* There are other eigenvalue choices for each problem which would also be correct *

## Ch8 Test Review

#1. Find the general solution of the system:

$$\frac{dx}{dt} = x + 2y$$

$$\frac{dy}{dt} = 4x + 3y$$

$\vec{X}' = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \vec{X}$

$\begin{vmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{vmatrix} = 0 \quad (1-\lambda)(3-\lambda) - 8 = 0$

$\lambda^2 - 4\lambda - 5 = 0$

$(\lambda - 5)(\lambda + 1) = 0$

$\lambda = 5 \quad \lambda = -1$

$\lambda = 5$

$\left[\begin{array}{cc|c} -4 & 2 & 0 \\ 4 & -2 & 0 \end{array}\right]$

rref

$\left[\begin{array}{cc|c} 1 & -1/2 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$k_1 - \frac{1}{2}k_2 = 0$

$k_1 = \frac{1}{2}k_2$

$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\lambda = -1$

$\left[\begin{array}{cc|c} 2 & 2 & 0 \\ 4 & 4 & 0 \end{array}\right]$

rref

$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$k_1 + k_2 = 0$

$k_1 = -k_2$

$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$$\boxed{\vec{X} = C_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{5t} + C_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-t}}$$

#2. Find the general solution of the system:

$$\vec{X}' = \begin{bmatrix} 10 & -5 \\ 8 & -12 \end{bmatrix} \vec{X}$$

$\begin{vmatrix} 10-\lambda & -5 \\ 8 & -12-\lambda \end{vmatrix} \quad (10-\lambda)(-12-\lambda) + 40 = 0$

$\lambda^2 + 2\lambda - 80 = 0$

$(\lambda - 8)(\lambda + 10) = 0$

$\lambda = 8 \quad \lambda = -10$

$\lambda = 8$

$\left[\begin{array}{cc|c} 2 & -5 & 0 \\ 8 & -20 & 0 \end{array}\right]$

rref

$\left[\begin{array}{cc|c} 1 & -5/2 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$k_1 - \frac{5}{2}k_2 = 0$

$k_1 = \frac{5}{2}k_2$

$\begin{bmatrix} 5 \\ 2 \end{bmatrix}$

$\lambda = -10$

$\left[\begin{array}{cc|c} 20 & -5 & 0 \\ 8 & -2 & 0 \end{array}\right]$

rref

$\left[\begin{array}{cc|c} 1 & -1/4 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$k_1 - \frac{1}{4}k_2 = 0$

$k_1 = \frac{1}{4}k_2$

$\begin{bmatrix} 1 \\ 4 \end{bmatrix}$

$$\boxed{\vec{X} = C_1 \begin{bmatrix} 5 \\ 2 \end{bmatrix} e^{8t} + C_2 \begin{bmatrix} 1 \\ 4 \end{bmatrix} e^{-10t}}$$

#3. Find the general solution of the system:

$\vec{X}' = \begin{bmatrix} -6 & 2 \\ -3 & 1 \end{bmatrix} \vec{X}$ $\begin{vmatrix} -6-\lambda & 2 \\ -3 & 1-\lambda \end{vmatrix} = 0$

$(-6-\lambda)(1-\lambda)+6=0$

$\lambda^2+5\lambda=0$

$\lambda(\lambda+5)=0$

$\lambda=0 \quad \underline{\lambda=-5}$

$\underline{\lambda=0}$

$\left[\begin{array}{cc|c} -6 & 2 & 0 \\ -3 & 1 & 0 \end{array}\right]$

rref

$\left[\begin{array}{cc|c} 1 & -1/3 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$k_1 - 1/3 k_2 = 0$

$k_1 = 1/3 k_2$

$\begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$\underline{\lambda=-5}$

$\left[\begin{array}{cc|c} -1 & 2 & 0 \\ -3 & 6 & 0 \end{array}\right]$

rref

$\left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$k_1 - 2k_2 = 0$

$k_1 = 2k_2$

$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\boxed{\vec{X} = c_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-5t}}$

#4. Find the general solution of the system:

