Derivations for DiffEq 4.6, Variation of Parameters

We'll use a second-order non-homogeneous DE with constant coefficients, but this derivation applies for higher-order constant coefficient DEs as well:

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x)$$

Divide by the leading coefficient to put in standard form:

$$y'' + P(x)y' + Q(x)y = f(x)$$
$$\text{where } f(x) = \frac{g(x)}{a_2(x)}$$

We know that the solution to the corresponding homogeneous DE is comprised of a linear combination of two solution terms:

$$y_C = C_1 y_1 + C_2 y_2$$

We need any particular solution y_P which satisfies the original non-homogeneous DE, and we've seen two general, related ideas for finding this. In the 'reduction of order' section, we postulated a function *u* took derivatives and substituted into the original DE to solve for the *u* that worked. Earlier, in our first-order solutions list, we included finding an integrating factor which when used to multiply the original DE produced the derivative of a solution which was then found by integrating.

In this new method, called 'variation of parameters' we'll use similar ideas, and for a second-order DE, we will need to find two functions, u_1 and u_2, and at the end of the procedure, we will end up integrating.

Once we've solved the corresponding homogeneous equation and have the two terms, we can postulate that the particular solution will involve two additional terms which each involve some new function, but multiplied by the form of a term in the homogeneous solution:

$$y_C = C_1 y_1 + C_2 y_2$$
$$y_P = u_1 y_1 + u_2 y_2$$

But here, the u_1 and u_2, are functions of *x*, not just constants. Working with just the particular solution, we can take derivatives….

$$y_P = u_1 y_1 + u_2 y_2$$
$$y_P' = u_1 y_1' + y_1 u_1' + u_2 y_2' + y_2 u_2'$$
$$y_P'' = u_1 y_1'' + y_1' u_1' + y_1 u_1'' + u_1' y_1' + u_2 y_2'' + y_2' u_2' + y_2 u_2'' + u_2' y_2'$$

Then we can substitute these into the non-homogeneous DE:

$$y'' + Py' + Qy = f(x)$$

$$\begin{aligned}&\left(u_1 y_1'' + y_1' u_1' + y_1 u_1'' + u_1' y_1' + u_2 y_2'' + y_2' u_2' + y_2 u_2'' + u_2' y_2'\right)\\&+P\left(u_1 y_1' + y_1 u_1' + u_2 y_2' + y_2 u_2'\right)\\&+Q\left(u_1 y_1 + u_2 y_2\right) = f(x)\end{aligned}$$

Regrouping terms…

$$\begin{aligned}&u_1\left[y_1'' + Py_1' + Qy_1\right] + u_2\left[y_2'' + Py_2' + Qy_2\right] + y_1 u_1'' + u_1' y_1' + y_2 u_2'' + u_2' y_2'\\&+P\left(y_1 u_1' + y_2 u_2'\right) + u_1' y_1' + u_2' y_2' = f(x)\end{aligned}$$

The portions in the square brackets are both zero because these are the two term solutions to the homogeneous DE, so they must equal zero. This leaves us with:

$$y_1 u_1'' + u_1' y_1' + y_2 u_2'' + u_2' y_2' + P\left(y_1 u_1' + y_2 u_2'\right) + u_1' y_1' + u_2' y_2' = f(x)$$

Now note that the first two pairs of terms are actually the product rule derivatives of pairs of values multiplied:

$$y_1 u_1'' + u_1' y_1' = \frac{d}{dx}\left[y_1 u_1'\right]$$

$$y_2 u_2'' + u_2' y_2' = \frac{d}{dx}\left[y_2 u_2'\right]$$

So we can rewrite as follows:

$$\frac{d}{dx}\left[y_1 u_1' + y_2 u_2'\right] + P\left(y_1 u_1' + y_2 u_2'\right) + u_1' y_1' + u_2' y_2' = f(x)$$

Okay, now looking at the result, every term contains a u_1' or a u_2' and if we could find values for these, we could integrate to find what we need to form the particular solution. To solve for two unknowns, we need two equations, so here is the non-intuitive part: we will assume that the expression in the square bracket above is zero. This will give us a first equation:

$$y_1 u_1' + y_2 u_2' = 0$$

…and then when we replace this expression with zero in the larger expression, we get the second equation:

$$\frac{d}{dx}\left[y_1u_1' + y_2u_2'\right] + P\left(y_1u_1' + y_2u_2'\right) + u_1'y_1' + u_2'y_2' = f\left(x\right)$$
$$\frac{d}{dx}[0] + P(0) + u_1'y_1' + u_2'y_2' = f\left(x\right)$$
$$u_1'y_1' + u_2'y_2' = f\left(x\right)$$

So now we have a system of two equations, where the unknowns are u_1' and u_2':

$$y_1u_1' + y_2u_2' = 0$$
$$y_1'u_1' + y_2'u_2' = f\left(x\right)$$

To solve for u_1' and u_2' we solve this system using Cramer's Rule. A quick review…

The system: $\begin{array}{l} ax+by=m \\ cx+dy=n \end{array}$ is solved by: $x = \dfrac{\begin{vmatrix} m & b \\ n & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}, \quad y = \dfrac{\begin{vmatrix} a & m \\ c & n \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$

So here, that means…

$$u_1' = \frac{\begin{vmatrix} 0 & y_2 \\ f\left(x\right) & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}}, \quad u_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & f\left(x\right) \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}}$$

Once we find u_1' and u_2', we integrate each to find u_1 and u_2

$$u_1 = \int u_1'\,dx, \qquad u_2 = \int u_2'\,dx$$

…and we use these to form the particular solution…

$$y_P = u_1y_1 + u_2y_2$$

…and finally, combine with the homogeneous solution for the complete general solution:

$$y = y_C + y_P = C_1y_1 + C_2y_2 + u_1y_1 + u_2y_2$$