Derivations for DiffEq 7.5, Laplace Transform of Dirac Delta Function

We'll start by considering the unit impulse function:

$$\delta_a(t-t_o)=\begin{cases} 0, & 0\le t<t_0-a \\ \dfrac{1}{2a}, & t_0-a\le t<t_0+a \\ 0 & t\ge t_0+a \end{cases}$$

…what can be written using Unit Step Functions:

$$\delta_a(t-t_o)=\frac{1}{2a}\left[u\left(t-(t_o-a)\right)-u\left(t+(t+t_0+a)\right)\right]$$

Taking the Laplace Transform:

$$L\{\delta_a(t-t_o)\}=\frac{1}{2a}\left[\frac{e^{-s(t_0-a)}}{s}-\frac{e^{-s(t_0+a)}}{s}\right]$$

$$=e^{-st_0}\left(\frac{e^{sa}-e^{-sa}}{2sa}\right)$$

To now work with the Dirac Delta Function $\delta(t-t_o)=\lim_{a\to 0}\delta_a(t-t_o)$ we take the limit:

$$L\{\delta(t-t_o)\}=\lim_{a\to 0}e^{-st_0}\left(\frac{e^{sa}-e^{-sa}}{2sa}\right)$$

$$=e^{-st_0}\lim_{a\to 0}\left(\frac{e^{sa}-e^{-sa}}{2sa}\right)=\frac{0}{0}$$

…which is an indeterminant form, so we can apply L'Hopital's Rule:

$$=e^{-st_0}\lim_{a\to 0}\left(\frac{ae^{sa}+ae^{-sa}}{2a}\right)$$

$$=e^{-st_0}\frac{2a}{2a}$$

$$=e^{-st_0}$$