$\dfrac{dx}{dt} = 3x - y$

$\dfrac{dy}{dt} = 9x - 3y$

$\vec{X}' = \begin{bmatrix} 3 & -1 \\ 9 & -3 \end{bmatrix} \vec{X}$

$\begin{vmatrix} 3-\lambda & -1 \\ 9 & -3-\lambda \end{vmatrix} = 0$

$(3-\lambda)(-3-\lambda)+9=0$

$\lambda^2=0 \quad \underline{\lambda=0 \text{ repeated}}$

$\underline{\lambda=0}$

$\left[\begin{array}{cc|c} 3 & -1 & 0 \\ 9 & -3 & 0 \end{array}\right]$

rref

$\left[\begin{array}{cc|c} 1 & -1/3 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$k_1 - 1/3 k_2 = 0$

$k_1 = 1/3 k_2$

$(1/3 k_2, k_2)$ $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$\underline{\text{repeat } \lambda=0}$

$\left[\begin{array}{cc|c} 3 & -1 & 1 \\ 9 & -3 & 3 \end{array}\right]$

rref

$\left[\begin{array}{cc|c} 1 & -1/3 & 1/3 \\ 0 & 0 & 0 \end{array}\right]$

$k_1 - 1/3 k_2 = 1/3$

$k_1 = 1/3 k_2 + 1/3$

$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\vec{X} = c_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{0t} + c_2 \left( \begin{bmatrix} 1 \\ 3 \end{bmatrix} t e^{0t} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{0t} \right)$

$\boxed{\vec{X} = c_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2 \left( \begin{bmatrix} 1 \\ 3 \end{bmatrix} t + \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right)}$

#5. Find the general solution of the system:

$\vec{X}' = \begin{bmatrix} -1 & 3 \\ -3 & 5 \end{bmatrix} \vec{X}$ $\quad \begin{vmatrix} -1-\lambda & 3 \\ -3 & 5-\lambda \end{vmatrix} = 0$ $\quad (-1-\lambda)(5-\lambda)+9=0$ $\quad (\lambda-2)(\lambda-2)=0$

$\lambda^2 - 4\lambda + 4 = 0$ $\quad \lambda = 2$ (repeated)

$\lambda = 2$

$\left[\begin{array}{cc|c} -3 & 3 & 0 \\ -3 & 3 & 0 \end{array}\right]$

rref

$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$k_1 - k_2 = 0$ $\quad \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$k_1 = k_2$

repeat $\lambda = 2$

$\left[\begin{array}{cc|c} -3 & 3 & 1 \\ -3 & 3 & 1 \end{array}\right]$

rref

$\left[\begin{array}{cc|c} 1 & -1 & -1/3 \\ 0 & 0 & 0 \end{array}\right]$

$k_1 - k_2 = -1/3$, $k_1 = k_2 - 1/3$ $\quad (k_2 - 1/3, k_2)$

$(-1/3, 0)$

$$\boxed{\vec{X} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t} + c_2\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} te^{2t} + \begin{bmatrix} -\frac{1}{3} \\ 0 \end{bmatrix} e^{2t}\right)}$$

#6. Solve the initial-value problem:

$\vec{X}' = \begin{bmatrix} 2 & 4 \\ -1 & 6 \end{bmatrix} \vec{X}, \quad \vec{X}(0) = \begin{bmatrix} -1 \\ 6 \end{bmatrix}$ $\quad \begin{vmatrix} 2-\lambda & 4 \\ -1 & 6-\lambda \end{vmatrix} = 0$ $\quad (2-\lambda)(6-\lambda)+4=0$ $\quad (\lambda-4)(\lambda-4)=0$

$\lambda^2 - 8\lambda + 16 = 0$ $\quad \lambda = 4$ (repeated)

