+c_1e) = 1$$

$$2 + 2c_1e = 1$$

$$2c_1e = -1$$

$$c_1 = \frac{-1}{2e}$$

$$\boxed{y = \frac{1}{1-\left(\frac{1}{2e}\right)e^{-x}}}$$

#2b. $y = \dfrac{1}{x^2+c}$ is a one-parameter family of solutions of the first-order DE $y' + 2xy^2 = 0$. Find a solution of the first-order Initial Value Problem given initial condition $y(0) = 1$. Also, give the largest interval I over which the solution is defined.

$$1 = \frac{1}{(0)^2+c} = \frac{1}{c}$$

$$c = 1$$

$$\boxed{y = \frac{1}{x^2+1}}$$

divide by zero?

$x^2+1 = 0$

(nowhere)

so for y, $(-\infty, \infty)$

for DE: $y = (x^2+1)^{-1}$

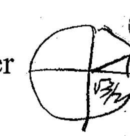
$$y' = -(x^2+1)^{-2}(2x) = \frac{-2x}{(x^2+1)^2}$$

DE: $\dfrac{-2x}{(x^2+1)^2} + 2x\left(\dfrac{1}{x^2+1}\right)^2 = 0$

undefined where $x^2+1=0$ (nowhere)

so $\boxed{I: (-\infty, \infty)}$

#3b. $x = C_1 \cos t + C_2 \sin t$ is a two-parameter family of solutions of the second-order DE $x'' + x = 0$. Find a solution of the second-order Initial Value Problem given initial conditions $x\left(\dfrac{\pi}{6}\right) = \dfrac{1}{2}$, $x'\left(\dfrac{\pi}{6}\right) = 0$.



$x = C_1\cos t + C_2\sin t$ $\qquad$ $x' = -C_1\sin t + C_2\cos t$

$x(\pi/6) = \frac{1}{2}$ $\qquad$ $x'(\pi/6) = 0$

$\frac{1}{2} = C_1\cos\pi/6 + C_2\sin\pi/6$ $\qquad$ $0 = -C_1\sin\pi/6 + C_2\cos\pi/6$

$\frac{1}{2} = C_1\left(\frac{\sqrt{3}}{2}\right) + C_2\left(\frac{1}{2}\right)$ $\qquad$ $0 = -C_1\left(\frac{1}{2}\right) + C_2\left(\frac{\sqrt{3}}{2}\right)$

$\sqrt{3}C_1 + C_2 = 1$ $\qquad$ $-\frac{1}{2}C_1 + \frac{\sqrt{3}}{2}C_2 = 0$

system:
$$\left[\begin{array}{cc|c} \sqrt{3} & 1 & 1 \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \end{array}\right]$$

rref
$$\left[\begin{array}{cc|c} 1 & 0 & .4330127019 \\ 0 & 1 & \frac{1}{4} \end{array}\right] \left(\frac{\sqrt{3}}{4}\right)$$

$C_1 = \frac{\sqrt{3}}{4}$, $C_2 = \frac{1}{4}$

$$\boxed{x = \frac{\sqrt{3}}{4}\cos t + \frac{1}{4}\sin t}$$

#4b. $y = C_1e^x + C_2e^{-x}$ is a two-parameter family of solutions of the second-order DE $y'' - y = 0$. Find a solution of the second-order Initial Value Problem given initial conditions $y(-1) = 5,\ y'(-1) = -5$.

$y = C_1e^x + C_2e^{-x} \qquad y' = C_1e^x - C_2e^{-x}$

$y(-1) = 5 \qquad y'(-1) = -5$

$5 = C_1e^{(-1)} + C_2e^{-(-1)} \qquad -5 = C_1e^{-1} - C_2e^{-(-1)}$

$5 = \frac{1}{e}C_1 + eC_2 \qquad -5 = \frac{1}{e}C_1 - eC_2$

$\frac{1}{e}C_1 + eC_2 = 5 \qquad \frac{1}{e}C_1 - eC_2 = -5$

System: $\left(\frac{1}{e}C_1 + eC_2 = 5\right)$
$-\left(\frac{1}{e}C_1 - eC_2 = -5\right)$

$2eC_2 = 10$

$C_2 = \frac{10}{2e} = \frac{5}{e}$

$\frac{1}{e}C_1 + e\left(\frac{5}{e}\right) = 5$

$\frac{1}{e}C_1 + 5 = 5$

$\frac{1}{e}C_1 = 0$

$C_1 = 0$

$y = (0)e^x + \left(\frac{5}{e}\right)e^{-x}$

$\boxed{y = \left(\frac{5}{e}\right)e^{-x}}$

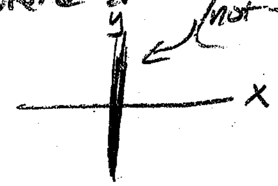
#5b. Determine a region of the xy-plane for which the given differential equation would have a unique solution whose graph passes through a point (x_0, y_0) in the region. $x\frac{dy}{dx} = y$

$\frac{dy}{dx} = \frac{y}{x} = f(x,y)$ Theorem: unique sol'n where $f(x,y)$ & $\frac{\partial f}{\partial y}$ are defined

$\frac{\partial f}{\partial y} = \frac{1}{x}$ avoid $x=0$

$\frac{dy}{dx} = f = \frac{y}{x}$ avoid $x=0$

There will be a unique solution everywhere except where $x=0$ (not to points on the y-axis)



#6b. Determine whether the differential equation $y' = \sqrt{y^2 - 9}$ possesses a unique solutions through the point $(2, -3)$.

$f = \sqrt{y^2-9} = (y^2-9)^{1/2}$

$\frac{\partial f}{\partial y} = \frac{1}{2}(y^2-9)^{-1/2}(2y) = \frac{y}{\sqrt{y^2-9}}$

at $(2,-3)$

$f = \sqrt{(-3)^2-9} = \sqrt{0}$ defined

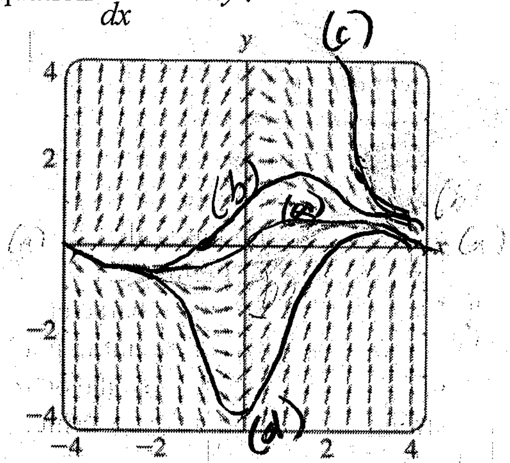
$\frac{\partial f}{\partial y} = \frac{-3}{\sqrt{(-3)^2-9}} = \frac{-3}{\sqrt{0}} = \frac{-3}{0}$ undefined

So there $\boxed{\text{is not}}$ guaranteed to be a unique solution through $(2,-3)$

(there may be no solution, one solution, or multiple solution curves)

2.1

#1b. A direction field is given for the differential equation $\frac{dy}{dx} = 1 - xy$:



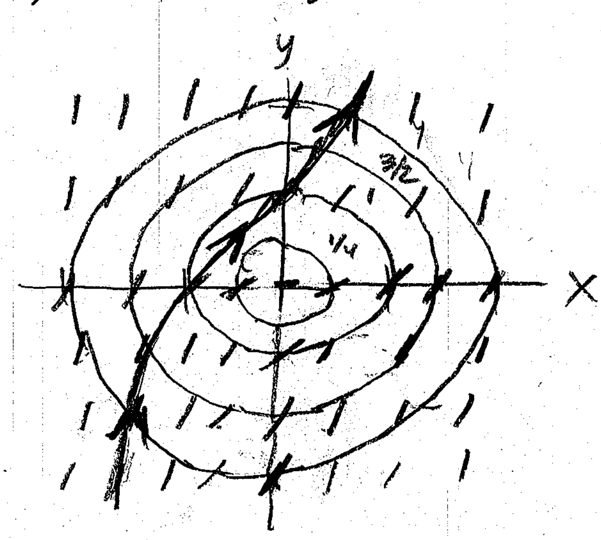
Without using a calculator, sketch an approximate solution curve on the direction field that passes through each of the indicated points.

(a) $y(0) = 0$ (b) $y(-1) = 0$

(c) $y(2) = 2$ (d) $y(0) = -4$

#2b. For the differential equation $\frac{dy}{dx} = x^2 + y^2$, sketch a few isoclines $\left(f(x,y) = c\right)$ for $c = \frac{1}{4}$, $c = 1$, $c = \frac{9}{4}$, $c = 4$. Then, construct a direction field by drawing lineal elements with the appropriate slope to match each isocline. Finally, use this rough direction field to sketch an approximate solution curve with the initial condition $y(0) = 1$.

