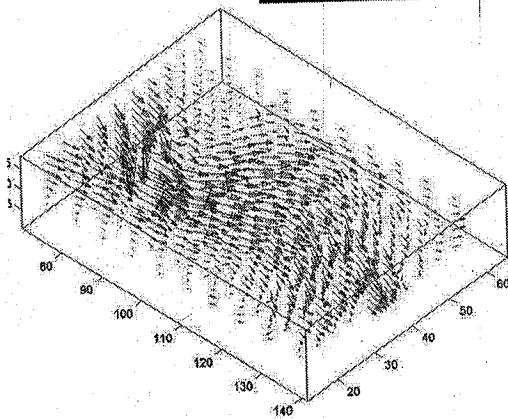
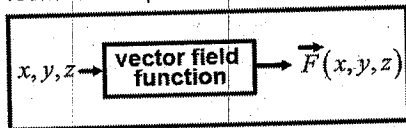


Calc3 – Lesson Notes - Chapter 16: Vector Calculus

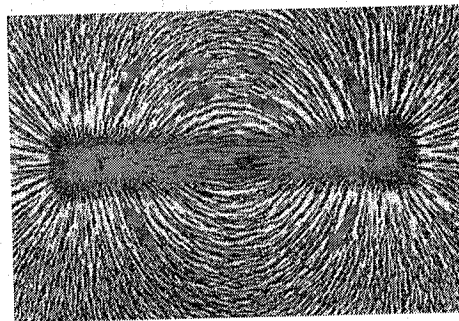
16.1: Vector Fields

What is a Vector Field?

A vector field assigns a vector to each point in the domain:



Air speed and direction



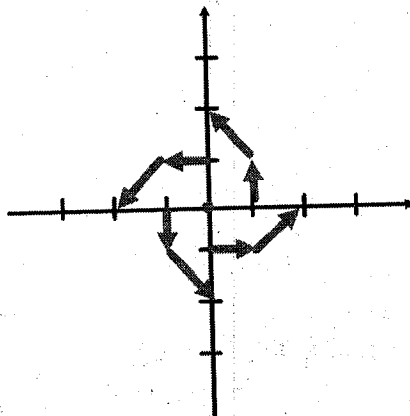
Magnetic field strength and direction

Sketching a vector field

To sketch, make a table of a few input values and sketch the resulting vectors at their locations:

Ex) Sketch $F(x, y) = \langle -y, x \rangle$

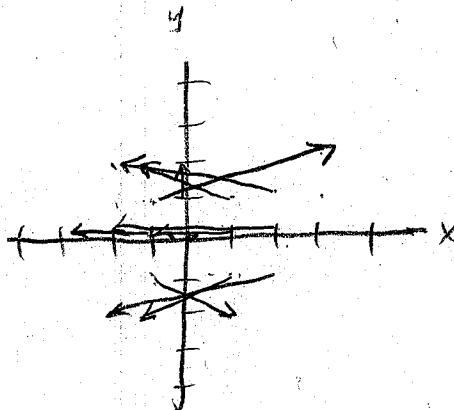
x	y	$F = \langle -y, x \rangle$
0	0	$\langle 0, 0 \rangle$
1	0	$\langle 0, 1 \rangle$
1	1	$\langle -1, 1 \rangle$
0	1	$\langle -1, 0 \rangle$
-1	1	$\langle -1, -1 \rangle$
-1	0	$\langle 0, -1 \rangle$
-1	-1	$\langle 1, -1 \rangle$
0	-1	$\langle 1, 0 \rangle$
1	-1	$\langle 1, 1 \rangle$



If this field represented current flow in water and you placed a particle in it, it would rotate about the origin.

