Derivations for DiffEq 4.7, Cauchy-Euler Equation solutions

Here, we will describe two methods for solving a Cauchy-Euler Equation, which is in the form

$$ax^2 \frac{d^2y}{dx^2} + bx\frac{dy}{dx} + cy = g(x)$$

We will only cover how to solve the corresponding homogeneous DE…

$$ax^2 \frac{d^2y}{dx^2} + bx\frac{dy}{dx} + cy = 0$$

…for the complementary solution y_C. (We use previously learned techniques to find y_P and the general solution.

Main method (method 1):

We start by postulating a solution of the form $y = x^m$. Taking derivatives…

$$y = x^m$$
$$y' = mx^{(m-1)}$$
$$y'' = m(m-1)x^{(m-2)}$$

…and substituting this into the DE, we get…

$$ax^2 \frac{d^2y}{dx^2} + bx\frac{dy}{dx} + cy = 0$$
$$ax^2\left[m(m-1)x^{(m-2)}\right] + bx\left[mx^{(m-1)}\right] + c\left[x^m\right] = 0$$
$$am(m-1)x^2x^{(m-2)} + bmxx^{(m-1)} + cx^m = 0$$
$$am(m-1)x^m + bmx^m + cx^m = 0$$
$$am^2x^m - amx^m + bmx^m + cx^m = 0$$
$$\left[am^2 + (b-a)m + c\right]x^m = 0$$

If we exclude x =0 (which means the interval will be $(0,\infty)$) the *m* for a solution will be given by the auxiliary equation:

$$am^2 + (b-a)m + c = 0$$

As in the earlier section, when we solve, there are three cases:

Case 1: two distinct, real roots:

In the event we get two real roots, $m_1 \text{ and } m_2$, the solution will be in the form:

$$y_C = C_1 x^{m_1} + C_2 x^{m_2}$$
$$\left(note: it\ is\ x^m, not\ e^{mx}\ as\ in\ the\ earlier\ procedure\right)$$

Case 2: one real, repeated root:

In the event we get one real root repeated, m, the first solution will be of the form: $y = C_1 x^m$
But to get the second solution, we'll need to use the Reduction of Order procedure.

Postulate a function *u* such that: $y = ux^m$ is also a solution to the DE. Because there are so many terms in when the product rule is applied, it will be faster for us to apply the formula that was developed in the Reduction of Order procedure to find the second solution, which is…

$$y_2 = y_1(x) \int \frac{e^{-\int P(x)dx}}{\left[y_1(x)\right]^2} dx$$

To use this, we need to put the DE into standard form:

$$ax^2 \frac{d^2y}{dx^2} + bx\frac{dy}{dx} + cy = 0$$
$$\frac{d^2y}{dx^2} + \frac{bx}{ax^2}\frac{dy}{dx} + \frac{c}{ax^2}y = \frac{0}{ax^2}$$
$$\frac{d^2y}{dx^2} + \frac{b}{ax}\frac{dy}{dx} + \frac{c}{ax^2}y = 0$$
$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0$$

so $P(x) = \dfrac{b}{ax}$

Substituting into the Reduction of order equation:

$$y_2 = x^m \int \frac{e^{-\int \frac{b}{a}\frac{1}{x}dx}}{\left[x^m\right]^2} dx = x^m \int \frac{e^{-\frac{b}{a}\ln|x|}}{\left[x^m\right]^2} dx = x^m \int \frac{e^{\ln x^{\frac{-b}{a}}}}{x^{2m}} dx$$

$$= x^m \int \frac{x^{\frac{-b}{a}}}{x^{2m}} dx = x^m \int x^{\left(-\frac{b}{a}-2m\right)} dx$$

Now it would be helpful if we could simplify the exponents in the integral, and we can, because in the auxiliary equation: $am^2 + (b-a)m + c = 0$ solved by quadratic formula:

$$am^2+(b-a)m+c=0$$

$$m=\frac{-(b-a)\pm\sqrt{(b-a)^2-4ac}}{2a}$$ the only way there is a single root is if the discriminant is zero, so :

$$m=\frac{-(b-a)}{2a}$$

$$-2m=\frac{(b-a)}{a}$$

so $x^{\left(-\frac{b}{a}-2m\right)}=x^{\left(-\frac{b}{a}+(-2m)\right)}=x^{\left(-\frac{b}{a}+\frac{(b-a)}{a}\right)}=x^{(-1)}$

…and then the second solution is:

$$\begin{aligned} y_2 &= x^m\int x^{\left(-\frac{b}{a}-2m\right)}dx \\ &= x^m\int x^{(-1)}dx \\ &= x^m\int \frac{1}{x}dx \\ &= x^m\ln x \end{aligned}$$

(We don't need the absolute value, because we've already restricted the solution to the interval of positive x only).