$\lambda = 4$

$\left[\begin{array}{cc|c} -2 & 4 & 0 \\ -1 & 2 & 0 \end{array}\right]$

rref

$\left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$k_1 - 2k_2 = 0$ $\quad \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$k_1 = 2k_2$

repeat $\lambda = 4$

$\left[\begin{array}{cc|c} -2 & 4 & 2 \\ -1 & 2 & 1 \end{array}\right]$

rref

$\left[\begin{array}{cc|c} 1 & -2 & -1 \\ 0 & 0 & 0 \end{array}\right]$

$k_1 - 2k_2 = -1$ $\quad \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$k_1 = 2k_2 - 1$

$$\vec{X} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{4t} + c_2\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} te^{4t} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t}\right)$$

$x = 2c_1 e^{4t} + 2c_2 te^{4t} + c_2 e^{4t}$

$-1 = 2c_1 e^0 + 2c_2(0)e^0 + c_2 e^0$

$-1 = 2c_1 + c_2$

$y = c_1 e^{4t} + c_2 te^{4t} + c_2 e^{4t}$

$6 = c_1 e^0 + c_2(0)e^0 + c_2 e^0$

$6 = c_1 + c_2$

$\begin{bmatrix} c_1 + c_2 = 6 \\ 2c_1 + c_2 = -1 \end{bmatrix}$ rref $\left[\begin{array}{cc|c} 1 & 0 & -7 \\ 0 & 1 & 13 \end{array}\right] \begin{matrix} = c_1 \\ = c_2 \end{matrix}$

$$\boxed{\vec{X} = -7\begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{4t} + 13\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} te^{4t} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t}\right)}$$

#7. Find the general solution of the system:

$\frac{dx}{dt} = x + y$

$\frac{dy}{dt} = -2x - y$

$\vec{X}' = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}\vec{X}$ $\begin{vmatrix} 1-\lambda & 1 \\ -2 & -1-\lambda \end{vmatrix} = 0$ $(1-\lambda)(-1-\lambda)+2=0$

$\lambda^2+1=0$ $\lambda = 0 \pm i$

$\lambda = 0+i$

$\left[\begin{array}{cc|c} 1-i & 1 & 0 \\ -2 & -1-i & 0 \end{array}\right]$ $(1-i)k_1 + k_2 = 0$ $k_2 = -(1-i)k_1$ $\begin{bmatrix} 1 \\ -1+i \end{bmatrix}$ $\vec{B_1} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\vec{B_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\vec{X} = C_1\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\cos t - \begin{bmatrix} 0 \\ 1 \end{bmatrix}\sin t\right)\cancel{e^{0t}} + C_2\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\cos t + \begin{bmatrix} 1 \\ -1 \end{bmatrix}\sin t\right)\cancel{e^{0t}}$

$$\boxed{\vec{X} = C_1\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\cos t - \begin{bmatrix} 0 \\ 1 \end{bmatrix}\sin t\right) + C_2\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\cos t + \begin{bmatrix} 1 \\ -1 \end{bmatrix}\sin t\right)}$$

#8. Find the general solution of the system:

$\vec{X}' = \begin{bmatrix} 4 & -5 \\ 5 & -4 \end{bmatrix}\vec{X}$ $\begin{vmatrix} 4-\lambda & -5 \\ 5 & -4-\lambda \end{vmatrix} = 0$ $(4-\lambda)(-4-\lambda)+25=0$

$\lambda^2+9=0$ $\lambda = 0 \pm 3i$

$\lambda = 0+3i$

$\left[\begin{array}{cc|c} 4-3i & -5 & 0 \\ 5 & -4-3i & 0 \end{array}\right]$ $(4-3i)k_1 - 5k_2 = 0$ $k_2 = \frac{(4-3i)}{5}k_1$ $\begin{bmatrix} 5 \\ 4-3i \end{bmatrix}$ $\vec{B_1} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$ $\vec{B_2} = \begin{bmatrix} 0 \\ -3 \end{bmatrix}$

$$\boxed{\vec{X} = C_1\left(\begin{bmatrix} 5 \\ 4 \end{bmatrix}\cos 3t - \begin{bmatrix} 0 \\ -3 \end{bmatrix}\sin 3t\right) + C_2\left(\begin{bmatrix} 0 \\ -3 \end{bmatrix}\cos 3t + \begin{bmatrix} 5 \\ 4 \end{bmatrix}\sin 3t\right)}$$

