Derivations for DiffEq 8.2 day 3, Complex Eigenvalues

It is very unwieldy to try to derive what is happening when we have complex eigenvalues in a general way, even for a small system (although mathematicians have, of course, proven this to be true). Instead, to give an intuitive understanding of what is happening, we will consider one specific example, but showing all details, and show how this is generalized to the procedural solution we are using in class.

Starting with this specific small system of differential equations…

$$\frac{dx}{dt} = 6x - y$$
$$\frac{dy}{dt} = 5x + 4y$$

…we find eigenvalues:

$$\begin{vmatrix} 6-\lambda & -1 \\ 5 & 4-\lambda \end{vmatrix} = 0$$
$$(6-\lambda)(4-\lambda)-(-1)(5)=0$$
$$24-10\lambda+\lambda^2+5=0$$
$$\lambda^2-10\lambda+29=0$$
$$\lambda=\frac{10\pm\sqrt{100-4(29)}}{2}$$
$$\lambda=\frac{10\pm\sqrt{-16}}{2}$$
$$\lambda=\frac{10\pm 4i}{2}$$
$$\lambda=5\pm 2i$$

So the two eigenvalues are a complex conjugate pair.

Next, we'll select the + case and find a corresponding eigenvector:

$$\begin{bmatrix} 6-(5+2i) & -1 & 0 \\ 5 & 4-(5+2i) & 0 \end{bmatrix}$$
$$\begin{bmatrix} 1-2i & -1 & 0 \\ 5 & -1-2i & 0 \end{bmatrix}$$

Because the values include imaginary numbers, we can't directly plug this into a calculator and do rref. Instead, we need to consider these two equations as equations and solve for all of the variables in terms of one of the variables as a 'parameter'. Since here we have two 'variables' k_1 and k_2 that means each equation will allow us to write one variable in terms of the other. We can use either equation, and doing so will end up producing different, but both correct, results. The value we use for one equation happens also to make the other equations in the system true as well (we'll demonstrate this in this example).

So the system now corresponds to a pair of equations:

$$\begin{bmatrix} 1-2i & -1 & 0 \\ 5 & -1-2i & 0 \end{bmatrix}$$
$$(1-2i)k_1 - k_2 = 0$$
$$5k_1 + (-1-2i)k_2 = 0$$

We pick one of these equations to work with…here, we will work with the top equation (later, we will redo this problem using the other equation to show that this also produces a valid solution). Using the top equation, we can solve for one variable in terms of the other. (If we had a larger system, we use multiple equations and solve all of them so that they use just one of the variables, usually the last one, like k_3 as the parameter).

$$(1-2i)k_1 - k_2 = 0$$
$$k_2 = (1-2i)k_1$$

So constants of the following form are solutions to the system:

$$(k_1,\ (1-2i)k_1)$$

We then choose any convenient value for k_1 and form the eigenvector:

$$(1,\ 1-2i)$$
$$\begin{bmatrix} 1 \\ 1-2i \end{bmatrix} \ \textit{for}\ \lambda = 5+2i$$

…which means this solution would be:

$$\vec{X}_1 = \begin{bmatrix} 1 \\ 1-2i \end{bmatrix} e^{(5+2i)t}$$
$$= \begin{bmatrix} 1 \\ 1-2i \end{bmatrix} e^{5t} e^{2it}$$

Using Euler's identity: $e^{i\theta} = \cos\theta + i\sin\theta$

$$\vec{X}_1 = \begin{bmatrix} 1 \\ 1-2i \end{bmatrix} e^{5t} e^{2it}$$
$$= \begin{bmatrix} 1 \\ 1-2i \end{bmatrix} e^{5t}(\cos 2t + i\sin 2t)$$

We can now form the general solution by combining this solution with another term made from the complex conjugate root from the pair:

$$\vec{X} = c_1 \vec{X}_1 + c_2 \vec{X}_2$$

$$\vec{X} = c_1 \begin{bmatrix} 1 \\ 1-2i \end{bmatrix} e^{5t} \left(\cos 2t + i \sin 2t\right) + c_2 \begin{bmatrix} 1 \\ 1+2i \end{bmatrix} e^{5t} \left(\cos(-2t) + i \sin(-2t)\right)$$

Using the fact that $\cos(-\theta) = \cos\theta \quad and \quad \sin(-\theta) = -\sin\theta$

$$\vec{X} = c_1 \begin{bmatrix} 1 \\ 1-2i \end{bmatrix} e^{5t} \left(\cos 2t + i \sin 2t\right) + c_2 \begin{bmatrix} 1 \\ 1+2i \end{bmatrix} e^{5t} \left(\cos 2t - i \sin 2t\right)$$

