## Ch 8 review problems

**324/1**

$$\vec{X}' = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \vec{X}$$

$$\begin{vmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(3-\lambda) - 8 = 0$$

$$\lambda^2 - 4\lambda - 5 = 0$$

$$(\lambda-5)(\lambda+1) = 0$$

$$\lambda = 5 \quad \lambda = -1$$

$\underline{\lambda = 5}$

$$\left[\begin{array}{cc|c} -4 & 2 & 0 \\ 4 & -2 & 0 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{cc|c} 1 & -1/2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$K_1 = 1/2\, K_2$

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$\underline{\lambda = -1}$

$$\left[\begin{array}{cc|c} 2 & 2 & 0 \\ 4 & 4 & 0 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$K_1 = -K_2$

$$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\boxed{\vec{X} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{5t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-t}}$$

* there are other valid eigenvector choices for each problem which would also be correct *

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**324/5**

$$\vec{X}' = \begin{bmatrix} 10 & -5 \\ 8 & -12 \end{bmatrix} \vec{X}$$

$$\begin{vmatrix} 10-\lambda & -5 \\ 8 & -12-\lambda \end{vmatrix} = 0$$

$$(10-\lambda)(-12-\lambda) + 40 = 0$$

$$\lambda^2 + 2\lambda - 80 = 0$$

$$(\lambda-8)(\lambda+10) = 0$$

$$\lambda = 8 \quad \lambda = -10$$

$\underline{\lambda = 8}$

$$\left[\begin{array}{cc|c} 2 & -5 & 0 \\ 8 & -20 & 0 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{cc|c} 1 & -5/2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$K_1 = 5/2\, K_2$

$$\begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

$\underline{\lambda = -10}$

$$\left[\begin{array}{cc|c} 20 & -5 & 0 \\ 8 & -2 & 0 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{cc|c} 1 & -1/4 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$K_1 = 1/4\, K_2$

$$\begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

$$\boxed{\vec{X} = c_1 \begin{bmatrix} 5 \\ 2 \end{bmatrix} e^{8t} + c_2 \begin{bmatrix} 1 \\ 4 \end{bmatrix} e^{-10t}}$$

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**324/6**

$$\vec{X}' = \begin{bmatrix} -6 & 2 \\ -3 & 1 \end{bmatrix} \vec{X}$$

$$\begin{vmatrix} -6-\lambda & 2 \\ -3 & 1-\lambda \end{vmatrix} = 0$$

$$(-6-\lambda)(1-\lambda) + 6 = 0$$

$$\lambda^2 + 5\lambda = 0$$

$$\lambda(\lambda+5) = 0$$

$$\lambda = 0 \quad \lambda = -5$$

$\underline{\lambda = 0}$

$$\left[\begin{array}{cc|c} -6 & 2 & 0 \\ -3 & 1 & 0 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{cc|c} 1 & -1/3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$K_1 = 1/3\, K_2$

$$\begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$\underline{\lambda = -5}$

$$\left[\begin{array}{cc|c} -1 & 2 & 0 \\ -3 & 6 & 0 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$K_1 = 2K_2$

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\vec{X} = c_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{0t} + c_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-5t}$$

$$\boxed{\vec{X} = c_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-5t}}$$

325/19

$\vec{X}' = \begin{bmatrix} 3 & -1 \\ 9 & -3 \end{bmatrix} \vec{X}$

$\begin{vmatrix} 3-\lambda & -1 \\ 9 & -3-\lambda \end{vmatrix} = 0$

$(3-\lambda)(-3-\lambda) + 9 = 0$

$\lambda^2 = 0$

$\lambda = 0$ repeated

$\lambda = 0$

$\left[\begin{array}{cc|c} 3 & -1 & 0 \\ 9 & -3 & 0 \end{array}\right]$

rref

$\left[\begin{array}{cc|c} 1 & -1/3 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$K_1 = 1/3\, K_2$