#9. Solve the initial-value problem:

$$\vec{X}' = \begin{bmatrix} 6 & -1 \\ 5 & 4 \end{bmatrix}\vec{X}, \qquad \vec{X}(0) = \begin{bmatrix} -2 \\ 8 \end{bmatrix}$$

$$\begin{vmatrix} 6-\lambda & -1 \\ 5 & 4-\lambda \end{vmatrix} = 0 \qquad (6-\lambda)(4-\lambda)+5=0$$

$$\lambda^2 - 10\lambda + 29 = 0$$

$$\lambda = \frac{10 \pm \sqrt{100-4(29)}}{2} = \frac{10 \pm 4i}{2} = 5 \pm 2i$$

$\underline{\lambda = 5+2i}$

$$\left[\begin{array}{cc|c} 6-(5+2i) & -1 & 0 \\ 5 & 4-(5+2i) & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} -1-2i & -1 & 0 \\ 5 & -1-2i & 0 \end{array}\right] \qquad (-1-2i)k_1 - k_2 = 0 \qquad k_2 = (1-2i)k_1$$

$$\begin{bmatrix} 1 \\ 1-2i \end{bmatrix} \qquad \vec{B}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \vec{B}_2 = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$

$$\vec{X} = c_1\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\cos 2t - \begin{bmatrix} 0 \\ -2 \end{bmatrix}\sin 2t\right)e^{5t} + c_2\left(\begin{bmatrix} 0 \\ -2 \end{bmatrix}\cos 2t + \begin{bmatrix} 1 \\ 1 \end{bmatrix}\sin 2t\right)e^{5t}$$

$$x = c_1(\cos 2t)e^{5t} + c_2(\sin 2t)e^{5t}$$

$$-2 = c_1(\cos 0)e^0 + c_2(\sin 0)e^0$$

$$-2 = c_1(1)(1) + c_2(0)(1)$$

$$\underline{c_1 = -2}$$

$$y = c_1(\cos 2t + 2\sin 2t)e^{5t} + c_2(-2\cos 2t + \sin 2t)e^{5t}$$

$$8 = c_1(\cos 0 + 2\sin 0)e^0 + c_2(-2(\cos 0) + \sin 0)e^0$$

$$8 = c_1(1+0)(1) + c_2(-2(1)+0)(1)$$

$$8 = c_1 - 2c_2$$

$$8 = (-2) - 2c_2 \qquad c_2 = \frac{10}{-2} = -5$$

$$\boxed{\vec{X} = -2\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\cos 2t - \begin{bmatrix} 0 \\ -2 \end{bmatrix}\sin 2t\right)e^{5t} - 5\left(\begin{bmatrix} 0 \\ -2 \end{bmatrix}\cos 2t + \begin{bmatrix} 1 \\ 1 \end{bmatrix}\sin 2t\right)e^{5t}}$$

#10. Solve both ways (method of undermined coefficients and variation of parameters):

$$\vec{X}' = \begin{bmatrix} 4 & 1/3 \\ 9 & 6 \end{bmatrix}\vec{X} + \begin{bmatrix} -3 \\ 10 \end{bmatrix} e^t$$

$\begin{vmatrix} 4-\lambda & 1/3 \\ 9 & 6-\lambda \end{vmatrix} = 0$ $\quad (4-\lambda)(6-\lambda) - 3 = 0 \quad (\lambda-7)(\lambda-3) = 0$

$\lambda^2 - 10\lambda + 21 = 0 \quad \lambda = 7,\ \lambda = 3$

$\lambda = 7$

$\left[\begin{array}{cc|c} -3 & 1/3 & 0 \\ 9 & -1 & 0 \end{array}\right]$

rref

$\left[\begin{array}{cc|c} 1 & -1/9 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$K_1 - 1/9 K_2 = 0$ $\quad \begin{bmatrix} 1 \\ 9 \end{bmatrix}$