Each term contains a mix of real and imaginary parts, so at this point we write the system out as separate equations…

$$x = c_1 e^{5t} \left(\cos 2t + i \sin 2t\right) + c_2 e^{5t} \left(\cos 2t - i \sin 2t\right)$$
$$y = c_1 \left(1-2i\right) e^{5t} \left(\cos 2t + i \sin 2t\right) + c_2 \left(1+2i\right) e^{5t} \left(\cos 2t - i \sin 2t\right)$$

…within each part, then grouping the sines and cosines together…

$$\begin{aligned}
x &= c_1 e^{5t} \cos 2t + i c_1 e^{5t} \sin 2t + c_2 e^{5t} \cos 2t - i c_2 e^{5t} \sin 2t \\
y &= c_1 e^{5t} \cos 2t + i c_1 e^{5t} \sin 2t - i2c_1 e^{5t} \cos 2t - i^2 2c_1 \sin 2t \\
&\qquad + c_2 e^{5t} \cos 2t - i c_2 e^{5t} \sin 2t + i2c_2 e^{5t} \cos 2t - i^2 2c_2 e^{5t} \sin 2t \\
&\quad and\ with\ i^2 = -1 \\
x &= c_1 e^{5t} \cos 2t + i c_1 e^{5t} \sin 2t + c_2 e^{5t} \cos 2t - i c_2 e^{5t} \sin 2t \\
y &= c_1 e^{5t} \cos 2t + i c_1 e^{5t} \sin 2t - i2c_1 e^{5t} \cos 2t + 2c_1 \sin 2t \\
&\qquad + c_2 e^{5t} \cos 2t - i c_2 e^{5t} \sin 2t + i2c_2 e^{5t} \cos 2t + 2c_2 e^{5t} \sin 2t \\
&\quad grouping\ cosines\ and\ sines\ together... \\
x &= \left[c_1 e^{5t} \cos 2t + c_2 e^{5t} \cos 2t\right] + \left[i c_1 e^{5t} \sin 2t - i c_2 e^{5t} \sin 2t\right] \\
y &= \left[c_1 e^{5t} \cos 2t - i2c_1 e^{5t} \cos 2t + c_2 e^{5t} \cos 2t + i2c_2 e^{5t} \cos 2t\right] \\
&\quad + \left[i c_1 e^{5t} \sin 2t + 2c_1 \sin 2t - i c_2 e^{5t} \sin 2t + 2c_2 e^{5t} \sin 2t\right]
\end{aligned}$$

$$x = \left[c_1 + c_2\right] e^{5t} \cos 2t + i\left[c_1 - c_2\right] e^{5t} \sin 2t$$
$$y = \left[c_1 - i2c_1 + c_2 + i2c_2\right] e^{5t} \cos 2t + \left[i c_1 + 2c_1 - i c_2 + 2c_2\right] e^{5t} \sin 2t$$

…separating the y equation's real and imaginary parts…

$$x=[c_1+c_2]e^{5t}\cos 2t+i[c_1-c_2]e^{5t}\sin 2t$$
$$y=[c_1+c_2]e^{5t}\cos 2t+[2c_1+2c_2]e^{5t}\sin 2t$$
$$+[-i2c_1+i2c_2]e^{5t}\cos 2t+[ic_1-ic_2]e^{5t}\sin 2t$$

$$x=[c_1+c_2]e^{5t}\cos 2t+i[c_1-c_2]e^{5t}\sin 2t$$
$$y=[c_1+c_2]e^{5t}\cos 2t+2[c_1+c_2]e^{5t}\sin 2t$$
$$-2i[c_1-c_2]e^{5t}\cos 2t+i[c_1-c_2]e^{5t}\sin 2t$$

We now define new constants: $C_1=c_1+c_2 \quad and \quad C_2=i[c_1-c_2]$

$$x=C_1e^{5t}\cos 2t+C_2e^{5t}\sin 2t$$
$$y=C_1e^{5t}\cos 2t+2C_1e^{5t}\sin 2t-2C_2e^{5t}\cos 2t+C_2e^{5t}\sin 2t$$

Now putting this back into matrix form…

$$x=C_1e^{5t}\cos 2t+C_2e^{5t}\sin 2t$$
$$y=C_1e^{5t}\cos 2t+2C_1e^{5t}\sin 2t-2C_2e^{5t}\cos 2t+C_2e^{5t}\sin 2t$$

$$\vec{X}=C_1\begin{bmatrix}\cos 2t\\ \cos 2t+2\sin 2t\end{bmatrix}e^{5t}+C_2\begin{bmatrix}\sin 2t\\ -2\cos 2t+\sin 2t\end{bmatrix}e^{5t}$$