$\begin{bmatrix} 1 \\ 3 \end{bmatrix}$

2nd

$\left[\begin{array}{cc|c} 3 & -1 & 1 \\ 9 & -3 & 3 \end{array}\right]$

rref

$\left[\begin{array}{cc|c} 1 & -1/3 & 1/3 \\ 0 & 0 & 0 \end{array}\right]$

$K_1 - 1/3\, K_2 = 1/3$

$K_1 = 1/3\, K_2 + 1/3$

$(1/3\, K_2 + 1/3,\ K_2)$

$\begin{bmatrix} 2/3 \\ 1 \end{bmatrix}$ or $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ or $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

I'm using this one

book uses this one

$\vec{X} = C_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{0t} + C_2 \left( \begin{bmatrix} 1 \\ 3 \end{bmatrix} t e^{0t} + \begin{bmatrix} 2/3 \\ 1 \end{bmatrix} e^{0t} \right)$

$$\boxed{\vec{X} = C_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + C_2 \left( \begin{bmatrix} 1 \\ 3 \end{bmatrix} t + \begin{bmatrix} 2/3 \\ 1 \end{bmatrix} \right)}$$

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325/21

$\vec{X}' = \begin{bmatrix} -1 & 3 \\ -3 & 5 \end{bmatrix} \vec{X}$

$\begin{vmatrix} -1-\lambda & 3 \\ -3 & 5-\lambda \end{vmatrix} = 0$

$(-1-\lambda)(5-\lambda) + 9 = 0$

$\lambda^2 - 4\lambda + 4 = 0$

$(\lambda - 2)(\lambda - 2) = 0$

$\lambda = 2$ repeated

$\lambda = 2$

$\left[\begin{array}{cc|c} -3 & 3 & 0 \\ -3 & 3 & 0 \end{array}\right]$

rref

$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$K_1 = K_2$

$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

2nd

$\left[\begin{array}{cc|c} -3 & 3 & 1 \\ -3 & 3 & 1 \end{array}\right]$

rref

$\left[\begin{array}{cc|c} 1 & -1 & -1/3 \\ 0 & 0 & 0 \end{array}\right]$

$K_1 - K_2 = -1/3$

$K_1 = K_2 - 1/3$

$(K_2 - 1/3,\ K_2)$

$\begin{bmatrix} -1/3 \\ 0 \end{bmatrix}$

$$\boxed{\vec{X} = C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t} + C_2 \left( \begin{bmatrix} 1 \\ 1 \end{bmatrix} t e^{2t} + \begin{bmatrix} -1/3 \\ 0 \end{bmatrix} e^{2t} \right)}$$

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325/29

$\vec{X}' = \begin{bmatrix} 2 & 4 \\ -1 & 6 \end{bmatrix} \vec{X} \quad \vec{X}(0) = \begin{bmatrix} -1 \\ 6 \end{bmatrix}$

$\begin{vmatrix} 2-\lambda & 4 \\ -1 & 6-\lambda \end{vmatrix} = 0$

$(2-\lambda)(6-\lambda) + 4 = 0$

$\lambda^2 - 8\lambda + 16 = 0$

$(\lambda - 4)(\lambda - 4) = 0$

$\lambda = 4$ repeated

$\lambda = 4$

$\left[\begin{array}{cc|c} -2 & 4 & 0 \\ -1 & 2 & 0 \end{array}\right]$

rref

$\left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$K_1 = 2K_2$

$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

2nd

$\left[\begin{array}{cc|c} -2 & 4 & 2 \\ -1 & 2 & 1 \end{array}\right]$

rref

$\left[\begin{array}{cc|c} 1 & -2 & -1 \\ 0 & 0 & 0 \end{array}\right]$

$K_1 = 2K_2 - 1$

$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\vec{X} = C_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{4t} + C_2 \left( \begin{bmatrix} 2 \\ 1 \end{bmatrix} t e^{4t} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t} \right)$

now $\vec{X}(0) = \begin{bmatrix} -1 \\ 6 \end{bmatrix}$

$\begin{bmatrix} -1 \\ 6 \end{bmatrix} = C_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^0 + C_2 \left( \begin{bmatrix} 2 \\ 1 \end{bmatrix} (0) e^0 + \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^0 \right)$

$\begin{bmatrix} -1 \\ 6 \end{bmatrix} = \begin{bmatrix} 2C_1 + C_2 \\ C_1 + C_2 \end{bmatrix}$