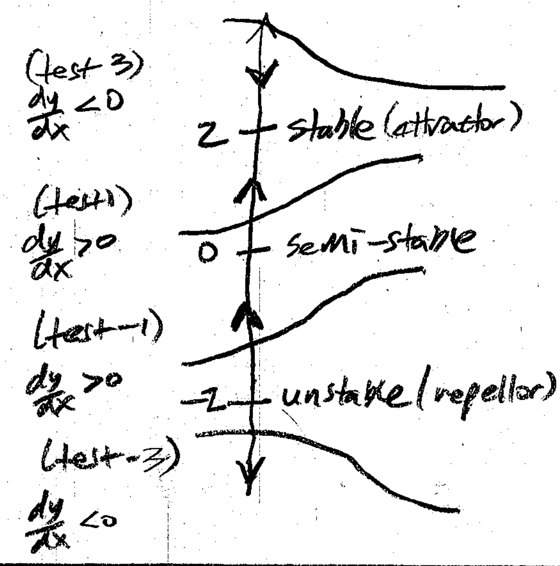
$x^2+y^2=\frac{1}{4}$ —— $x^2+y^2=4$



#3b. For the autonomous, first-order differential equation $\frac{dy}{dx} = y^2\left(4 - y^2\right)$, find the critical points and phase portrait. Then, classify each critical point as asymptotically stable, unstable, or semi-stable. Finally, sketch typical solution curves in the regions in the xy-plan determined by the graphs of the equilibrium solutions.

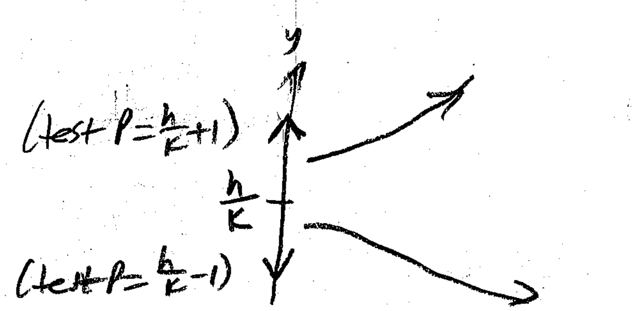
$y^2(4-y^2) = -y^2(y^2-4) = -y^2(y-2)(y+2)$

critical: $y=0$, $y=2$, $y=-2$



Extra #4b. A population model is given by $\frac{dP}{dt} = kP - h$, where h and k are positive constants. For what initial values $P(0) = P_0$ does this model predict that the population will go extinct?

critical: $kP-h=0$, $kP=h \Rightarrow P=\frac{h}{k}$



$\left.\frac{dy}{dx}\right|_{P=\frac{h}{k}+1} = k\left(\frac{h}{k}+1\right)-h = h+k-h = k > 0$

$\left.\frac{dy}{dx}\right|_{P=\frac{h}{k}-1} = k\left(\frac{h}{k}-1\right)-h = h-k-h = -k < 0$

$\boxed{\text{Extinction for any } P_0 < \frac{h}{k}}$

2.2

#1b. Solve by separation of variables:

$$\frac{dy}{dx} = (x+1)^2 \qquad \int dy = \int (x+1)^2 dx \qquad u = x+1$$

$$\frac{du}{dx} = 1$$

$$\int dy = \int u^2 du \qquad du = dx$$

$$y = \frac{1}{3}u^3 + c$$

$$\boxed{y = \frac{1}{3}(x+1)^3 + c}$$

#2b. Solve by separation of variables:

$$dy - (y-1)^2 dx = 0$$

$$dy = (y-1)^2 dx$$

$$\int (y-1)^{-2} dy = \int dx$$

$$u = y-1 \qquad \int u^{-2} du = \int dx$$

$$\frac{du}{dy} = 1 \qquad -\frac{1}{u} = x + c$$

$$du = dy \qquad -\frac{1}{y-1} = x + c$$

$$\frac{1}{y-1} = -x - c, \quad y - 1 = \frac{1}{-x-c} = -\frac{1}{x+c}$$

$$\boxed{y = 1 - \frac{1}{x+c}}$$

#3b. Solve by separation of variables:

$$\frac{dy}{dx} + 2xy^2 = 0$$

$$dy = -2xy^2 dx$$

$$\int y^{-2} dy = \int -2x\,dx$$

$$-\frac{1}{y} = -x^2 + C_1$$

$$\frac{1}{y} = x^2 + c$$

$$\boxed{y = \frac{1}{x^2 + c}}$$

#4b. Solve by separation of variables:

$$e^x y \frac{dy}{dx} = e^{-y} + e^{-2x-y}$$

$$e^x y\,dy = e^{-y} dx + e^{-2x} e^{-y} dx$$

$$e^x y\,dy = e^{-y}(1 + e^{-2x}) dx$$

$$\frac{y}{e^{-y}} dy = \frac{1 + e^{-2x}}{e^x} dx$$

$$\int y e^y dy = \int (e^{-x} + e^{-3x}) dx$$

continued ...

#5b. Solve by separation of variables:

$$\frac{dy}{dx} = \left(\frac{2y+3}{4x+5}\right)^2 = \frac{(2y+3)^2}{(4x+5)^2}$$

$$\int (2y+3)^{-2} dy = \int (4x+5)^{-2} dx$$

$$u = 2y+3 \qquad u = 4x+5$$

$$\frac{du}{dy} = 2 \qquad \frac{du}{dx} = 4$$

$$du = 2dy \qquad du = 4dx$$

$$dy = \frac{1}{2} du \qquad dx = \frac{1}{4} du$$

$$\frac{1}{2}\int u^{-2} du = \frac{1}{4}\int u^{-2} du$$

$$\boxed{-\frac{1}{2(2y+3)} = -\frac{1}{4(4x+5)} + C_1}$$

could solve this one if we wanted to...

$$\frac{2}{2y+3} = \frac{1}{4x+5} + \frac{c(4x+5)}{4x+5} = \frac{4cx+5c}{4x+5}$$

$$\frac{2y+3}{2} = \frac{4x+5}{4cx+5c}$$

$$2y+3 = \frac{8x+10}{4cx+5c}$$

$$2y = \frac{8x+10}{4cx+5c} - 3$$

$$\boxed{y = \frac{4x+5}{4cx+5c} - \frac{3}{2}}$$

4b continued...

$\int y e^y dy = \int e^{-x} dx + \int e^{-3x} dx$

by parts

$u = y \quad dv = e^y dy$

$\frac{du}{dy} = 1 \quad \int dv = \int e^y dy$

$du = dy \quad v = e^y$

$uv - \int v du$

$ye^y - \int e^y dy$

$$\boxed{ye^y - e^y = -e^{-x} - \frac{1}{3}e^{-3x} + C}$$

#6b. Solve by separation of variables:
$\sin 3x\, dx + 2y \cos^3(3x)\, dy = 0$

$$2y\cos^3(3x)dy = -\sin(3x)dx$$

$$\int 2y\, dy = -\int \frac{\sin(3x)}{\cos^3(3x)}dx \qquad u = \cos(3x)$$

$$\frac{du}{dx} = -\sin(3x)\cdot 3$$

$$du = -3\sin(3x)dx$$

$$-\sin(3x)dx = \tfrac{1}{3}du$$

$$2\int y\,dy = \tfrac{1}{3}\int u^{-3}du$$

$$y^2 = -\tfrac{1}{6}u^{-2} + c$$

$$\boxed{y^2 = \frac{-1}{6\cos^2(3x)} + c}$$

#7b. Find an explicit solution of the given initial-value problem: $x^2\dfrac{dy}{dx} = y - xy,\quad y(-1) = -1$

$$x^2 dy = y(1-x)dx$$

$$\tfrac{1}{y}dy = \frac{1-x}{x^2}dx = (x^{-2} - \tfrac{1}{x})dx$$

$$\int \tfrac{1}{y}dy = \int x^{-2}dx - \int \tfrac{1}{x}dx$$

$$\ln|y| = -\tfrac{1}{x} - \ln|x| + c$$

$$\ln|-1| = -\tfrac{1}{(-1)} - \ln|-1| + c$$

$$\ln(1) = 1 - \ln(1) + c$$

$$0 = 1 - 0 + c$$

$$c = -1$$

$$\ln|y| = -\tfrac{1}{x} - \ln|x| - 1$$

$$e^{\ln|y|} = e^{(-\frac{1}{x} - \ln|x| - 1)} = e^{-\frac{1}{x}}e^{-\ln|x|}e^{-1}$$

$$|y| = e^{-1/x}e^{\ln|x^{-1}|}e^{-1} = e^{-1/x}x^{-1}e^{-1}$$

$$|y| = e^{(-\frac{1}{x}-1)}\tfrac{1}{x}$$

$$\boxed{y = \frac{e^{(-\frac{1}{x}-1)}}{x}}$$

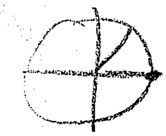
#7c. Find an explicit solution of the given initial-value problem:

$\sqrt{1-y^2}\,dx - \sqrt{1-x^2}\,dy = 0,\quad y(0) = \dfrac{\sqrt{3}}{2}$ $\qquad x = 0,\ y = \frac{\sqrt{3}}{2}$

$$\sqrt{1-y^2}\,dx = \sqrt{1-x^2}\,dy$$

$$\int \frac{1}{\sqrt{1-x^2}}dx = \int \frac{1}{\sqrt{1-y^2}}dy$$

$$\arcsin x = \arcsin y + c$$



$$\arcsin(0) = \arcsin(\tfrac{\sqrt{3}}{2}) + c$$

$$0 = \tfrac{\pi}{3} + c$$

$$c = -\pi/3$$

$$\arcsin x = \arcsin y - \pi/3$$

$$\arcsin y = \arcsin x + \pi/3$$

$$\boxed{y = \sin(\arcsin x + \pi/3)}$$

can simplify with a trig identity:

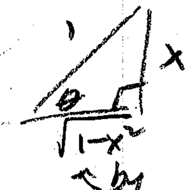
$$\sin(u+v) = \sin u\cos v + \cos u \sin v$$

$$y = \sin(\arcsin x + \tfrac{\pi}{3})$$

$$y = \sin(\arcsin x)\cos(\pi/3) + \cos(\arcsin x)\sin(\pi/3)$$

for $\cos(\underbrace{\arcsin x}_{})$ draw a triangle:

$$\sin\theta = \frac{x}{1}\ \frac{opp}{hyp}$$



$$\cos(\theta) = \frac{adj}{hyp}$$

$$\cos(\arcsin x) = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}$$

So:

$$\boxed{y = x(\tfrac{1}{2}) + \sqrt{1-x^2}(\tfrac{\sqrt{3}}{2})}$$

2.3

#1b. Find the general solutions of the differential equation $\frac{dy}{dx}+y=e^{3x}$. Then, give the largest interval I over which the general solution is defined. Finally, determine whether there are any transient terms in the general solution.

$$\frac{dy}{dx}+1y=e^{3x} \qquad P(x)=1$$

$$\text{I.F.}=e^{\int 1dx}=e^{x}$$

$$\frac{d}{dx}[e^{x}y]=e^{3x}e^{x}=e^{4x}$$

$$e^{x}y=\int e^{4x}dx \qquad y=\frac{1}{4}\frac{e^{4x}}{e^{x}}+\frac{c}{e^{x}}$$

$$e^{x}y=\frac{1}{4}e^{4x}+c \qquad \boxed{y=\frac{1}{4}e^{3x}+Ce^{-x}}$$

$\boxed{I: (-\infty,\infty)}$ $\boxed{\text{transient term: } Ce^{-x}}$

#2b. Find the general solutions of the differential equation $x^2y'+x(x+2)y=e^{x}$. Then, give the largest interval I over which the general solution is defined. Finally, determine whether there are any transient terms in the general solution.

$$y'+\frac{x+2}{x}y=x^{-2}e^{x} \qquad P(x)=\frac{x+2}{x}=1+2\frac{1}{x}$$

$$\text{I.F.}=e^{\int(1+2\frac{1}{x})dx}$$

$$=e^{x+2\ln x}=e^{x}e^{\ln x^2}$$

$$=x^2e^{x}$$

$$x^2e^{x}y=\int x^{-2}e^{x}x^2e^{x}dx=\int e^{2x}dx$$

$$x^2e^{x}y=\frac{1}{2}e^{2x}+c$$

$$y=\frac{1}{2}\frac{e^{2x}}{x^2e^{x}}+\frac{c}{x^2e^{x}}$$

$$\boxed{y=\frac{1}{2x^2}e^{x}+Cx^{-2}e^{-x}}$$

$\boxed{I: (0,\infty)}$

$\boxed{Cx^{-2}e^{-x} \text{ is a transient term}}$

#3b. Find the general solutions of the differential equation $\frac{dP}{dt}+2tP=P+4t-2$. Then, give the largest interval I over which the general solution is defined. Finally, determine whether there are any transient terms in the general solution.

$$\frac{dP}{dt}+(2t-1)P=4t-2 \qquad P(t)=2t-1$$

$$\text{I.F.}=e^{\int 2t-1dt}$$

$$=e^{t^2-t}$$

$$e^{(t^2-t)}P=\int(4t-2)e^{(t^2-t)}dt$$

$$=2\int(2t-1)e^{(t^2-t)}dt$$

$$u=t^2-t$$

$$\frac{du}{dt}=2t-1$$

$$du=(2t-1)dt$$

$$=2\int e^{u}du$$

$$=2e^{u}+c$$

$$e^{(t^2-t)}P=2e^{(t^2-t)}+c$$

$$P=2\frac{e^{(t^2-t)}}{e^{(t^2-t)}}+\frac{c}{e^{(t^2-t)}}$$

$$\boxed{P=2+Ce^{-(t^2-t)}}$$

$\boxed{I: (-\infty,\infty)}$

$\boxed{\text{transient term: } Ce^{-(t^2-t)}}$

#4b. Solve the initial-value problem:

$$y\frac{dx}{dy}-x=2y^2, \quad y(1)=5.$$

$$\frac{dx}{dy}-\frac{1}{y}x=2y \qquad P(y)=-\frac{1}{y}$$

$$I.F.=e^{-\int\frac{1}{y}dy}=e^{-\ln y}$$

$$=e^{\ln(y^{-1})}=y^{-1}=\frac{1}{y}$$

$$\frac{1}{y}x=\int 2y\frac{1}{y}\,dy=\int 2\,dy$$

$$\frac{1}{y}x=2y+c$$

$$x=2y^2+Cy$$

$$y(1)=5 \qquad x=1, \; y=5$$

$$(1)=2(5)^2+C(5)$$

$$5C+50=1$$

$$5C=-49$$

$$C=\frac{-49}{5}$$

$$\boxed{x=2y^2-\frac{49}{5}y}$$

#5b. Solve the initial-value problem:

$$\frac{dT}{dt}=k(T-T_m), \quad T(0)=T_0.$$

$[k, T_m, \text{ and } T_0 \text{ are constants}]$

$$\frac{dT}{dt}=kT-kT_m$$

$$\frac{dT}{dt}-kT=-kT_m \qquad P(t)=-k$$

$$I.F.=e^{\int -k\,dt}$$

$$=e^{-kt}$$

$$e^{-kt}T=\int -kT_m e^{-kt}\,dt$$

$$=-kT_m\frac{e^{-kt}}{-k}+c$$

$$e^{-kt}T=T_m e^{-kt}+c$$

$$T=T_m\frac{e^{-kt}}{e^{-kt}}+\frac{c}{e^{-kt}}$$

$$T=T_m+Ce^{kt}$$

$$T(0)=T_0 \qquad t=0,\; T=T_0$$

$$T_0=T_m+Ce^0, \qquad C=T_0-T_m$$

$$\boxed{T(t)=T_m+(T_0-T_m)e^{kt}}$$

2.4

#1b. Determine whether the given differential equation is exact. If it is exact, then solve it.

$\underbrace{(5x+4y)}_{M}\,dx+\underbrace{(4x-8y^3)}_{N}\,dy=0$

$\frac{\partial M}{\partial y}=4, \quad \frac{\partial N}{\partial x}=4$ exact ✓

$\frac{\partial f}{\partial x}=5x+4y$

$f=\int(5x+4y)dx=\frac{5}{2}x^2+4xy+g(y)$

$\frac{\partial f}{\partial y}=4x+g'(y)\overset{\text{must}}{=}4x-8y^3$

$g'(y)=-8y^3$

$g(y)=\int-8y^3dy=-2y^4$

$\boxed{\frac{5}{2}x^2+4xy-2y^4=c}$

#2b. Determine whether the given differential equation is exact. If it is exact, then solve it.