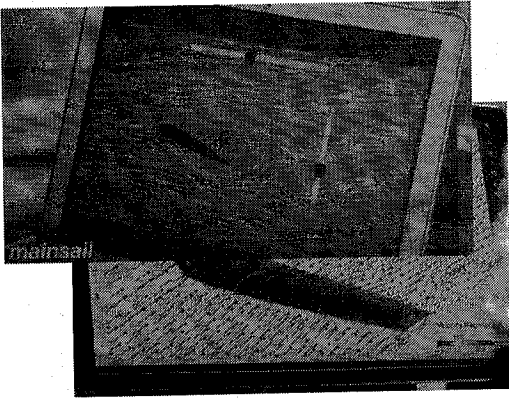
Ex) Sketch $F(x, y) = \langle -2x, y \rangle$

x	y	\vec{F}
0	0	$\langle 0, 0 \rangle$
1	0	$\langle -2, 0 \rangle$
0	1	$\langle 0, 1 \rangle$
1	1	$\langle -2, 1 \rangle$
1	-1	$\langle -2, -1 \rangle$
2	0	$\langle -4, 0 \rangle$
2	1	$\langle -4, 1 \rangle$
2	-1	$\langle -4, -1 \rangle$
-1	0	$\langle 2, 0 \rangle$
-1	-1	$\langle 2, -1 \rangle$
-1	1	$\langle 2, 1 \rangle$

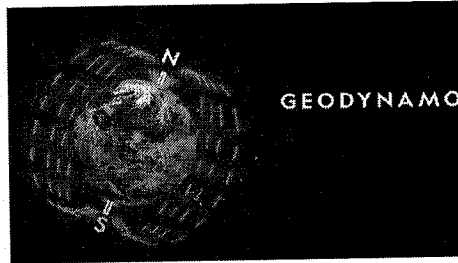


Vector Fields are very useful

Much of the universe can be modeled using vectors which vary at locations...vector fields are extremely useful in many field.



<https://www.youtube.com/watch?v=BVuWXApdfdE>



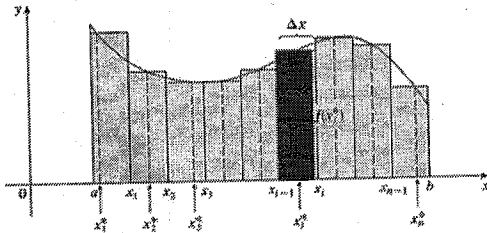
<https://www.youtube.com/watch?v=Elv3WpL32UE>

16.2: Line Integrals

Integral 'sum' things...

Recall that we first defined integrals as smoothed Riemann Sums, which means that integrals are 'summing' things...

If we have a numerical function of 1 input variable, we can use an integral to sum the areas of rectangles to compute the total area under the curve:

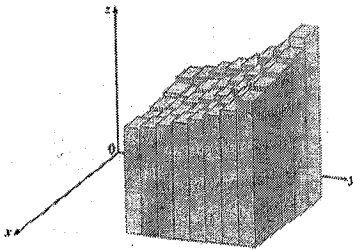


$$A \approx \sum_{i=1}^n f(x_i^*) \Delta x$$

$$A = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x$$

$$A = \int_a^b f(x) dx$$

If we have a numerical function of 2 input variables, we can use a double integral over the 2D domain region to sum the values of the function over this domain area...



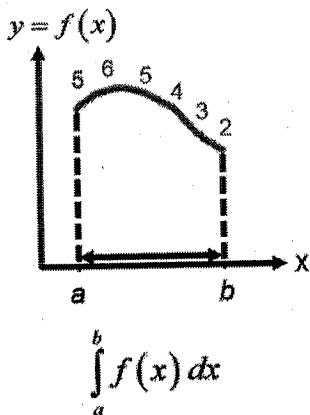
$$V = \iint_R f(x, y) dA$$

...and if we treat the numerical values of the function as the 'height', then we are also computing the volume under the function surface.

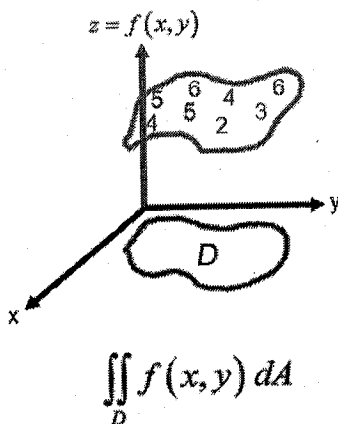
Integral 'sum' things...