$\begin{cases} 2C_1 + C_2 = -1 \\ C_1 + C_2 = 6 \end{cases}$

$\left[\begin{array}{cc|c} 2 & 1 & -1 \\ 1 & 1 & 6 \end{array}\right]$

rref

$\left[\begin{array}{cc|c} 1 & 0 & -7 \\ 0 & 1 & 13 \end{array}\right]$ $C_1 = -7$, $C_2 = 13$

$$\boxed{\vec{X} = -7 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{4t} + 13 \left( \begin{bmatrix} 2 \\ 1 \end{bmatrix} t e^{4t} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t} \right)}$$

325/34

$$\vec{X}' = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} \vec{X}$$

$$\begin{vmatrix} 1-\lambda & 1 \\ -2 & -1-\lambda \end{vmatrix} = 0$$

$(1-\lambda)(-1-\lambda) + 2 = 0$

$\lambda^2 + 1 = 0$

$\lambda^2 = -1$

$\lambda = \pm\sqrt{-1} = \pm i$

$\lambda = i$

$$\left[\begin{array}{cc|c} 1-i & 1 & 0 \\ -2 & -1-i & 0 \end{array}\right]$$

$(1-i)k_1 + k_2 = 0$

$-2k_1 + (-1-i)k_2 = 0$

using $(1-i)k_1 + k_2 = 0$

$k_2 = (-1+i)k_1$

$(k_1, (-1+i)k_1)$

$$\begin{bmatrix} 1 \\ -1+i \end{bmatrix}$$

$\vec{B}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\vec{B}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

using $-2k_1 + (-1-i)k_2 = 0$

$2k_1 + (1+i)k_2 = 0$

$k_1 = -\frac{1}{2}(1+i)k_2$

$(-\frac{1}{2}(1+i)k_2, k_2)$

$$\begin{bmatrix} -1-i \\ 2 \end{bmatrix}$$

$\vec{B}_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ $\vec{B}_2 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

$$\vec{X} = c_1\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\cos t - \begin{bmatrix} 0 \\ 1 \end{bmatrix}\sin t\right) + c_2\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\cos t + \begin{bmatrix} 1 \\ -1 \end{bmatrix}\sin t\right)$$

$$\boxed{\vec{X} = c_1\begin{bmatrix} \cos t \\ -\cos t - \sin t \end{bmatrix} + c_2\begin{bmatrix} \sin t \\ \cos t - \sin t \end{bmatrix}}$$

either is correct

$$\vec{X} = c_1\left(\begin{bmatrix} -1 \\ 2 \end{bmatrix}\cos t - \begin{bmatrix} -1 \\ 0 \end{bmatrix}\sin t\right) + c_2\left(\begin{bmatrix} -1 \\ 0 \end{bmatrix}\cos t + \begin{bmatrix} -1 \\ 2 \end{bmatrix}\sin t\right)$$

$$\boxed{\vec{X} = c_1\begin{bmatrix} -\cos t + \sin t \\ 2\cos t \end{bmatrix} + c_2\begin{bmatrix} -\cos t - \sin t \\ 2\sin t \end{bmatrix}}$$

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325/37

$$\vec{X}' = \begin{bmatrix} 4 & -5 \\ 5 & -4 \end{bmatrix} \vec{X}$$

$$\begin{vmatrix} 4-\lambda & -5 \\ 5 & -4-\lambda \end{vmatrix} = 0$$

$(4-\lambda)(-4-\lambda) + 25 = 0$

$\lambda^2 + 9 = 0$

$\lambda^2 = -9$ $\lambda = \pm 3i$

$\lambda = 3i$

$$\left[\begin{array}{cc|c} 4-3i & -5 & 0 \\ 5 & -4-3i & 0 \end{array}\right]$$

using $(4-3i)k_1 - 5k_2 = 0$

$5k_1 - (4+3i)k_2 = 0$

using $(4-3i)k_1 - 5k_2 = 0$

$k_2 = \frac{1}{5}(4-3i)k_1$

$(k_1, \frac{1}{5}(4-3i)k_1)$

$$\begin{bmatrix} 5 \\ 4-3i \end{bmatrix}$$

$\vec{B}_1 = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$ $\vec{B}_2 = \begin{bmatrix} 0 \\ -3 \end{bmatrix}$