$\underbrace{(\sin y-y\sin x)}_{M}\,dx+\underbrace{(\cos x+x\cos y-y)}_{N}\,dy=0$

$\frac{\partial M}{\partial y}=\cos y-\sin x, \quad \frac{\partial N}{\partial x}=-\sin x+\cos y$, exact ✓

$\frac{\partial f}{\partial x}=\sin y-y\sin x$

$f=\int(\sin y-y\sin x)dx=x\sin y+y\cos x+g(y)$

$\frac{\partial f}{\partial y}=x\cos y+\cos x+g'(y)\overset{\text{must}}{=}\cos x+x\cos y-y$

$g'(y)=-y$

$g(y)=\int-y\,dy=-\frac{1}{2}y^2$

$\boxed{x\sin y+y\cos x-\frac{1}{2}y^2=c}$

#3b. Determine whether the given differential equation is exact. If it is exact, then solve it.

$\underbrace{\left(y\ln y-e^{-xy}\right)}_{M}dx+\underbrace{\left(\frac{1}{y}+x\ln y\right)}_{N}dy=0$

$\frac{\partial M}{\partial y}=y\frac{1}{y}+\ln y(1)-e^{-xy}(-x)$

$=1+\ln y+xe^{-xy}$

$\frac{\partial N}{\partial x}=\ln y \quad \neq$

$\boxed{\text{not exact}}$

#4b. Determine whether the given differential equation is exact. If it is exact, then solve it.

$\frac{dy}{dx}=\frac{-x^3-y^3}{3xy^2}$

$(3xy^2)dy=(-x^3-y^3)dx$

$\underbrace{(x^3+y^3)}_{M}dx+\underbrace{(3xy^2)}_{N}dy=0$

$\frac{\partial M}{\partial y}=3y^2, \quad \frac{\partial N}{\partial x}=3y^2$ exact ✓

$\frac{\partial f}{\partial x}=x^3+y^3$

$f=\int(x^3+y^3)dx=\frac{1}{4}x^3+y^3x+g(y)$

$\frac{\partial f}{\partial y}=3y^2x+g'(y)\overset{\text{must}}{=}3xy^2$

$g'(y)=0$

$g(y)=\int 0\,dy=0$

$\boxed{\frac{1}{4}x^3+y^3x=c}$

#5b. Solve the initial-value problem:

$(4y+2t-5)\,dt+(6y+4t-1)\,dy=0, \quad y(-1)=2$

$M \qquad N$

$\frac{\partial M}{\partial y}=4, \quad \frac{\partial N}{\partial t}=4$ exact ✓

$\frac{\partial f}{\partial t}=4y+2t-5$

$f=\int(4y+2t-5)\,dt=4yt+t^2-5t+g(y)$

$\frac{\partial f}{\partial y}=4t+g'(y)\overset{\text{must}}{=}6y+4t-1$

$g'(y)=6y+4t-1-4t$

$g'(y)=6y-1$

$g(y)=\int(6y-1)\,dy=3y^2-y$

$4yt+t^2-5t+3y^2-y=C$

$y(-1)=2$

$t=-1, y=2$

$4(2)(-1)+(-1)^2-5(-1)+3(2)^2-(2)=C$

$-8+1+5+12-2=C$

$C=8$

$$\boxed{4yt+t^2-5t+3y^2-y=8}$$

#6b. Solve the given differential equation by first finding an integrating factor, then using it to convert the differential equation to exact form:

$(6xy)\,dx+(4y+9x^2)\,dy=0$

$M \qquad N$

$\frac{\partial M}{\partial y}=6x, \quad \frac{\partial N}{\partial x}=18x \neq$ not exact (yet)

$\frac{M_y-N_x}{N}=\frac{6x-18x}{4y+9x^2}=\frac{-12x}{4y+9x^2}$ ☹

$\frac{N_x-M_y}{M}=\frac{18x-6x}{6xy}=\frac{12x}{6xy}=\frac{2}{y}$ ☺

use $\frac{2}{y}$ ... $IF=e^{\int\frac{2}{y}dy}=e^{2\ln y}$

$=e^{\ln(y^2)}=y^2$

$y^2(6xy)dx+y^2(4y+9x^2)dy=0$

$(6xy^3)dx+(4y^3+9x^2y^2)dy=0$

$M \qquad N$

$\frac{\partial M}{\partial y}=18xy^2, \quad \frac{\partial N}{\partial x}=18xy^2$ Now exact ✓

$\frac{\partial f}{\partial x}=6xy^3$

$f=\int 6xy^3\,dx=3x^2y^3+g(y)$

$\frac{\partial f}{\partial y}=9x^2y^2+g'(y)\overset{\text{must}}{=}4y^3+9x^2y^2$

$g'(y)=4y^3$

$g(y)=\int 4y^3\,dy=y^4$

$$\boxed{3x^2y^3+y^4=C}$$

2.5

#1b. Solve the Bernoulli form differential equation by using an appropriate substitution: $\frac{dy}{dx} - y = e^x y^2$

$\frac{dy}{dx} - y = e^x y^2 \quad n=2$

$u = y^{1-n} = y^{1-2} = y^{-1} = \frac{1}{y} \rightarrow y = \frac{1}{u}$

$\frac{du}{dx} = \frac{du}{dy}\frac{dy}{dx}$

$\frac{du}{dx} = -y^{-2}\frac{dy}{dx}$

$\frac{du}{dx} = \frac{-1}{y^2}\frac{dy}{dx}, \quad \frac{dy}{dx} = -y^2\frac{du}{dx}$

$\frac{dy}{dx} - y = e^x y^2$

$-y^2\frac{du}{dx} - y = e^x y^2$

$(y = \frac{1}{u})$

$-\frac{1}{u^2}\frac{du}{dx} - \frac{1}{u} = e^x\left(\frac{1}{u^2}\right)$

$(-u^2)\left[-\frac{1}{u^2}\frac{du}{dx} - \frac{1}{u}\right] = \left[e^x\frac{1}{u^2}\right](-u^2)$

$\frac{du}{dx} + u = -e^x$

Linear, $P(x) = 1$

I.F. $= e^{\int 1dx} = e^x$

$e^x u = \int (-e^x)e^x dx = -\int e^{2x}dx$

$e^x u = -\frac{1}{2}e^{2x} + C$

$u = -\frac{1}{2}\frac{e^{2x}}{e^x} + \frac{C}{e^x} = -\frac{1}{2}e^x + Ce^{-x}$

$\frac{1}{y} = -\frac{1}{2}e^x + Ce^{-x}$

$$\boxed{y = \frac{1}{-\frac{1}{2}e^x + Ce^{-x}}}$$

#2b. Solve the Bernoulli form differential equation by using an appropriate substitution:

$x\frac{dy}{dx} - (1+x)y = xy^2$

$\frac{dy}{dx} - \frac{1+x}{x}y = y^2 \quad n=2$

$u = y^{1-n} = y^{1-2} = y^{-1} = \frac{1}{y}, \quad y = \frac{1}{u}$

$\frac{du}{dx} = \frac{du}{dy}\frac{dy}{dx}$

$\frac{du}{dx} = -y^{-2}\frac{dy}{dx} = \frac{-1}{y^2}\frac{dy}{dx}, \quad \frac{dy}{dx} = -y^2\frac{du}{dx}$

$\frac{dy}{dx} - \frac{1+x}{x}y = y^2$

$\frac{-y^2\frac{du}{dx}}{-y^2} - \frac{\frac{1+x}{x}y}{-y^2} = \frac{y^2}{-y^2}$

$\frac{du}{dx} + \frac{1+x}{x}\frac{1}{y} = -1$

Linear, $P(x) = \frac{1+x}{x} = \frac{1}{x} + 1$

I.F. $= e^{\int(\frac{1}{x}+1)dx} = e^{\ln x + x} = e^{\ln x}e^x$

$= xe^x$

$xe^x u = \int (-1)xe^x dx$

by parts: $w = -x \quad dv = e^x dx$

$\frac{dw}{dx} = -1 \quad \int dv = \int e^x dx$

$dw = -dx \quad v = e^x$

$wv - \int v\,dw$

$(-x)e^x - \int e^x(-dx)$

$-xe^x + \int e^x dx$

$xe^x u = -xe^x + e^x + C$

$u = \frac{-xe^x}{xe^x} + \frac{e^x}{xe^x} + \frac{C}{xe^x}$

$u = -1 + \frac{1}{x} + \frac{C}{xe^x}$

$\frac{1}{y} = -1 + \frac{1}{x} + \frac{C}{xe^x}$

$$\boxed{y = \frac{1}{-1 + \frac{1}{x} + \frac{C}{xe^x}}}$$

#3b. Solve the differential equation by using an appropriate substitution: $\frac{dy}{dx} = 2+\sqrt{y-2x+3}$

