Derivations for DiffEq 4.3, solving homogeneous linear DEs with constant coefficients

First-order

We'll start by considering a first-order DE with constant coefficients: $ay' + by = 0$

...and solving for *y'*: $y' = \frac{b}{a}y \quad or \quad y' = ky$

The function which is the solution to this DE must be one where the derivative of the function is the same as the original function, just multiplied by a constant. The only non-trivial function for which this is true is the exponential function: $y = e^{mx}$.

Taking the derivative: $y = e^{mx}, \quad y' = me^{mx}$ and substituting this into the DE: $ay' + by = 0$ gives:

$$a\left(me^{mx}\right) + b\left(e^{mx}\right) = 0 \quad or \quad e^{mx}\left(am + b\right) = 0$$

Since e^{mx} can never be zero, the differential equation will be solved for any value of *m* which makes $am + b = 0$ true.

The equation $am + b = 0$ is called the **auxiliary equation** for the DE, and by solving it to obtain *m*, the solution to the DE is then $y = C_1 e^{mx}$.

Second-order

We can do something similar for a second-order linear DE: $ay'' + by' + c = 0$

There is a corresponding auxiliary equation: $am^2 + bm + c = 0$ which, in general, has two *m* values, m_1 and m_2, as solutions, and if you postulate that the two solutions: $y_1 = e^{m_1 x}$ and $y_2 = e^{m_2 x}$ follow the same pattern, if you take derivatives of each and plug in to the DE, you'll find that both are also solutions to the DE (we won't show this here, it would require using the quadratic formula to compute m and then substituting the + case in for m_1 and the – case in for m_2 and then trying each of these solutions into the original DE, but it does work).

For a 2nd-order DE, which has a 2nd-degree auxiliary equation $am^2 + bm + c = 0$ there are three possible cases which may result when computing the two *m* values:

a) You may get two distinct real values for m_1 and m_2.
b) You may get a single (repeated) real value for m_1 and m_2.
a) You may get two complex values for m_1 and m_2 which are complex-conjugates of each other.

We'll examine each of these cases…

a) Two distinct real values: If you get two different real values for m_1 and $m2$, then each is a solution of the form $y = e^{mx}$ and if you check the Wronskian, you'll find that these two solutions for a fundamental solution set, so this means that, by the superposition principle, the solution for the 2nd-order DE would be:

$$y = C_1 e^{m_1 x} + C_2 e^{m_2 x}$$

b) Repeated real values: If both m_1 and $m2$ are the same real value m then one solution is $y_1 = e^{mx}$. If you use the procedures for reduction of order in section 4.2, you can then find that the second solution is $y_2 = xe^{mx}$ (the first solution but multiplied by an extra *x*). Taking derivatives and plugging in reveals that this is also a solution to the original 2nd-order DE. Using the Wronksian, these two solutions also form a fundamental solution set, so by the superposition principle, the solution in the case of a single repeated-value is:

$$y = C_1 e^{mx} + C_2 x e^{mx}$$

c) Complex-conjugate pair m values: If the quadratic formula used to solve the auxiliary equation has a negative under the square root, the m values will be complex numbers. In this case, we express the values using this notation…

$$m_1 = \alpha + i\beta, \quad m_2 = \alpha - i\beta$$

…and the solution would be:

$$y = C_1 e^{(\alpha + i\beta)x} + C_2 e^{(\alpha - i\beta)x}$$

However, because we prefer to work with real numbers, rather than complex exponential, we can use something called **Euler's Formula** to rewrite the result:

$$Euler's\ Formula: \quad e^{i\theta} = \cos\theta + i\sin\theta$$

(Euler's Formula can be shown to be true by Maclaurin series expansion of the expressions on each side, but here we accept it as a given.)

Using Euler's Formula for the exponential terms in our solution here:

$$e^{i\beta x} = \cos\beta x + i\sin\beta x \quad and$$

$$e^{-i\beta x} = \cos(-\beta x) + i\sin(-\beta x) = \cos\beta x - i\sin\beta x \quad (due\ to\ even/odd\ nature\ of\ \cos\ and\ \sin)$$

If you add and subtract these two equations, you get:

$$\begin{aligned} e^{i\beta x} + e^{-i\beta x} &= \cos\beta x + i\sin\beta x + \cos\beta x - i\sin\beta x \\ &= 2\cos\beta x \\ e^{i\beta x} - e^{-i\beta x} &= \cos\beta x + i\sin\beta x - \cos\beta x + i\sin\beta x \\ &= 2i\sin\beta x \end{aligned}$$

…and since our solution is in the form: $y = C_1 e^{(\alpha+i\beta)x} + C_2 e^{(\alpha-i\beta)x}$ if you chose C1=1 and C2 to be either 1 or -1, this would give two solutions:

$$y_1 = e^{(\alpha+i\beta)x} + e^{(\alpha-i\beta)x} \qquad y_2 = e^{(\alpha+i\beta)x} - e^{(\alpha-i\beta)x}$$
$$= e^{\alpha x}e^{i\beta x} + e^{\alpha x}e^{-i\beta x} \qquad = e^{\alpha x}e^{i\beta x} - e^{\alpha x}e^{-i\beta x}$$
$$= e^{\alpha x}\left(e^{i\beta x} + e^{-i\beta x}\right) \qquad = e^{\alpha x}\left(e^{i\beta x} - e^{-i\beta x}\right)$$
$$= e^{\alpha x} 2\cos\beta x \qquad = e^{\alpha x} 2i\sin\beta x$$

In the first solution, the 2 is a constant and in the second, the 2 and the *i* are constants, so the form of the two solutions is

$$y_1 = C_1 e^{\alpha x}\cos\beta x \qquad y_2 = C_2 e^{\alpha x}\sin\beta x$$

…and these two solutions can be shown not only to be solutions of the original 2nd-order DE, but also linearly independent, so they form a fundamental solution set. Therefore, the general solution in the complex-conjugate pairs case is:

$$y = C_1 e^{\alpha x}\cos\beta x + C_2 e^{\alpha x}\sin\beta x$$

<u>Higher-order DEs</u>

We can use the same methods to solve homogeneous linear differential equations of 3rd or higher order. We form the auxiliary equation, solve for the *m* values (there will be 3 or more), and how we handle each depends upon whether each *m* value is real, is repeated, or is complex (always appearing in complex-conjugate pairs).

Any real value *m* will include a term of the form: $y = Ce^{mx}$

Any real value which is repeated, will include multiple terms, each repeat multiplying by an additional *x*. For example, if the auxiliary equation has solution *m=4* but with multiplicity of 3, then we would add the following terms to the solution: $y = C_1 e^{mx} + C_2 x e^{mx} + C_3 x^2 e^{mx}$. Complex pair roots are handled the same way…with the form $y = C_1 e^{\alpha x}\cos\beta x + C_2 e^{\alpha x}\sin\beta x$ for each pair.

(Proof of these assertions for higher-order DEs is very involved because of the large number of combinations of single real, repeated real, and complex-conjugate *m* value possibilities, so is not presented here or in our textbook).