using $5k_1 - (4+3i)k_2 = 0$

$k_1 = \frac{1}{5}(4+3i)k_2$

$(\frac{1}{5}(4+3i)k_2, k_2)$

$$\begin{bmatrix} 4+3i \\ 5 \end{bmatrix}$$

$\vec{B}_1 = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$ $\vec{B}_2 = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$

$$\vec{X} = c_1\left(\begin{bmatrix} 5 \\ 4 \end{bmatrix}\cos 3t - \begin{bmatrix} 0 \\ -3 \end{bmatrix}\sin 3t\right) + c_2\left(\begin{bmatrix} 0 \\ -3 \end{bmatrix}\cos 3t + \begin{bmatrix} 5 \\ 4 \end{bmatrix}\sin 3t\right)$$

$$\boxed{\vec{X} = c_1\begin{bmatrix} 5\cos 3t \\ 4\cos 3t + 3\sin 3t \end{bmatrix} + c_2\begin{bmatrix} 5\sin 3t \\ -3\cos 3t + 4\sin 3t \end{bmatrix}}$$

$$\vec{X} = c_1\left(\begin{bmatrix} 4 \\ 5 \end{bmatrix}\cos 3t - \begin{bmatrix} 3 \\ 0 \end{bmatrix}\sin 3t\right) + c_2\left(\begin{bmatrix} 3 \\ 0 \end{bmatrix}\cos 3t + \begin{bmatrix} 4 \\ 5 \end{bmatrix}\sin 3t\right)$$

$$\boxed{\vec{X} = c_1\begin{bmatrix} 4\cos 3t - 3\sin 3t \\ 5\cos 3t \end{bmatrix} + c_2\begin{bmatrix} 3\cos 3t + 4\sin 3t \\ 5\sin 3t \end{bmatrix}}$$

326 / 46

$\vec{X}' = \begin{bmatrix} 6 & -1 \\ 5 & 4 \end{bmatrix} \vec{X} \quad \vec{X}(0) = \begin{bmatrix} -2 \\ 8 \end{bmatrix}$

$\begin{vmatrix} 6-\lambda & -1 \\ 5 & 4-\lambda \end{vmatrix} = 0$

$(6-\lambda)(4-\lambda) + 5 = 0$

$\lambda^2 - 10\lambda + 29 = 0$

$\lambda = \dfrac{10 \pm \sqrt{100 - 4(29)}}{2}$

$\lambda = \dfrac{10 \pm \sqrt{-16}}{2} = \dfrac{10 \pm 4i}{2}$

$\lambda = 5 \pm 2i$

$\lambda = 5 + 2i$

$\left[\begin{array}{cc|c} 6-(5+2i) & -1 & 0 \\ 5 & 4-(5+2i) & 0 \end{array}\right)$

$(1-2i)k_1 - k_2 = 0$

$5k_1 + (-1-2i)k_2 = 0$

using

using $(1-2i)k_1 - k_2 = 0$

$k_2 = (1-2i)k_1 \quad \begin{bmatrix} 1 \\ 1-2i \end{bmatrix}$

$(k_1, (1-2i)k_1)$

$\vec{B_1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \vec{B_2} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$

$\vec{X} = c_1\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\cos 2t - \begin{pmatrix} 0 \\ -2 \end{pmatrix}\sin 2t\right)e^{5t}$

$+ c_2\left(\begin{bmatrix} 0 \\ -2 \end{bmatrix}\cos 2t + \begin{pmatrix} 1 \\ 1 \end{pmatrix}\sin 2t\right)e^{5t}$

$\vec{X} = c_1\begin{bmatrix} \cos 2t \\ \cos 2t + 2\sin 2t \end{bmatrix}e^{5t} + c_2\begin{bmatrix} \sin 2t \\ -2\cos 2t + \sin 2t \end{bmatrix}e^{5t}$

$\vec{X}(0) = \begin{bmatrix} -2 \\ 8 \end{bmatrix}$

$\begin{bmatrix} -2 \\ 8 \end{bmatrix} = \begin{bmatrix} c_1 \cos(0) \\ c_1(\cos 0 + 2\sin 0) \end{bmatrix}e^0 + \begin{bmatrix} c_2 \sin 0 \\ c_2(-2\cos 0 + \sin 0) \end{bmatrix}e^0$

