Higher-Order DE procedures
<u>for solving homogeneous DEs</u>

2nd-order linear, have one solution and need another?

$$a(x)y'' + b(x)y' + c(x)y = 0$$

yes — **Reduction of Order**

$$y_2(x) = u(x)y_1(x)$$

(take derivatives and substitute w=u', solve resulting 1st-order DE, then integrate to get u).

no — **homogeneous with constant coefficients?**

$$ay'' + by' + cy = 0$$

yes — **use auxiliary equation**

$$am^2 + bm + c = 0$$

single root:	$y = C_1 e^{mx}$
repeated roots:	$y = C_1 e^{mx} + C_2 x e^{mx} + C_3 x^2 e^{mx} \ldots$
complex pairs:	$y = C_1 e^{\alpha x}\cos\beta x + C_2 e^{\alpha x}\sin\beta x$

no — **2nd-order, Cauchy-Euler form?**

$$ax^2 y'' + bxy' + cy = 0$$

no — **We'll need methods from future chapters...**

yes — **use auxiliary equation**

$$am^2 + (b-a)m + c = 0$$

distinct roots:	$y = C_1 x^{m_1} + C_2 x^{m_2}$
repeated roots:	$y = C_1 x^m + C_2 x^m \ln x$
complex pairs:	$y = C_1 x^\alpha \cos(\beta \ln x) + C_2 x^\alpha \sin(\beta \ln x)$

Check if solutions are Linearly Independent (form a fundamental solution set) using Wronskian:

$$W(f_1(x), f_2(x) \ldots f_n(x)) = \begin{vmatrix} f_1 & f_2 & \ldots & f_n \\ f_1' & f_2' & \ldots & f_n' \\ \ldots & \ldots & \ldots & \ldots \\ f_1^{(n-1)} & f_2^{(n-1)} & \ldots & f_n^{(n-1)} \end{vmatrix}$$

Linearly Independent if Wronskian is non-zero:

Linear non-homogeneous DE with constant coefficients?

$$ay'' + by' + cy = g(x)$$

(can be higher order too)

yes — **Undetermined Coefficients**

use table to find form...

$g(x)$	Form of y_p
1. 1 (any constant)	A
2. $5x + 7$	$Ax + B$
3. $3x^2 - 2$	$Ax^2 + Bx + C$
4. $x^3 - x + 1$	$Ax^3 + Bx^2 + Cx + E$
5. $\sin 4x$	$A\cos 4x + B\sin 4x$
6. $\cos 4x$	$A\cos 4x + B\sin 4x$
7. e^{5x}	Ae^{5x}
8. $(9x - 2)e^{5x}$	$(Ax + B)e^{5x}$
9. $x^2 e^{5x}$	$(Ax^2 + Bx + C)e^{5x}$
10. $e^{3x}\sin 4x$	$Ae^{3x}\cos 4x + Be^{3x}\sin 4x$
11. $5x^2 \sin 4x$	$(Ax^2 + Bx + C)\cos 4x + (Ex^2 + Fx + G)\sin 4x$
12. $xe^{3x}\cos 4x$	$(Ax + B)e^{3x}\cos 4x + (Cx + E)e^{3x}\sin 4x$

if a y_p term matches a y_c term (use smallest *n* that isn't absorbed) = 'absorbed' so multiply term by x^n

...take derivatives, plug into DE and solve for constants, then: $y = y_C + y_P$

no — **Variation of Parameters**

assume terms have same form as homogeneous solution multiplied by *u* functions:

$$y_C = C_1 y_1 + C_2 y_2$$

$$y_P = u_1(x)y_1 + u_2(x)y_2$$

Divide by leading term to get $f(x) = \dfrac{g(x)}{a_n(x)}$

then find *u* function derivatives with Cramer's rule:

$$u_1' = \frac{\begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}}, \quad u_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}}$$

then integrate to find *u* functions, and solution:

$$u_1 = \int u_1'\,dx, \qquad u_2 = \int u_2'\,dx$$

$$y_P = u_1 y_1 + u_2 y_2$$

$$y = y_C + y_P$$