Rearranging…

$$\vec{X}=C_1\left(\begin{bmatrix}1\\1\end{bmatrix}\cos 2t+\begin{bmatrix}0\\2\end{bmatrix}\sin 2t\right)e^{5t}+C_2\left(\begin{bmatrix}0\\-2\end{bmatrix}\cos 2t+\begin{bmatrix}1\\1\end{bmatrix}\sin 2t\right)e^{5t}$$
$$\vec{X}=C_1\left(\begin{bmatrix}1\\1\end{bmatrix}\cos 2t-\underset{\substack{\uparrow\\ double-negative\ so\ vector\ matches\ 2nd\ term}}{\begin{bmatrix}0\\-2\end{bmatrix}}\sin 2t\right)e^{5t}+C_2\left(\begin{bmatrix}0\\-2\end{bmatrix}\cos 2t+\begin{bmatrix}1\\1\end{bmatrix}\sin 2t\right)e^{5t}$$

These column vectors are the real and imaginary parts of the eigenvector we found earlier associated with the positive complex conjugate eigenvalue, and we will define these as follows:

$$\begin{bmatrix}1\\1-2i\end{bmatrix}\ for\ \lambda=5+2i$$
$$\vec{B}_1=\operatorname{Re}\left\{\begin{bmatrix}1\\1-2i\end{bmatrix}\right\}=\begin{bmatrix}1\\1\end{bmatrix} \qquad \vec{B}_2=\operatorname{Im}\left\{\begin{bmatrix}1\\1-2i\end{bmatrix}\right\}=\begin{bmatrix}0\\-2\end{bmatrix}$$

Rewriting the solution using these *B* vectors:

$$\vec{X} = C_1\left(\vec{B}_1 \cos 2t - \vec{B}_2 \sin 2t\right)e^{5t} + C_2\left(\vec{B}_2 \cos 2t + \vec{B}_1 \sin 2t\right)e^{5t}$$

Or we could write it this way:

$$\vec{X} = C_1 \vec{X}_1 + C_2 \vec{X}_2$$

$$\vec{X}_1 = \left(\vec{B}_1 \cos 2t - \vec{B}_2 \sin 2t\right)e^{5t}$$

$$\vec{X}_2 = \left(\vec{B}_2 \cos 2t + \vec{B}_1 \sin 2t\right)e^{5t}$$

This is the solution for this specific example, but this is generalizable to any system, made into a procedure as follows:

1) Express the system in matrix form and find the eigenvalues using $\left|\vec{A} - \lambda \vec{I}\right| = 0$.

2) For any eigenvalues which appear as complex conjugate pairs, write them in the form $\lambda = \alpha \pm \beta i$

3) Using the positive case eigenvalue $\lambda = \alpha + \beta i$ use $\left[\vec{A} - \lambda \vec{I}\right] = \vec{0}$ to find the system and write out the equations (you won't be able to solve using calculator rref because of the imaginary values).

4) This will result in a system of equations for the eigenvector constants $k_1, k_2, \ldots$ Use the equations to express all of the constants in terms of one (parameter) constant, then choose a convenient value for this constant to form the eigenvector $\vec{K}$.

5) Find vectors for the real and imaginary parts of the eigenvector: $\vec{B}_1 = \mathrm{Re}\left\{\vec{K}\right\}, \quad \vec{B}_2 = \mathrm{Im}\left\{\vec{K}\right\}$

6) The system solution is then given by:

$$\vec{X} = C_1 \vec{X}_1 + C_2 \vec{X}_2$$

$$\vec{X}_1 = \left(\vec{B}_1 \cos \beta t - \vec{B}_2 \sin \beta t\right)e^{\alpha t}$$

$$\vec{X}_2 = \left(\vec{B}_2 \cos \beta t + \vec{B}_1 \sin \beta t\right)e^{\alpha t}$$

A few more things to explore…

Did it bother you that we only worked with one of the equations in the system to produce the eigenvector?
Perhaps it bothered you that we only worked with the first equation, then just proceeded without check if our general solution also made the rest of the system true. We can check this now:

$$\begin{bmatrix} 1-2i & -1 & 0 \\ 5 & -1-2i & 0 \end{bmatrix}$$
$$(1-2i)k_1 - k_2 = 0$$
$$5k_1 + (-1-2i)k_2 = 0$$

We chose to work with the first equation, and ended up with this solution for the constants:

$$(1,\ 1-2i)$$
$$\begin{bmatrix} 1 \\ 1-2i \end{bmatrix} \ for\ \lambda = 5+2i$$

If we plug this solution into the 2nd equation in the system…

$$5k_1 + (-1-2i)k_2 = 0$$
$$5(1) + (-1-2i)(1-2i) = 0$$
$$5 + \left(-1+2i-2i+4i^2\right) = 0 \qquad i^2 = -1$$
$$5 - 1 + 4(-1) = 0$$
$$0 = 0$$

So our first equation solution is also a solution for the whole system (and this can be shown to be always true).