$u = y - 2x + 3$

$\frac{du}{dx} = \frac{\partial u}{\partial x}\frac{dx}{dx} + \frac{\partial u}{\partial y}\frac{dy}{dx}$

$\frac{du}{dx} = (-2)(1) + (1)\frac{dy}{dx} = -2 + \frac{dy}{dx}$

$\frac{dy}{dx} = \frac{du}{dx} + 2$

$\frac{dy}{dx} = 2 + (y-2x+3)^{1/2}$

$\frac{du}{dx} + 2 = 2 + u^{1/2}$

$\frac{du}{dx} = u^{1/2}$

separable

$\int u^{-1/2}du = \int dx$

$2u^{1/2} = x + C$

$\boxed{2\sqrt{y-2x+3} = x + C}$

#4b. Solve the differential equation by using an appropriate substitution: $\frac{dy}{dx} = \frac{1-x-y}{x+y} = \frac{1-(x+y)}{x+y}$

$u = x + y$

$\frac{du}{dx} = \frac{\partial u}{\partial x}\frac{dx}{dx} + \frac{\partial u}{\partial y}\frac{dy}{dx}$

$\frac{du}{dx} = (1)(1) + (1)\frac{dy}{dx} = 1 + \frac{dy}{dx}$

$\frac{dy}{dx} = \frac{du}{dx} - 1$

$\frac{dy}{dx} = \frac{1-(x+y)}{(x+y)}$

$\frac{du}{dx} - 1 = \frac{1-u}{u} = \frac{1}{u} - 1$

$\frac{du}{dx} = \frac{1}{u}$

separable

$\int u\,du = \int dx$

$\frac{1}{2}u^2 = x + C$

$\boxed{\frac{1}{2}(x+y)^2 = x + C}$

2.6

#1b. Use Euler's method to obtain a four-decimal approximation of $y(1.5)$ if $y(1)=1.4$ and $y'=(x-y)^2$. Use x-increments of 0.1.

(x,y)	$y_{n+1}=y_n+(x-y)^2 0.1$
$(1, 1.4)$	$y=1.4+(1-1.4)^2(0.1)$ $=1.416$
$(1.1, 1.416)$	$y=1.416+(1.1-1.416)^2(0.1)$ $=1.4259856$
$(1.2, 1.4259856)$	$y=1.426..+(1.2-1.426)^2 0.1$ $=1.431092549$
$(1.3, 1.431092549)$	$y=1.43109+(1.3-1.43109)^2 0.1$ $=1.4328110875$
$(1.4, 1.4328110875)$	$y=1.43281+(1.4-1.43281)^2 0.1$ $=1.432918732$
$(1.5, 1.432918732)$	

so $y(1.5)\approx 1.432918732$

$\boxed{1.4329}$

#2. Use Euler's method to obtain a four-decimal approximation of $y(1.5)$ if $y(1)=1$ and $y'=xy^2-\frac{y}{x}$. Use x-increments of 0.05.

(x,y)	$y_{n+1}=y_n+(xy^2-\frac{y}{x})0.05$
$(1,1)$	$y=1+(1\cdot1^2-\frac{1}{1})(0.05)$ $=1$
$(1.05, 1)$	$y=1+((1.05)(1)^2-\frac{1}{1.05})0.05$ $=1.004880952$
$(1.1, 1.004880952)$	$y=1.00488...+(1.1(1.00488...)^2-\frac{1.0488}{1.1})0.05$ $=1.01474276$
$(1.15, 1.01474276)$	$y=1.0147...+(1.15(1.0147)^2-\frac{1.0147}{1.15})0.05$ $=1.029831425$
$(1.2, 1.029831425)$	$y=1.0298..+(1.2(1.0298)^2-\frac{1.0298}{1.2})0.05$ $=1.050554948$
$(1.25, 1.050554948)$	$y=1.05055+(1.25(1.05055)^2-\frac{1.05055}{1.25})0.05$ $=1.077511856$
$(1.3, 1.077511856)$	$y=1.0775+(1.3(1.0775)^2-\frac{1.0775}{1.3})0.05$ $=1.111536159$
$(1.35, 1.111536159)$	$y=1.1115+(1.35(1.1115)^2-\frac{1.1115}{1.35})0.05$ $=1.153765256$
$(1.4, 1.153765256)$	$y=1.1537+(1.4(1.1537)^2-\frac{1.1537}{1.4})0.05$ $=1.205741553$
$(1.45, 1.205741553$	$y=1.2057+(1.45(1.2057)^2-\frac{1.2057}{1.45})0.05$ $=1.269565678$
$(1.5, 1.269565678)$	

so $y(1.5)\approx 1.269565678$

$\boxed{1.2696}$

Diff Eq Ch1-2 Test Review

#1. Classify each differential equation as 'separable', 'exact', 'linear' or 'Bernoulli', or 'Composite' form. Some equations may be more than one kind. <u>Do not solve the differential equations, just classify them so you can identify which methods could use used to solve them.</u>

(a) $\frac{dy}{dx} = \frac{x-y}{x}$

separable? $x\,dy = (x-y)dx$

not separable

linear?

in y: $\frac{dy}{dx} = \frac{x-y}{x} = 1 - \frac{y}{x} = 1 - \frac{1}{x}y$

$\frac{dy}{dx} + \frac{1}{x}y = 1$ $\boxed{\text{linear in } y}$

in x: $\frac{dx}{dy} = \frac{x}{x-y}$ can't write in form $\frac{dx}{dy} + P(y)x = Q(y)$

not linear in y

Bernoulli (in y)?

form $\frac{dx}{dy} + P(y)x = Q(y)x^n$ no

not Bernoulli

exact? $x\,dy = (x-y)dx$

$(x-y)dx + (-1)dy = 0$

$M \quad N$

$\frac{\partial M}{\partial y} = -1, \quad \frac{\partial N}{\partial x} = 0$ not exact

composite? $\frac{dy}{dx} = \frac{x-y}{x} = \frac{(x)-y}{(x)}$

not composite

$\boxed{\text{only linear in } y}$

(b) $\frac{dy}{dx} = \frac{1}{y-x}$

separable? $(y-x)dy = dx$ not separable

linear?

in y: $\frac{dy}{dx} = \frac{1}{y-x}$ can't put in form $\frac{dy}{dx} + P(x)y = Q(x)$

not linear in y

in x: $\frac{dx}{dy} = \frac{y-x}{1} = y - x$

$\frac{dx}{dy} + (1)x = y$ $\boxed{\text{linear in } x}$

Bernoulli in y? (no, Bernoulli is)

$\frac{dy}{dx} + P(x)y = Q(x)y^n$

not Bernoulli

exact?

$(y-x)dy = dx$

$(1)dx - (y-x)dy = 0$

$(1)dx + (x-y)dy = 0$

$M \quad N$

$\frac{\partial M}{\partial y} = 0, \quad \frac{\partial N}{\partial x} = 1$ not exact

composite? $\frac{dy}{dx} = \frac{1}{(y-x)}$

$u = y - x$

composite w $u = y - x$

$\boxed{\text{linear in } x\text{, or composite w/ } u = y - x}$

#1 continued...

(c) $(x+1)\frac{dy}{dx} = -y+10$

Separable?

$\frac{1}{-y+10}dy = \frac{1}{x+1}dx$

$\boxed{\text{separable}}$

linear?

in y: $\frac{dy}{dx} = \frac{-y+10}{x+1} = -\frac{1}{x+1}y + \frac{10}{x+1}$

$\frac{dy}{dx} + \left(\frac{1}{x+1}\right)y = \frac{10}{x+1}$ $\boxed{\text{linear in y}}$

in x: $\frac{dx}{dy} = \frac{x+1}{-y+10} = \left(\frac{1}{-y+10}\right)x + \frac{1}{-y+10}$

$\frac{dx}{dy} + \left(\frac{-1}{-y+10}\right)x = \frac{1}{-y+10}$ $\boxed{\text{linear in x}}$

Bernoulli? (not needed, linear in x & y)

exact? $(x+1)dy = (-y+10)dx$

$(x+1)dy - (-y+10)dx = 0$

$\underset{M}{(x+1)}dy + \underset{N}{(y-10)}dx = 0$

$\frac{\partial M}{\partial y} = 0, \frac{\partial N}{\partial x} = 0$ $\boxed{\text{exact}}$

Composite? $\frac{dy}{dx} = \frac{-y+10}{x+1}$

not composite

$\boxed{\text{Separable, exact, and linear in both x \& y}}$

(d) $\frac{dy}{dx} = \frac{1}{x(x-y)}$

Separable?