$\begin{bmatrix} -2 \\ 8 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_1 - 2c_2 \end{bmatrix}$

$c_1 = -2$

$8 = (-2) - 2c_2$

$10 = -2c_2$

$c_2 = -5$

$$\boxed{\vec{X} = -2\begin{bmatrix} \cos 2t \\ \cos 2t + 2\sin 2t \end{bmatrix}e^{5t} - 5\begin{bmatrix} \sin 2t \\ -2\cos 2t + \sin 2t \end{bmatrix}e^{5t}}$$

— or —

using $5k_1 + (-1-2i)k_2 = 0$

$k_1 = \frac{1}{5}(1+2i)k_2 \quad \begin{bmatrix} 1+2i \\ 5 \end{bmatrix}$

$(\frac{1}{5}(1+2i)k_2, k_2)$

$\vec{B_1} = \begin{bmatrix} 1 \\ 5 \end{bmatrix} \quad \vec{B_2} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

$\vec{X} = c_1\left(\begin{bmatrix} 1 \\ 5 \end{bmatrix}\cos 2t - \begin{pmatrix} 2 \\ 0 \end{pmatrix}\sin 2t\right)e^{5t} + c_2\left(\begin{bmatrix} 2 \\ 0 \end{bmatrix}\cos 2t + \begin{pmatrix} 1 \\ 5 \end{pmatrix}\sin 2t\right)e^{5t}$

$\vec{X} = c_1\begin{bmatrix} \cos 2t - 2\sin 2t \\ 5\cos 2t \end{bmatrix}e^{5t} + c_2\begin{bmatrix} 2\cos 2t + \sin 2t \\ 5\sin 2t \end{bmatrix}e^{5t}$

$\vec{X}(0) = \begin{bmatrix} -2 \\ 8 \end{bmatrix}$

$\begin{bmatrix} -2 \\ 8 \end{bmatrix} = \begin{bmatrix} c_1(\cos 0 - 2\sin 0) \\ 5c_1 \cos 0 \end{bmatrix}e^0 + \begin{bmatrix} c_2(2\cos 0 + \sin 0) \\ 5c_2 \sin 0 \end{bmatrix}e^0$

$\begin{bmatrix} -2 \\ 8 \end{bmatrix} = \begin{bmatrix} c_1 + 2c_2 \\ 5c_1 \end{bmatrix}$

$8 = 5c_1 \quad c_1 = \frac{8}{5}$

$-2 = c_1 + 2c_2$

$-2 = (\frac{8}{5}) + 2c_2$

$-10 = 8 + 10c_2$

$-18 = 10c_2$

$c_2 = \frac{-18}{10} = \frac{-9}{5}$

$$\boxed{\vec{X} = \frac{8}{5}\begin{bmatrix} \cos 2t - 2\sin 2t \\ 5\cos 2t \end{bmatrix}e^{5t} - \frac{9}{5}\begin{bmatrix} 2\cos 2t + \sin 2t \\ 5\sin 2t \end{bmatrix}e^{5t}}$$

332/5 $\vec{X}' = \begin{bmatrix} 4 & 1/3 \\ 9 & 6 \end{bmatrix}\vec{X} + \begin{bmatrix} -3 \\ -10 \end{bmatrix} e^t$

$\begin{vmatrix} 4-\lambda & 1/3 \\ 9 & 6-\lambda \end{vmatrix} = 0$

$(4-\lambda)(6-\lambda) - 3 = 0$

$\lambda^2 - 10\lambda + 21 = 0$

$(\lambda-7)(\lambda-3) = 0$

$\lambda = 7 \quad \lambda = 3$

$\lambda = 7$

$\left[\begin{array}{cc|c} -3 & 1/3 & 0 \\ 9 & -1 & 0 \end{array}\right]$

rref

$\left[\begin{array}{cc|c} 1 & -1/9 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$K_1 = 1/9 K_2$