Multiple solutions are possible depending upon how the eigenvector is developed

At the step in the procedure where we go from the system equations to the specific choice of constant and a particular eigenvector, there is often more than one way to choose the eigenvector, and choosing differently produces a different, but also correct, solution.

In the example explored at this step…

$$\begin{bmatrix} 1-2i & -1 & 0 \\ 5 & -1-2i & 0 \end{bmatrix}$$
$$(1-2i)k_1 - k_2 = 0$$
$$5k_1 + (-1-2i)k_2 = 0$$

…we chose to use the top equation, but we could have used the bottom equation instead:

$$5k_1 + (-1-2i)k_2 = 0$$

Following the remaining steps would produce a different solution:

$$5k_1 = (1+2i)k_2$$

$$k_1 = \frac{1}{5}(1+2i)k_2$$

$$\left(\frac{1}{5}(1+2i)k_2,\ \ k_2\right)$$

$$\textit{then choose } k_2 = 5:$$

$$(1+2i,\ \ 5)$$

$$\vec{K} = \begin{bmatrix} 1+2i \\ 5 \end{bmatrix}$$

$$\vec{B_1} = \operatorname{Re}\left\{\vec{K}\right\},\ \ \vec{B_2} = \operatorname{Im}\left\{\vec{K}\right\}$$

$$\vec{B_1} = \begin{bmatrix} 1 \\ 5 \end{bmatrix},\ \ \vec{B_2} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$\vec{X_1} = \left(\begin{bmatrix} 1 \\ 5 \end{bmatrix}\cos 2t - \begin{bmatrix} 2 \\ 0 \end{bmatrix}\sin 2t\right)e^{5t}$$

$$\vec{X_2} = \left(\begin{bmatrix} 2 \\ 0 \end{bmatrix}\cos 2t + \begin{bmatrix} 1 \\ 5 \end{bmatrix}\sin 2t\right)e^{6t}$$

$$\vec{X} = C_1\vec{X_1} + C_2\vec{X_2}$$

…and this is also a fully correct solution to the system.

Both of the terms are developed from the same Eigenvector, also is it really valid to just rename a constant with an imaginary value C_2?

Ultimately, for a system with two differential equations we just need two linearly-independent solutions to combine using the superposition principle to form the general solution, so we could check using the system Wronskian if the resulting two terms are, indeed, linearly-independent. Using the ones from the alternate solution just computed above:

$$\vec{X}_1 = \left(\begin{bmatrix} 1 \\ 5 \end{bmatrix} \cos 2t - \begin{bmatrix} 2 \\ 0 \end{bmatrix} \sin 2t \right) e^{5t} = \begin{bmatrix} e^{5t} \cos 2t - 2e^{5t} \sin 2t \\ 5e^{5t} \cos 2t \end{bmatrix}$$

$$\vec{X}_2 = \left(\begin{bmatrix} 2 \\ 0 \end{bmatrix} \cos 2t + \begin{bmatrix} 1 \\ 5 \end{bmatrix} \sin 2t \right) e^{5t} = \begin{bmatrix} 2e^{5t} \cos 2t + e^{5t} \sin 2t \\ 5e^{5t} \sin 2t \end{bmatrix}$$

$$W = \begin{vmatrix} e^{5t} \cos 2t - 2e^{5t} \sin 2t & 2e^{5t} \cos 2t + e^{5t} \sin 2t \\ 5e^{5t} \cos 2t & 5e^{5t} \sin 2t \end{vmatrix}$$

$$\left(e^{5t} \cos 2t - 2e^{5t} \sin 2t \right) 5e^{5t} \sin 2t - \left(2e^{5t} \cos 2t + e^{5t} \sin 2t \right) 5e^{5t} \cos 2t$$

$$5e^{10t} \cos 2t \sin 2t - 10e^{10t} \sin^2 2t - 10e^{10t} \cos^2 2t - 5e^{10t} \cos 2t \sin 2t$$

$$-10e^{10t} \sin^2 2t - 10e^{10t} \cos^2 2t$$

$$-10e^{10t} \left(\sin^2 2t + \cos^2 2t \right)$$

$$-10e^{10t} \neq 0$$

Because the Wronskian is non-zero, these solutions are linearly-independent and form a fundamental solution set. If you did this same calculation on the original solution found earlier in this document, the Wronskian is also non-zero, so both of these are correct, general solutions for the system.