$K_1 = 1/9 K_2$

$\lambda = 3$

$\left[\begin{array}{cc|c} 1 & 1/3 & 0 \\ 9 & 3 & 0 \end{array}\right]$

rref

$\left[\begin{array}{cc|c} 1 & 1/3 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$K_1 + 1/3 K_2 = 0$ $\quad \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

$K_1 = -1/3 K_2$

$\vec{X}_c = C_1 \begin{bmatrix} 1 \\ 9 \end{bmatrix} e^{7t} + C_2 \begin{bmatrix} -1 \\ 3 \end{bmatrix} e^{3t}$

undetermined coefficients

for $\begin{bmatrix} -3e^t \\ 10e^t \end{bmatrix}$ $\quad \vec{X}_p = \begin{bmatrix} Ae^t \\ Be^t \end{bmatrix}$

(no absorption)

$\vec{X}' = \begin{bmatrix} Ae^t \\ Be^t \end{bmatrix}$

into DE system:

$\begin{bmatrix} Ae^t \\ Be^t \end{bmatrix} = \begin{bmatrix} 4 & 1/3 \\ 9 & 6 \end{bmatrix}\begin{bmatrix} Ae^t \\ Be^t \end{bmatrix} + \begin{bmatrix} -3e^t \\ 10e^t \end{bmatrix}$

$\begin{cases} 4Ae^t + \frac{1}{3}Be^t - 3e^t = Ae^t \\ 9Ae^t + 6Be^t + 10e^t = Be^t \end{cases}$

$\begin{cases} 4A + \frac{1}{3}B - 3 = A \\ 9A + 6B + 10 = B \end{cases}$

$\begin{cases} 3A + \frac{1}{3}B = 3 \\ 9A + 5B = -10 \end{cases}$

$\left[\begin{array}{cc|c} 3 & 1/3 & 3 \\ 9 & 5 & -10 \end{array}\right]$

rref

$\left[\begin{array}{cc|c} 1 & 0 & 55/36 \\ 0 & 1 & -19/4 \end{array}\right]$

$\vec{X}_p = \begin{bmatrix} 55/36 \\ -19/4 \end{bmatrix} e^t$

$$\boxed{\vec{X} = C_1 \begin{bmatrix} 1 \\ 9 \end{bmatrix} e^{7t} + C_2 \begin{bmatrix} -1 \\ 3 \end{bmatrix} e^{3t} + \begin{bmatrix} 55/36 \\ -19/4 \end{bmatrix} e^t}$$

variation of parameters

$\Phi = \begin{bmatrix} e^{7t} & -e^{3t} \\ 9e^{7t} & 3e^{3t} \end{bmatrix}$ $\quad \det\Phi = 3e^{10t} + 9e^{10t} = 12e^{10t}$

$\Phi^{-1} = \frac{1}{12e^{10t}} \begin{bmatrix} 3e^{3t} & e^{3t} \\ -9e^{7t} & e^{7t} \end{bmatrix} = \begin{bmatrix} 1/4 e^{-7t} & 1/12 e^{-7t} \\ -3/4 e^{-3t} & 1/12 e^{-3t} \end{bmatrix}$

$\vec{X}_p = \Phi \int \Phi^{-1} F\, dt$

$= \begin{bmatrix} e^{7t} & -e^{3t} \\ 9e^{7t} & 3e^{3t} \end{bmatrix} \int \begin{bmatrix} 1/4 e^{-7t} & 1/12 e^{-7t} \\ -3/4 e^{-3t} & 1/12 e^{-3t} \end{bmatrix}\begin{bmatrix} -3e^t \\ 10e^t \end{bmatrix} dt$

$\int \begin{bmatrix} -3/4 e^{-6t} + 5/6 e^{-6t} \\ 9/4 e^{-2t} + 5/6 e^{-2t} \end{bmatrix} dt$