It is also valid to think of single and double integrals as simply summing up the numerical values of the function over a 1D or 2D domain region, and we extended this idea to 3D with triple integrals:

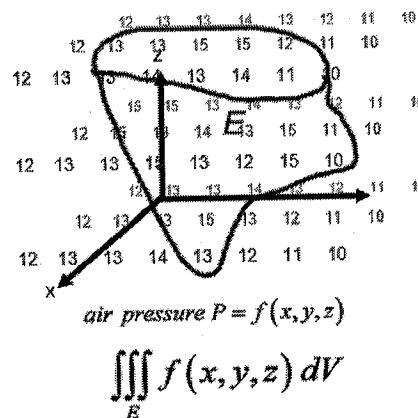
Single integral



Double integral



Triple integral

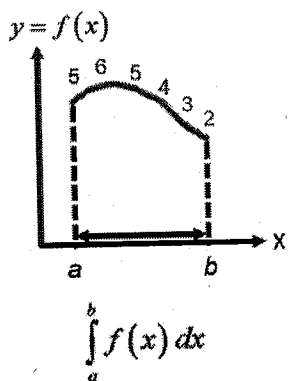


Most of the rest of this chapter will be about using integrals to sum up quantities, but instead of the quantities being numbers, the quantities will be vectors.

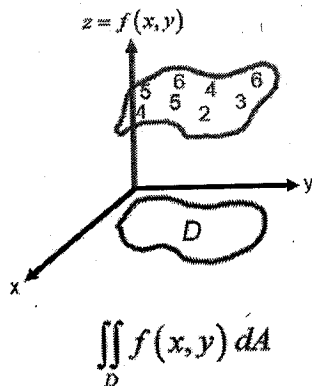
Line integrals along a curve

However, the first thing we'll do is to work with summing a numerical function along a single path, similar to a single integral, except that we will sum the numerical values of a 2D function along a single path through 2D. This is called a **line integral of f along curve C** .

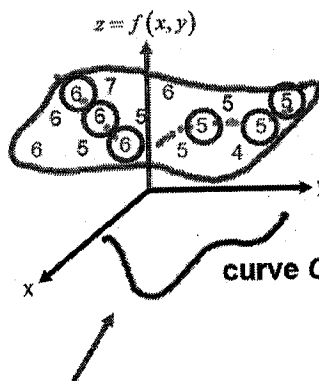
Single Integral



Double Integral



Line integral along curve C



Line integrals should really be called 'curve integrals' but line integrals is the historic name for this structure.

We are summing single function values from a 2D function, but along a single dimension curved path, C , in the 2D domain space (of course, there would be an infinite number of values being summed, not just the few example values circled here).

Review - parametric curves

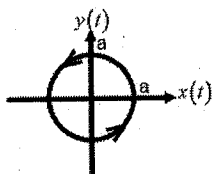
In order to move along this curved path, C , but sum in a single dimension (a distance along the path) we need to first parametrize the curve in the domain. Recall the following about parametrizing curves:

1) Circle in 2D:

$$x(t) = a \cos t$$

$$y(t) = a \sin t$$

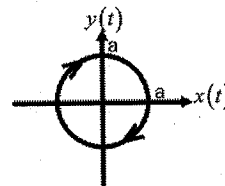
where $0 \leq t \leq 2\pi$



$$x(t) = -a \cos t \text{ --or-- } x(t) = a \sin t$$

$$y(t) = a \sin t \quad y(t) = a \cos t$$

where $0 \leq t \leq 2\pi$

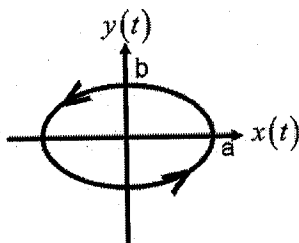


2) Ellipse in 2D:

$$x(t) = a \cos t$$

$$y(t) = b \cos t$$

where $0 \leq t \leq 2\pi$



$$\cos t = \frac{x}{a}$$

$$\sin t = \frac{y}{b}$$

$$\cos^2 t + \sin^2 t = 1$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

3) line in 2D (line segments):

from (x_0, y_0) to (x_1, y_1)