$x(x-y)dy = dx$

$(x^2-xy)dy = dx$

Not separable

linear?

in y: $\frac{dy}{dx} = \frac{1}{x(x-y)}$ can't put in form $\frac{dy}{dx} + P(x)y = Q(x)$

not linear in y

in x: $\frac{dx}{dy} = \frac{x(x-y)}{1} = x^2 - xy$

$\frac{dx}{dy} + yx = x^2$ not linear in x

but...

Bernoulli: $\frac{dx}{dy} + yx = (1)x^2$

is $\boxed{\text{Bernoulli-form in x with n=2}}$

exact?

$x(x-y)dy = dx$

$dx - (x^2-xy)dy = 0$

$\underset{M}{(1)}dx + \underset{N}{(xy-x^2)}dy = 0$

$\frac{\partial M}{\partial y} = 0, \frac{\partial N}{\partial x} = y-2x$ not exact

composite?

$\frac{dy}{dx} = \frac{1}{x(x-y)}$

not composite

$\boxed{\text{only Bernoulli form in x}}$

#1 continued...

(e) $\frac{dy}{dx}=\frac{y^2+y}{x^2+x}$

Separable? $\frac{1}{y^2+y}dy=\frac{1}{x^2+x}dx$

$\boxed{\text{Separable}}$

linear/Bernoulli?

in y: $\frac{dy}{dx}=\frac{y^2+y}{x^2+x}=\left(\frac{1}{x^2+x}\right)y^2+\left(\frac{1}{x^2+x}\right)y$

$\frac{dy}{dx}+\left(\frac{-1}{x^2+x}\right)y=\left(\frac{1}{x^2+x}\right)y^2$

not linear in y

but $\boxed{\text{Bernoulli in } y \text{ w/ } n=2}$

in x: $\frac{dx}{dy}=\frac{x^2+x}{y^2+y}=\left(\frac{1}{y^2+y}\right)x^2+\left(\frac{1}{y^2+y}\right)x$

$\frac{dx}{dy}+\left(\frac{-1}{y^2+y}\right)x=\left(\frac{1}{y^2+y}\right)x^2$

not linear in x

but $\boxed{\text{Bernoulli in } x \text{ w/ } n=2}$

exact? $(x^2+x)dy=(y^2+y)dx$

$(y^2+y)dx-(x^2+x)dy=0$

$\underbrace{(y^2+y)}_{M}dx+\underbrace{(-x^2-x)}_{N}dy=0$

$\frac{\partial M}{\partial y}=2y+1,\ \frac{\partial N}{\partial x}=-2x-1$

not exact

composite? $\frac{dy}{dx}=\frac{y^2+y}{x^2+x}$

not composite

$\boxed{\text{Separable, Bernoulli in x and y}}$

(f) $\frac{dy}{dx}=5y+y^2$

Separable?

$\frac{1}{5y+y^2}dy=dx$ $\boxed{\text{Separable}}$

linear/Bernoulli?

in y: $\frac{dy}{dx}=5y+y^2$

$\frac{dy}{dx}+(-5)y=(1)y^2$

not linear in y

but $\boxed{\text{Bernoulli in } y \text{ w/ } n=2}$

in x: $\frac{dx}{dy}=\frac{1}{5y+y^2}$

$\frac{dx}{dy}+(0)x=\frac{1}{5y+y^2}$ $\boxed{\text{linear in x}}$

exact? $dy=(5y+y^2)dx$

$\underbrace{(5y+y^2)}_{M}dx+\underbrace{(-1)}_{N}dy=0$

$\frac{\partial M}{\partial y}=5+2y,\ \frac{\partial N}{\partial x}=0$ not exact

composite?

$\frac{dy}{dx}=5y+y^2$ (no inside function)

not composite

$\boxed{\text{Separable, linear in x, Bernoulli in y}}$

#1 continued...

(g) $y\,dx = \left(y - xy^2\right) dy$

separable?

$dx = \frac{y-xy^2}{y} dy$

$dx = (1-xy)dy$

not separable

linear/Bernoulli?

in y: $\frac{dy}{dx} = \frac{y}{y-xy^2} = \frac{1}{1-xy}$

not linear or Bernoulli in y

in x: $\frac{dx}{dy} = \frac{y-xy^2}{y} = 1-xy$

$\frac{dx}{dy} + (y)x = 1$ $\boxed{\text{linear in x}}$

exact?

$(y)dx - (y-xy^2)dy = 0$

$\underbrace{(y)}_{M} dx + \underbrace{(xy^2-y)}_{N} dy = 0$

$\frac{\partial M}{\partial y} = 1, \quad \frac{\partial N}{\partial x} = y^2$ not exact

composite?

$\frac{dy}{dx} = \frac{y}{y-xy^2} = \frac{1}{1-xy}$

$1-xy$ is inside function

but u must be $u = Ax+By+C$

not composite

$\boxed{\text{linear in x}}$

(h) $xyy' + y^2 = 2x$

separable? $xy\frac{dy}{dx} + y^2 = 2x$

$xy\,dy + y^2 dx = 2x\,dx$

$xy\,dy = (2x-y^2)dx$ not separable

linear/Bernoulli?

in y: $\frac{dy}{dx} = \frac{2x-y^2}{xy} = \frac{2x}{xy} - \frac{y^2}{xy}$

$\frac{dy}{dx} = \frac{2}{y} - \frac{1}{x}y, \quad \frac{dy}{dx} + \left(\frac{1}{x}\right)y = \frac{2}{y} = 2y^{-1}$

not linear in y,

but $\boxed{\text{Bernoulli in y w/ n = -1}}$

in x: $\frac{dx}{dy} = \frac{xy}{2x-y^2}$ not linear, nor Bernoulli in x

exact? $xy\,dy = (2x-y^2)dx$

$\underbrace{(2x-y^2)}_{M}dx + \underbrace{(-xy)}_{N}dy = 0$

$\frac{\partial M}{\partial y} = -2y, \quad \frac{\partial N}{\partial x} = -y$ not exact

composite?

$\frac{dy}{dx} = \frac{2x-y^2}{xy}$ not composite

$\boxed{\text{Bernoulli in y w/ n = -1}}$

#1 continued...

(i) $y\,dx + x\,dy = 0$

separable?

$x\,dy = -y\,dx$

$\frac{1}{y}dy = -\frac{1}{x}dx$ $\boxed{\text{separable}}$

linear/Bernoulli?

in y: $\frac{dy}{dx} = \frac{-y}{x} = -\left(\frac{1}{x}\right)y$

$\frac{dy}{dx} + \left(\frac{1}{x}\right)y = 0$ $\boxed{\text{linear in y}}$

in x: $\frac{dx}{dy} = -\frac{x}{y} = -\left(\frac{1}{y}\right)x$

$\frac{dx}{dy} + \left(\frac{1}{y}\right)x = 0$ $\boxed{\text{linear in x}}$

exact?

$\underset{M}{(y)}\,dx + \underset{N}{(x)}\,dy = 0$

$\frac{\partial M}{\partial y} = 1, \quad \frac{\partial N}{\partial x} = 1$ $\boxed{\text{exact}}$

composite?

$\frac{dy}{dx} = \frac{-x}{y}$ not composite

$\boxed{\text{separable, linear in both x \& y, and exact}}$

(j) $\left(x^2 + \frac{2y}{x}\right)dx = \left(3 - \ln\left(x^2\right)\right)dy$

separable? can't factor $x^2 + \frac{2y}{x}$

not separable

linear/Bernoulli?

in y: $\frac{dy}{dx} = \frac{x^2 + \frac{2y}{x}}{3 - \ln(x^2)} = \frac{x^2}{3 - \ln x^2} + \frac{\frac{2y}{x}}{3 - \ln(x^2)}$

$\frac{dy}{dx} = \frac{x^2}{3 - \ln x^2} + \left(\frac{2}{x(3 - \ln(x^2))}\right)y$

$\frac{dy}{dx} + \left(\frac{-2}{x(3 - \ln(x^2))}\right)y = \frac{x^2}{3 - \ln(x^2)}$ $\boxed{\text{linear in y}}$

in x: $\frac{dx}{dy} = \frac{3 - \ln(x^2)}{x^2 + \frac{2y}{x}} = \frac{3x - x\ln(x^2)}{x^3 + 2y}$

$\frac{dx}{dy} = \frac{3x}{x^3 + 2y} - \frac{x\ln(x^2)}{x^3 + 2y}$ not linear nor Bernoulli in x

exact? $\underset{M}{\left(x^2 + \frac{2y}{x}\right)}dx + \underset{N}{\left(\ln(x^2) - 3\right)}dy = 0$

$\frac{\partial M}{\partial y} = \frac{2}{x}, \quad \frac{\partial N}{\partial x} = \frac{1}{x^2}(2x) = \frac{2}{x}$ $\boxed{\text{exact}}$

composite?