$\begin{bmatrix} 1 \\ 9 \end{bmatrix}$

$\lambda = 3$

$\left[\begin{array}{cc|c} 1 & 1/3 & 0 \\ 9 & 3 & 0 \end{array}\right]$

rref

$\left[\begin{array}{cc|c} 1 & 1/3 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$K_1 = -1/3 K_2$

$\begin{bmatrix} -1 \\ 3 \end{bmatrix}$

$$\boxed{\vec{X}_c = C_1 \begin{bmatrix} 1 \\ 9 \end{bmatrix} e^{7t} + C_2 \begin{bmatrix} -1 \\ 3 \end{bmatrix} e^{3t}}$$

method of undetermined coefficients

$\vec{F} = \begin{bmatrix} -3e^t \\ 10e^t \end{bmatrix}$ table $\vec{X}_p = \begin{bmatrix} Ae^t \\ Be^t \end{bmatrix}$

(no absorption w/ $\vec{X}_c$ terms)

$\vec{X}_p = \begin{bmatrix} Ae^t \\ Be^t \end{bmatrix} \quad \vec{X}' = \begin{bmatrix} Ae^t \\ Be^t \end{bmatrix}$

into DE system:

$\begin{bmatrix} Ae^t \\ Be^t \end{bmatrix} = \begin{bmatrix} 4 & 1/3 \\ 9 & 6 \end{bmatrix}\begin{bmatrix} Ae^t \\ Be^t \end{bmatrix} + \begin{bmatrix} -3e^t \\ 10e^t \end{bmatrix}$

$\begin{bmatrix} Ae^t \\ Be^t \end{bmatrix} = \begin{bmatrix} 4Ae^t + \frac{1}{3}Be^t - 3e^t \\ 9Ae^t + 6Be^t + 10e^t \end{bmatrix}$

$\begin{cases} 4A + 1/3 B - 3 = A \\ 9A + 6B + 10 = B \end{cases}$

$\begin{cases} 3A + 1/3 B = 3 \\ 9A + 5B = -10 \end{cases}$

$\left[\begin{array}{cc|c} 3 & 1/3 & 3 \\ 9 & 5 & -10 \end{array}\right]$

rref

$\left[\begin{array}{cc|c} 1 & 0 & 55/36 \\ 0 & 1 & -19/4 \end{array}\right]$

so $\vec{X}_p = \begin{bmatrix} 55/36 \\ -19/4 \end{bmatrix} e^t$

$$\boxed{\vec{X} = C_1 \begin{bmatrix} 1 \\ 9 \end{bmatrix} e^{7t} + C_2 \begin{bmatrix} -1 \\ 3 \end{bmatrix} e^{3t} + \begin{bmatrix} 55/36 \\ -19/4 \end{bmatrix} e^t}$$

method of variation of parameters

$\Phi = \begin{bmatrix} e^{7t} & -e^{3t} \\ 9e^{7t} & 3e^{3t} \end{bmatrix}$ $\det\Phi = (e^{7t})3e^{3t} + e^{3t}9e^{7t}$
$= 3e^{10t} + 9e^{10t} = 12e^{10t}$

$\Phi^{-1} = \frac{1}{12e^{10t}}\begin{bmatrix} 3e^{3t} & e^{3t} \\ -9e^{7t} & e^{7t} \end{bmatrix} = \begin{bmatrix} 1/4 e^{-7t} & 1/12 e^{-7t} \\ -3/4 e^{-3t} & 1/12 e^{-3t} \end{bmatrix}$

$\vec{X}_p = \Phi \int \Phi^{-1} F \, dt$

$= \begin{bmatrix} e^{7t} & -e^{3t} \\ 9e^{7t} & 3e^{3t} \end{bmatrix} \int \begin{bmatrix} 1/4 e^{-7t} & 1/12 e^{-7t} \\ -3/4 e^{-3t} & 1/12 e^{-3t} \end{bmatrix}\begin{bmatrix} -3e^t \\ 10e^t \end{bmatrix} dt$

" $\int \begin{bmatrix} -\frac{3}{4}e^{-6t} + \frac{5}{6}e^{-6t} \\ \frac{9}{4}e^{-2t} + \frac{5}{6}e^{-2t} \end{bmatrix} dt$