$\int \begin{bmatrix} 1/12 e^{-6t} \\ 37/12 e^{-2t} \end{bmatrix} dt$

$\begin{bmatrix} e^{7t} & -e^{3t} \\ 9e^{7t} & 3e^{3t} \end{bmatrix}\begin{bmatrix} -\frac{1}{72} e^{-6t} \\ -37/24 e^{-2t} \end{bmatrix}$

$\begin{bmatrix} -\frac{1}{72} e^t + \frac{37}{24} e^t \\ -\frac{9}{72} e^t - \frac{111}{24} e^t \end{bmatrix} = \begin{bmatrix} 55/36 \\ -19/4 \end{bmatrix} e^t$

$$\boxed{\vec{X} = C_1 \begin{bmatrix} 1 \\ 9 \end{bmatrix} e^{7t} + C_2 \begin{bmatrix} -1 \\ 3 \end{bmatrix} e^{3t} + \begin{bmatrix} 55/36 \\ -19/4 \end{bmatrix} e^t}$$

#11. Solve both ways (method of undermined coefficients and variation of parameters):

$$\vec{X}' = \begin{bmatrix} 3 & 2 \\ -2 & -1 \end{bmatrix}\vec{X} + \begin{bmatrix} 2e^{-t} \\ e^{-t} \end{bmatrix}$$

$\begin{vmatrix} 3-\lambda & 2 \\ -2 & -1-\lambda \end{vmatrix} = 0$ $\quad (3-\lambda)(-1-\lambda)+4=0 \quad \lambda^2-2\lambda+1=0 \quad (\lambda-1)(\lambda-1)=0 \quad \underline{\lambda=1 \text{ repeated}}$

$\underline{\lambda=1}$

$\left[\begin{array}{cc|c} 2 & 2 & 0 \\ -2 & -2 & 0 \end{array}\right]$

rref

$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$k_1+k_2=0$
$k_1=-k_2$ $\quad \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

repeat $\lambda=1$

$\left[\begin{array}{cc|c} 2 & 2 & 1 \\ -2 & -2 & -1 \end{array}\right]$

rref

$\left[\begin{array}{cc|c} 1 & 1 & 1/2 \\ 0 & 0 & 0 \end{array}\right]$

$k_1+k_2=1/2$
$k_2=1/2-k_1$ $\quad \begin{bmatrix} 0 \\ 1/2 \end{bmatrix}$

$$\vec{X}_c = c_1\begin{bmatrix} 1 \\ -1 \end{bmatrix}e^t + c_2\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}te^t + \begin{bmatrix} 0 \\ 1/2 \end{bmatrix}e^t\right)$$

## undetermined coefficients

for $\begin{bmatrix} 2e^{-t} \\ e^{-t} \end{bmatrix}$ table:

$\vec{X}_p = \begin{bmatrix} Ae^{-t} \\ Be^{-t} \end{bmatrix} \quad \vec{X}' = \begin{bmatrix} -Ae^{-t} \\ -Be^{-t} \end{bmatrix}$

(no absorption)

into DE system:

$\begin{bmatrix} -Ae^{-t} \\ -Be^{-t} \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} Ae^{-t} \\ Be^{-t} \end{bmatrix} + \begin{bmatrix} 2e^{-t} \\ e^{-t} \end{bmatrix}$

$3Ae^{-t}+2Be^{-t}+2e^{-t}=-Ae^{-t}$
$-2Ae^{-t}-Be^{-t}+e^{-t}=-Be^{-t}$

$\begin{cases} 3A+2B+2=-A \\ -2A-B+1=-B \end{cases}$

$\begin{cases} 4A+2B=-2 \\ -2A+0B=-1 \end{cases}$

$\left[\begin{array}{cc|c} 4 & 2 & -2 \\ -2 & 0 & -1 \end{array}\right]$

rref

$\left[\begin{array}{cc|c} 1 & 0 & 1/2 \\ 0 & 1 & -2 \end{array}\right] \begin{matrix} =A \\ =B \end{matrix}$