Therefore, for case 2 with repeated real roots, the solution is:

$$y_C=C_1x^m+C_2x^m\ln x$$

Case3: a complex-conjugate pair of roots:

In the event we get a complex-conjugate pair of roots from the auxiliary equation the solution will be of the form: $y=C_1x^{(\alpha+i\beta)}+C_2x^{(\alpha-i\beta)}$

In the previous section, we used Euler's identity to write this using only real values, so we will need something similar but for base *x* instead of base *e*:

$$\begin{aligned} &Euler: e^{i\theta}=\cos\theta+i\sin\theta \\ &also\ note\ that\ x^{i\beta}=\left(e^{\ln x}\right)^{i\beta}=e^{i\beta\ln x} \\ &then\ x^{i\beta}=\cos(\beta\ln x)+i\sin(\beta\ln x) \\ &and\ x^{-i\beta}=\cos(\beta\ln x)-i\sin(\beta\ln x) \end{aligned}$$

Adding and subtracting the last two equations gives us…

$$x^{i\beta}+x^{-i\beta}=2\cos(\beta\ln x)$$
$$x^{i\beta}-x^{-i\beta}=2i\sin(\beta\ln x)$$

…and since our solution is in the form: $y=C_1x^{(\alpha+i\beta)}+C_2x^{(\alpha-i\beta)}$ if you chose C1=1 and C2 to be either 1 or -1, this would give two solutions:

$$\begin{aligned} y_1 &= x^{(\alpha+i\beta)}+x^{(\alpha-i\beta)} \\ &= x^{\alpha}x^{i\beta}+x^{\alpha}x^{-i\beta} \\ &= x^{\alpha}\left(x^{i\beta}+x^{-i\beta}\right) \\ &= x^{\alpha}2\cos(\beta\ln x) \end{aligned} \qquad \begin{aligned} y_2 &= x^{(\alpha+i\beta)}-x^{(\alpha-i\beta)} \\ &= x^{\alpha}x^{i\beta}-x^{\alpha}x^{-i\beta} \\ &= x^{\alpha}\left(x^{i\beta}-x^{-i\beta}\right) \\ &= x^{\alpha}2i\sin(\beta\ln x) \end{aligned}$$

In the first solution, the 2 is a constant and in the second, the 2 and the *i* are constants, so the form of the two solutions is

$$y_1=C_1x^{\alpha}\cos(\beta\ln x) \qquad y_2=C_2x^{\alpha}\sin(\beta\ln x)$$

…and these two solutions can be shown not only to be solutions of the original 2nd-order DE, but also linearly independent, so they form a fundamental solution set. Therefore, the general solution in the complex-conjugate pairs case is:

$$y=C_1x^{\alpha}\cos(\beta\ln x)+C_2x^{\alpha}\sin(\beta\ln x)$$

Substitute to Change to Constant Coefficients method (method 2):

There is another method which can be used which produces the same results. Any Cauchy-Euler equation can be re-expressed as a linear differential equation with constant coefficients by means of a specific substitution:

Substitute: $x = e^t \qquad \textit{which means that}: \quad t = \ln x$

In practice, you also need to substitute expressions for y'' and y', so we'll compute those as well using the chain rule and product rule:

$$\textit{if } x = e^t, \textit{ then } \frac{dx}{dt} = e^t \qquad \textit{and } t = \ln x, \textit{ so } \frac{dt}{dx} = \frac{1}{x}$$

This means that...

$$y' = \frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{1}{x}\frac{dy}{dt} \qquad \textit{and} \qquad y'' = \frac{1}{x}\frac{d}{dx}\left[\frac{dy}{dt}\right] + \frac{dy}{dt}\left(-\frac{1}{x^2}\right)$$

The 2nd derivative contains this: $\frac{d}{dx}\left[\frac{dy}{dt}\right]$ which, since y is a function of t and t is a function of x requires

using the Chain Rule: $\frac{d}{dx}\left[\frac{dy}{dt}\right] = \frac{d^2y}{dt^2}\frac{dt}{dx} = \frac{d^2y}{dt^2}\frac{1}{x}$

Therefore...

$$y' = \frac{1}{x}\frac{dy}{dt} \qquad \textit{and} \qquad y'' = \frac{1}{x}\frac{d^2y}{dt^2}\frac{1}{x} + \frac{dy}{dt}\left(-\frac{1}{x^2}\right)$$

$$= \frac{1}{x^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)$$

To show how this substitution works, let's consider a specific example.

$$x^2y'' - xy' + y = \ln x$$

Using the following substitutions: $x = e^t, \quad t = \ln x, \quad y' = \frac{1}{x}\frac{dy}{dt}, \quad y'' = \frac{1}{x^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)$, the DE

becomes:

$$x^2\left[\frac{1}{x^2}\left(\frac{d^2y}{dt^2}-\frac{dy}{dt}\right)\right]-x\left[\frac{1}{x}\frac{dy}{dt}\right]+y=t$$

$$\frac{d^2y}{dt^2}-\frac{dy}{dt}-\frac{dy}{dt}+y=t$$

$$\frac{d^2y}{dt^2}-2\frac{dy}{dt}+y=t$$

This is a linear 2nd order differential equation with all constant coefficients, so earlier methods can be used to solve. We could now write the equation:

$$y''-2y'+y=t$$

But we need to be careful to remember that the independent variable now is *t* not *x* so if we were to solve this and get the general solution:

$$y=C_1e^t+C_2te^t+2+t$$

…we need to remember to replace *t* with $t=\ln x$:

$$y=C_1e^{\ln x}+C_2\ln xe^{\ln x}+2+\ln x$$

$$y=C_1x+C_2x\ln x+2+\ln x$$