$\begin{bmatrix} e^{7t} & -e^{3t} \\ 9e^{7t} & 3e^{3t} \end{bmatrix}\begin{bmatrix} \frac{1}{8}e^{-6t} - \frac{5}{36}e^{-6t} \\ -\frac{9}{8}e^{-2t} - \frac{5}{12}e^{-2t} \end{bmatrix}$

$\vec{X}_p = \begin{bmatrix} 1/8 e^t - \frac{5}{36}e^t + \frac{9}{8}e^t + \frac{5}{12}e^t \\ \frac{9}{8}e^t - \frac{5}{4}e^t - \frac{27}{8}e^t - \frac{5}{4}e^t \end{bmatrix}$

$\vec{X}_p = \begin{bmatrix} 55/36 \\ -19/4 \end{bmatrix} e^t$

so

$$\boxed{\vec{X} = C_1 \begin{bmatrix} 1 \\ 9 \end{bmatrix} e^{7t} + C_2 \begin{bmatrix} -1 \\ 3 \end{bmatrix} e^{3t} + \begin{bmatrix} 55/36 \\ -19/4 \end{bmatrix} e^t}$$

333/19 $\vec{X}' = \begin{bmatrix} 3 & 2 \\ -2 & -1 \end{bmatrix}\vec{X} + \begin{bmatrix} 2e^{-t} \\ e^{-t} \end{bmatrix}$

$\begin{vmatrix} 3-\lambda & 2 \\ -2 & -1-\lambda \end{vmatrix} = 0$

$(3-\lambda)(-1-\lambda)+4=0$

$\lambda^2-2\lambda+1=0$

$(\lambda-1)(\lambda-1)$

$\lambda=1$ repeated

$\lambda=1$

$\left[\begin{array}{cc|c} 2 & 2 & 0 \\ -2 & -2 & 0 \end{array}\right]$

rref

$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$K_1=-K_2$

$\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

2nd

$\left[\begin{array}{cc|c} 2 & 2 & 1 \\ -2 & -2 & -1 \end{array}\right]$

rref

$\left[\begin{array}{cc|c} 1 & 1 & 1/2 \\ 0 & 0 & 0 \end{array}\right]$

$K_1+K_2=1/2$

$K_2=-K_1+1/2$

$\begin{bmatrix} 0 \\ 1/2 \end{bmatrix}$

$$\boxed{\vec{X}_c = c_1\begin{bmatrix} 1 \\ -1 \end{bmatrix}e^t + c_2\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}te^t + \begin{bmatrix} 0 \\ 1/2 \end{bmatrix}e^t\right)}$$

undetermined coefficients

$\vec{X}_p = \begin{bmatrix} Ae^{-t} \\ Be^{-t} \end{bmatrix}$ (no absorption w/ $X_c$ term)

$X' = \begin{bmatrix} -Ae^{-t} \\ -Be^{-t} \end{bmatrix}$

$\begin{bmatrix} -Ae^{-t} \\ -Be^{-t} \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} Ae^{-t} \\ Be^{-t} \end{bmatrix} + \begin{bmatrix} 2e^{-t} \\ e^{-t} \end{bmatrix}$

$\begin{bmatrix} -Ae^{-t} \\ -Be^{-t} \end{bmatrix} = \begin{bmatrix} 3Ae^{-t}+2Be^{-t}+2e^{-t} \\ -2Ae^{-t}-Be^{-t}+e^{-t} \end{bmatrix}$

$\begin{cases} 3A+2B+2=-A \\ -2A-B+1=-B \end{cases}$

$\begin{cases} 4A+2B=-2 \\ -2A \quad\;\; =-1 \end{cases}$

$\left[\begin{array}{cc|c} 4 & 2 & -2 \\ -2 & 0 & -1 \end{array}\right]$

rref

$\left[\begin{array}{cc|c} 1 & 0 & 1/2 \\ 0 & 1 & -2 \end{array}\right]$ $A=1/2$, $B=-2$

$\vec{X}_p = \begin{bmatrix} 1/2\,e^{-t} \\ -2e^{-t} \end{bmatrix}$

$$\boxed{\vec{X} = c_1\begin{bmatrix} 1 \\ -1 \end{bmatrix}e^t + c_2\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}te^t + \begin{bmatrix} 0 \\ 1/2 \end{bmatrix}e^t\right) + \begin{bmatrix} 1/2 \\ -2 \end{bmatrix}e^{-t}}$$