$\frac{dy}{dx} = \frac{3x - x\ln(x^2)}{x^3 + 2y}$ not composite

$\boxed{\text{linear in y, exact}}$

#1 continued...

(k) $\frac{y}{x^2}\frac{dy}{dx} + e^{(2x^3+y^2)} = 0$

Separable? $\frac{y}{x^2}dy = -e^{(2x^3+y^2)}dx$

$\frac{y}{x^2}dy = -e^{2x^3}e^{y^2}dx$

$\frac{y}{e^{y^2}}dy = -x^2e^{2x^3}dx$ $\boxed{\text{Separable}}$

Linear/Bernoulli?

in y: $\frac{dy}{dx} = \frac{-x^2e^{2x^3}e^{y^2}}{y}$

not linear, nor Bernoulli in y

in x: $\frac{dx}{dy} = \frac{-ye^{-2x^3}e^{-y^2}}{x^2}$

$\frac{dx}{dy} = \left(\frac{-e^{-2x^3}}{x^2}\right)ye^{-y^2}$

not linear, nor Bernoulli in x

exact? (yx^{-2})

$(\underbrace{e^{(2x^3+y^2)}}_{M})dx + (\underbrace{\tfrac{y}{x^2}}_{N})dy = 0$

$\frac{\partial M}{\partial y} = e^{(2x^3+y^2)}(2y)$, $\frac{\partial N}{\partial x} = (-2yx^{-3})$

not exact

composite?

$\frac{dy}{dx} = \frac{-x^2e^{(2x^3+y^2)}}{y}$ not composite

$\boxed{\text{only Separable}}$

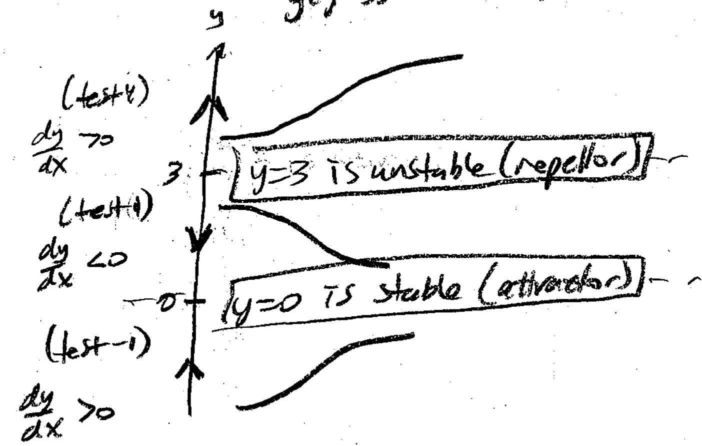
For #2 and #3, Given the autonomous, first-order differential equation, find the critical points and phase portrait. Then, classify each critical point as asymptotically stable, unstable, or semi-stable. Finally, sketch typical solution curves in the regions in the *xy*-plan determined by the graphs of the equilibrium solutions.

#2. $\frac{dy}{dx} = y^2 - 3y$

critical pts:

$y^2 - 3y = 0$

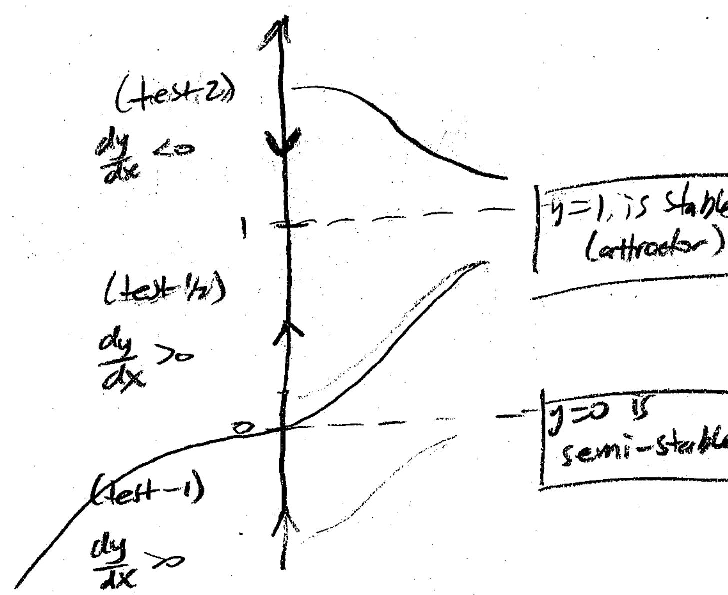
$y(y-3) = 0$ $y = 0, y = 3$



#3. $\frac{dy}{dx} = y^2 - y^3$ critical points:

$y^2 - y^3 = 0$

$y^2(1-y) = 0$ $y = 0$, $y = 1$



#4. Solve and write the solution in explicit form:

$dy-(y-1)^2\,dx=0$

$dy=(y-1)^2\,dx$

$\int \frac{1}{(y-1)^2}\,dy=\int dx$

$u=y-1$

$\frac{du}{dy}=1,\ du=dy$

$\int u^{-2}\,du=\int dx$

$-\frac{1}{u}$

$-\frac{1}{y-1}=x+C_1$

$\frac{1}{y-1}=-x+C$

$y-1=\frac{1}{-x+C}$

$\boxed{y=\frac{1}{-x+c}+1}$

#5. Solve and write the solution in explicit form:

$\csc y\,dx+\sec^2 x\,dy=0$

$\sec^2 x\,dy=-\csc y\,dx$

$\frac{1}{\csc y}\,dy=-\frac{1}{\sec^2 x}\,dx$

$\int \sin y\,dy=-\int \cos^2 x\,dx$

trig identity: $\cos^2 x=\frac{1}{2}+\frac{1}{2}\cos 2x$

$\int \sin y\,dy=-\int\left(\frac{1}{2}+\frac{1}{2}\cos(2x)\right)dx$

$-\cos y=-\int \frac{1}{2}\,dx-\frac{1}{2}\int \cos(2x)\,dx$

$u=2x$

$\frac{du}{dx}=2,\ du=2dx$

$dx=\frac{1}{2}du$

$-\cos y=-\frac{1}{2}x-\frac{1}{2}\left(\frac{1}{2}\right)\sin(2x)+C_1$

$\cos y=\frac{1}{2}x+\frac{1}{4}\sin(2x)+c$

$\boxed{y=\cos^{-1}\left(\frac{1}{2}x+\frac{1}{4}\sin(2x)+c\right)}$

#6. Solve and write the solution in explicit form:

$3\frac{dy}{dx}+12y=4$ $\qquad \frac{dy}{dx}+4y=\frac{4}{3}$

linear w/ $P(x)=4$

$I.F.=e^{\int P(x)dx}=e^{\int 4dx}=e^{4x}$

$e^{4x}y=\int \frac{4}{3}e^{4x}\,dx$

$e^{4x}y=\frac{4}{3}\left(\frac{1}{4}e^{4x}\right)+c$

$\frac{e^{4x}y}{e^{4x}}=\frac{\frac{1}{3}e^{4x}}{e^{4x}}+\frac{c}{e^{4x}}$

$\boxed{y=\frac{1}{3}+Ce^{-4x}}$

#7. Solve and write the solution in explicit form:

$x\frac{dy}{dx}-y=x^2\sin x$ $\qquad \frac{dy}{dx}-\left(\frac{1}{x}\right)y=x\sin x$

linear, $P(x)=-\frac{1}{x}$

$I.F.=e^{\int -\frac{1}{x}dx}=e^{-\ln(x)}=e^{\ln(x^{-1})}$

$=e^{\ln\left(\frac{1}{x}\right)}=\frac{1}{x}$

$\frac{1}{x}y=\int x\sin x\,\frac{1}{x}\,dx=\int \sin x\,dx$

$\frac{1}{x}y=-\cos x+C$

$\boxed{y=-x\cos x+Cx}$

#8. Solve and write the solution in ~~explicit~~ implicit form:

$$\underbrace{(3x^2y+e^y)}_{M}\,dx+\underbrace{(x^3+xe^y-2y)}_{N}\,dy=0$$

$$\frac{\partial M}{\partial y}=3x^2+e^y, \quad \frac{\partial N}{\partial x}=3x^2+e^y$$

exact, sol'n: $f=c$

$$\frac{\partial f}{\partial x} \overset{\text{must}}{=} 3x^2y+e^y$$

$$f=\int(3x^2y+e^y)\,dx = x^3y+e^yx+g(y)$$

$$\frac{\partial f}{\partial y}=x^3+e^yx+g'(y) \overset{\text{must}}{=} x^3+xe^y-2y$$

$$g'(y)=-2y$$

$$g(y)=\int -2y\,dy=-y^2$$

$$f=x^3y+xe^y-y^2$$

Solution:

$$\boxed{x^3y+xe^y-y^2=C}$$

#9. Solve (leave answer in implicit form):

$$\frac{dy}{dx}=\frac{1-x-y}{x+y}=\frac{1-(x+y)}{(x+y)}$$

composite fcn w/ $u=x+y$

$$\frac{du}{dx}=\frac{\partial u}{\partial x}\frac{dx}{dx}+\frac{\partial u}{\partial y}\frac{dy}{dx}$$

$$\frac{du}{dx}=(1)(1)+(1)\frac{dy}{dx}=1+\frac{dy}{dx}$$

so $\frac{dy}{dx}=\frac{du}{dx}-1$, $u=x+y$

Sub into DE:

$$\frac{dy}{dx}=\frac{1-(x+y)}{(x+y)}$$

$$\frac{du}{dx}-1=\frac{1-u}{u}$$

$$\frac{du}{dx}-1=\frac{1}{u}-1$$

$$\frac{du}{dx}=\frac{1}{u} \quad \text{(separable)}$$

$$\int u\,du=\int dx$$

$$\frac{1}{2}u^2=x+C$$

$$\boxed{\frac{1}{2}(x+y)^2=x+C}$$

#10. Find the particular solution of the initial value problem (you can leave your answer in implicit form):

$$\left(\frac{3y^2-t^2}{y^5}\right)\frac{dy}{dt}+\frac{t}{2y^4}=0; \qquad y(1)=1$$

$$\left(\frac{3y^2-t^2}{y^5}\right)dy+\left(\frac{t}{2y^4}\right)dt=0$$

$$\underbrace{\left(\frac{t}{2y^4}\right)}_{M}dt+\underbrace{\left(\frac{3y^2-t^2}{y^5}\right)}_{N}dy=0$$

exact? $\frac{\partial M}{\partial y}=\frac{\partial}{\partial y}\left[\frac{t}{2}y^{-4}\right]=-\frac{4t}{2}y^{-5}=-2ty^{-5}$

$\frac{\partial N}{\partial t}=\frac{\partial}{\partial t}\left[\frac{3y^2-t^2}{y^5}\right]=\frac{\partial}{\partial t}\left[3y^{-3}-y^{-5}t^2\right]$

$=-2ty^{-5}$

yes, exact ✓

sol'n: $f=C$

$$\frac{\partial f}{\partial t}\overset{\text{must}}{=}\frac{t}{2y^4}=\frac{1}{2}ty^{-4}$$

$$f=\int\frac{1}{2}ty^{-4}dt=\frac{1}{4}t^2y^{-4}+g(y)$$

$$\frac{\partial f}{\partial y}=-t^2y^{-5}+g'(y)\overset{\text{must}}{=}\frac{3y^2-t^2}{y^5}=3y^{-3}-t^2y^{-5}$$

so $g'(y)=3y^{-3}$

$g(y)=\int 3y^{-3}dy=-\frac{3}{2}y^{-2}$

$f=\frac{1}{4}t^2y^{-4}-\frac{3}{2}y^{-2}$

Soln: $\frac{1}{4}t^2y^{-4}-\frac{3}{2}y^{-2}=C_1$ (multiply everything by 4)

$$\frac{t^2}{y^4}-\frac{6}{y^2}=4C_1$$

now, use initial condition: $y(1)=1$

$t=1,\ y=1$

$$\frac{t^2}{y^4}-\frac{6}{y^2}=C$$

$$\frac{1^2}{1^4}-\frac{6}{1^2}=C, \quad C=1-6=-5$$

$$\boxed{\frac{t^2}{y^4}-\frac{6}{y^2}=-5}$$

#11. Solve and write the solution in explicit form:

$\frac{dy}{dx} = y\left(xy^3 - 1\right) = xy^4 - y$

$\frac{dy}{dx} + y = xy^4$ Bernoulli, w/ $n = 4$
Substitute $u = y^{1-n} = y^{1-4}$
$u = y^{-3} = \frac{1}{y^3}$

$\frac{du}{dx} = \frac{\partial u}{\partial x}\frac{dx}{dx} + \frac{\partial u}{\partial y}\frac{dy}{dx}$

$\frac{du}{dx} = (0)(1) + (-3y^{-4})\frac{dy}{dx} = \frac{-3}{y^4}\frac{dy}{dx}$

So $\frac{dy}{dx} = \frac{-y^4}{3}\frac{du}{dx}$, $u = \frac{1}{y^3}$

Sub into DE:

$\frac{dy}{dx} + y = xy^4$

$\frac{-\frac{y^4}{3}\frac{du}{dx}}{\left(\frac{-y^4}{3}\right)} + \frac{y}{\left(\frac{-y^4}{3}\right)} = \frac{xy^4}{\left(\frac{-y^4}{3}\right)}$

$\frac{du}{dx} - 3\frac{1}{y^3} = -3x \quad \left(\frac{1}{y^3} = u\right)$

$\frac{du}{dx} - 3u = -3x$ linear in u w/ $P(x) = -3$)

I.F. $= e^{\int -3dx} = e^{-3x}$

$e^{-3x}u = \int -3xe^{-3x}dx$

by parts: $u = -3x$ $dv = e^{-3x}$
$\frac{du}{dx} = -3$ $\int dv = \int e^{-3x}dx$
$du = -3dx$ $v = -\frac{1}{3}e^{-3x}$

$uv - \int v\,du$

$xe^{-3x} + \int\left(-\frac{1}{3}e^{-3x}\right)(-3)dx$

$xe^{-3x} - \int e^{-3x}dx$

$e^{-3x}u = xe^{-3x} + \frac{1}{3}e^{-3x} + c$

$\frac{e^{-3x}\frac{1}{y^3}}{e^{-3x}} = \frac{xe^{-3x}}{e^{-3x}} + \frac{\frac{1}{3}e^{-3x}}{e^{-3x}} + \frac{c}{e^{-3x}}$

$\frac{1}{y^3} = x + \frac{1}{3} + ce^{3x}$

$\boxed{y = \sqrt[3]{\frac{1}{x + \frac{1}{3} + ce^{3x}}}}$

#12. Solve and write the solution in explicit form:

$t^2\frac{dy}{dt} + y^2 = ty \qquad \frac{dy}{dt} + \frac{1}{t^2}y^2 = \frac{1}{t}y$

$\frac{dy}{dt} + \left(-\frac{1}{t}\right)y = \left(\frac{-1}{t^2}\right)y^2$ Bernoulli w/ $n = 2$
$u = y^{1-n} = y^{1-2} = y^{-1} = \frac{1}{y}$

$\frac{du}{dx} = \frac{\partial u}{\partial x}\frac{dx}{dx} + \frac{\partial u}{\partial y}\frac{dy}{dx}$

$\frac{du}{dx} = (0)(1) + (-y^{-2})\frac{dy}{dt} = -\frac{1}{y^2}\frac{dy}{dt}$

So $\frac{dy}{dt} = -y^2\frac{du}{dt}$, $u = \frac{1}{y}$, $y = \frac{1}{u}$

Sub into DE

$\frac{dy}{dt} + \left(-\frac{1}{t}\right)y = \left(-\frac{1}{t^2}\right)y^2$

$\frac{-y^2\frac{du}{dt}}{-y^2} - \frac{\frac{1}{t}y}{-y^2} = \frac{-\frac{1}{t^2}y^2}{-y^2}$

$\frac{du}{dt} + \frac{1}{t}\frac{1}{y} = \frac{1}{t^2}$

$\frac{du}{dt} + \frac{1}{t}u = \frac{1}{t^2}$ linear in u w/ $P(t) = \frac{1}{t}$

I.F. $= e^{\int \frac{1}{t}dt} = e^{\ln(t)} = t$

$tu = \int \frac{1}{t^2}t\,dt = \int \frac{1}{t}dt$

$\frac{tu}{t} = \frac{\ln t}{t} + \frac{c}{t}$

$u = \frac{\ln t}{t} + \frac{c}{t}$

$\frac{1}{y} = \frac{\ln t + c}{t}$

$\boxed{y = \frac{t}{\ln t + c}}$