$\vec{X}_p = \begin{bmatrix} 1/2\,e^{-t} \\ -2e^{-t} \end{bmatrix}$

$$\boxed{\vec{X} = c_1\begin{bmatrix} 1 \\ -1 \end{bmatrix}e^t + c_2\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}te^t + \begin{bmatrix} 0 \\ 1/2 \end{bmatrix}e^t\right) + \begin{bmatrix} 1/2 \\ -2 \end{bmatrix}e^{-t}}$$

## variation of parameters

$\Phi = \begin{bmatrix} e^t & te^t \\ -e^t & -te^t+\frac{1}{2}e^t \end{bmatrix}$ $\quad \det\Phi = -te^{2t}+\frac{1}{2}e^{2t}+te^{2t} = \frac{1}{2}e^{2t}$

$\Phi^{-1} = \frac{1}{\frac{1}{2}e^{2t}}\begin{bmatrix} -te^t+\frac{1}{2}e^t & -te^t \\ e^t & e^t \end{bmatrix} = \begin{bmatrix} -2te^{-t}+e^{-t} & -2te^{-t} \\ 2e^{-t} & 2e^{-t} \end{bmatrix}$

$\vec{X}_p = \Phi\int\Phi^{-1}F\,dt$

$= \begin{bmatrix} e^t & te^t \\ -e^t & -te^t+\frac{1}{2}e^t \end{bmatrix}\int\begin{bmatrix} -2te^{-t}+e^{-t} & -2te^{-t} \\ 2e^{-t} & 2e^{-t} \end{bmatrix}\begin{bmatrix} 2e^{-t} \\ e^{-t} \end{bmatrix}dt$

$\int\begin{bmatrix} -4te^{-2t}+2e^{-2t}-2te^{-2t} \\ 4e^{-2t}+2e^{-2t} \end{bmatrix}dt$

$\int\begin{bmatrix} -6te^{-2t}+2e^{-2t} \\ 6e^{-2t} \end{bmatrix}dt$

$\begin{bmatrix} 3te^{-2t}+\frac{3}{2}e^{-2t}-e^{-2t} \\ -3e^{-2t} \end{bmatrix}$

$\begin{bmatrix} e^t & te^t \\ -e^t & -te^t+\frac{1}{2}e^t \end{bmatrix}\begin{bmatrix} 3te^{-2t}+\frac{1}{2}e^{-2t} \\ -3e^{-2t} \end{bmatrix}$

$\vec{X}_p = \begin{bmatrix} 3te^{-t}+\frac{1}{2}e^{-t}-3te^{-t} \\ -3te^{-t}-\frac{1}{2}e^{-t}+3te^{-t}-\frac{3}{2}e^{-t} \end{bmatrix}$

$= \begin{bmatrix} \frac{1}{2}e^{-t} \\ -2e^{-t} \end{bmatrix}$

$$\boxed{\vec{X} = c_1\begin{bmatrix} 1 \\ -1 \end{bmatrix}e^t + c_2\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}te^t + \begin{bmatrix} 0 \\ 1/2 \end{bmatrix}e^t\right) + \begin{bmatrix} 1/2 \\ -2 \end{bmatrix}e^{-t}}$$

#12. Rewrite the following $3^{rd}$-order differential equation as a system of 3 $1^{st}$-order differential equations (and also write the initial conditions in matrix form):

$$y''' - y'' + 2y' - 3y = t^3 - 4t$$

$$y(0) = 3, \quad y'(0) = 7, \quad y''(0) = 5$$

(do not solve the system)

$$\begin{cases} y' = 0y + y' + 0y'' + 0 \\ y'' = 0y + 0y' + y'' + 0 \\ y''' = 3y - 2y' + y'' + t^3 - 4t \end{cases}$$

$$\boxed{\begin{aligned} &\vec{X}' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 3 & -2 & 1 \end{bmatrix} \vec{X} + \begin{bmatrix} 0 \\ 0 \\ t^3 - 4t \end{bmatrix} \qquad \vec{X}(0) = \begin{bmatrix} 3 \\ 7 \\ 5 \end{bmatrix} \\ &\text{where } \vec{X} = \begin{bmatrix} y \\ y' \\ y'' \end{bmatrix} \end{aligned}}$$