* $\int -6te^{-2t}\,dt$

$u=-6t \quad dv=e^{-2t}dt$

$du=-6dt \quad v=-\frac{1}{2}e^{-2t}$

$uv-\int v\,du$

$3te^{-2t}-\int -\frac{1}{2}e^{-2t}(-6)dt$

$-3\int e^{-2t}dt$

$3te^{-2t}+\frac{3}{2}e^{-2t}$

variation of parameters

$\Phi = \begin{bmatrix} e^t & te^t \\ -e^t & -te^t+1/2e^t \end{bmatrix}$ $\det\Phi = -te^{2t}+\frac{1}{2}e^{2t}+te^{2t}$
$= \frac{1}{2}e^{2t}$

$\Phi^{-1} = \frac{1}{\frac{1}{2}e^{2t}}\begin{bmatrix} -te^t+1/2e^t & -te^t \\ e^t & e^t \end{bmatrix} = \begin{bmatrix} -2te^{-t}+e^{-t} & -2te^{-t} \\ 2e^{-t} & 2e^{-t} \end{bmatrix}$

$\vec{X}_p = \Phi\int\Phi^{-1}F\,dt$

$= \begin{bmatrix} e^t & te^t \\ -e^t & -te^t+\frac{1}{2}e^t \end{bmatrix}\int\begin{bmatrix} -2te^{-t}+e^{-t} & -2te^{-t} \\ 2e^{-t} & 2e^{-t} \end{bmatrix}\begin{bmatrix} 2e^{-t} \\ e^{-t} \end{bmatrix}dt$

$\int\begin{bmatrix} -4te^{-2t}+2e^{-2t}-2te^{-2t} \\ 4e^{-2t}+2e^{-2t} \end{bmatrix}dt$

$\int\begin{bmatrix} ^{*}-6te^{-2t}+2e^{-2t} \\ 6e^{-2t} \end{bmatrix}dt$

$\begin{bmatrix} 3te^{-2t}+\frac{3}{2}e^{-2t}-e^{-2t} \\ -3e^{-2t} \end{bmatrix}$

$\begin{bmatrix} e^t & te^t \\ -e^t & -te^t+\frac{1}{2}e^t \end{bmatrix}\begin{bmatrix} 3te^{-2t}+\frac{1}{2}e^{-2t} \\ -3e^{-2t} \end{bmatrix}$

$\vec{X}_p = \begin{bmatrix} 3te^{-t}+\frac{1}{2}e^{-t}-3te^{-t} \\ -3te^{-t}-\frac{1}{2}e^{-t}+3te^{-t}-\frac{3}{2}e^{-t} \end{bmatrix}$

$= \begin{bmatrix} \frac{1}{2}e^{-t} \\ -2e^{-t} \end{bmatrix}$

so

$$\boxed{\vec{X} = c_1\begin{bmatrix} 1 \\ -1 \end{bmatrix}e^t + c_2\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}te^t + \begin{bmatrix} 0 \\ 1/2 \end{bmatrix}e^t\right) + \begin{bmatrix} 1/2 \\ -2 \end{bmatrix}e^{-t}}$$

added problem: write as a system:

$$y''' - y'' + 2y' - 3y = t^3 - 4t \qquad y(0)=3,\ y'(0)=7,\ y''(0)=5$$

$$y''' = 3y - 2y' + y'' + t^3 - 4t$$

so

$$\begin{cases} y' = 0y + 1y' + 0y'' + 0 \\ y'' = 0y + 0y' + 1y'' + 0 \\ y''' = 3y - 2y' + y'' + t^3 - 4t \end{cases}$$

$$\boxed{\text{if } \vec{X} = \begin{bmatrix} y \\ y' \\ y'' \end{bmatrix} \text{ then } \vec{X}' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 3 & -2 & 1 \end{bmatrix} \vec{X} + \begin{bmatrix} 0 \\ 0 \\ t^3 - 4t \end{bmatrix} \text{ and } \vec{X}(0) = \begin{bmatrix} 3 \\ 7 \\ 5 \end{bmatrix}}$$