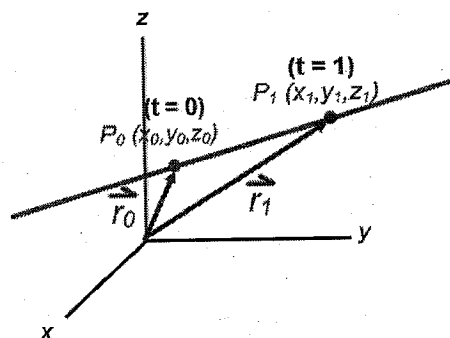
$$(\vec{r}_0 \text{ to } \vec{r}_1)$$

$$r(t) = (1-t)r_0 + t r_1$$

$$x(t) = (1-t)x_0 + t x_1$$

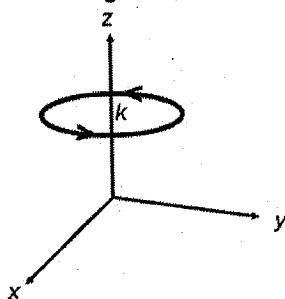
$$y(t) = (1-t)y_0 + t y_1$$

where $0 \leq t \leq 1$



4) circular paths in 3D (we'll need this when we add a dimension):

floating circle



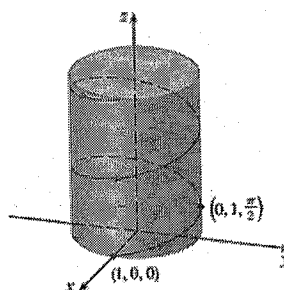
$$x(t) = a \cos t$$

$$y(t) = b \sin t$$

$$z(t) = k$$

where $0 \leq t \leq 2\pi$

helix



$$x(t) = a \cos t$$

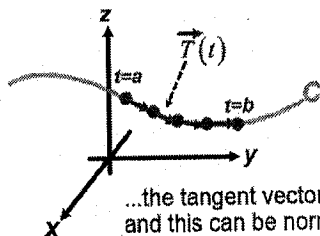
$$y(t) = b \sin t$$

$$z(t) = t$$

where $0 \leq t \leq 2\pi$

Moving along the curve

Back in chapter 13, we explored arc length and concluded that for a path C...



$$\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$$

...the tangent vector at a point is found by taking the derivative of the position vector and this can be normalized to form the unit tangent function.

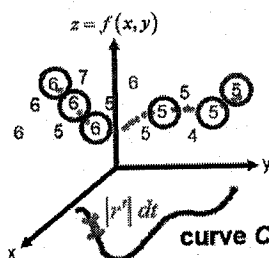
Then the length moving along the curve is given by:

$$L = \int_a^b |\vec{r}'(t)| dt$$

Line Integrals along C for Scalar Functions

We can be working with a scalar function with has numerical values, or we can be working with a vector function which defines a vector output at each input. In that case,

Line Integral along curve C



...which moves along a small distance, given by:

$$|\vec{r}'| dt$$

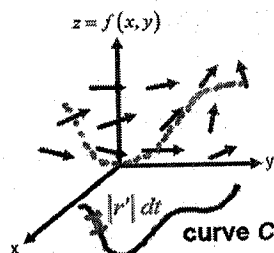
Then to integrate the numerical values of the function, we use the following integral:

$$\int_C f(x, y) ds = \int_a^b f(x(t), y(t)) |\vec{r}'| dt$$

Line Integral along C for Vector Functions

We can also define line integrals for Vector Functions (where the output at each point in the domain is a vector instead of a numerical quantity). Summing with an integral in this situation can be interpreted as finding the summation of the vectors' actions in the direction of the curve, along the curve...

Line Integral along curve C



...and that means that we need to sum only the portion of the vector at a particular point that is in the direction of the curved path at that point.

This can be found by taking the dot product of the Vector Function, F, and the tangent vector, r':



The line integral for a vector function is then defined as:

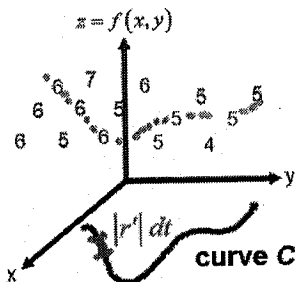
$$\int_C \vec{F}(x, y) dr = \int_a^b \vec{F}(r) \cdot \vec{r}' dt$$

Note: since we are integrating the result of a dot product, a line integral of a vector function still produces a scalar result.

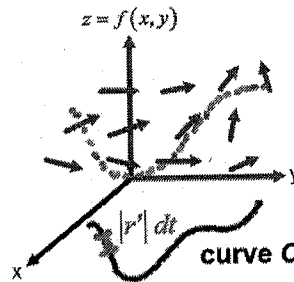
Line Integrals for Scalar versus Vector Functions

So there are two cases for line (curve) integrals, one for scalar functions, and one for vector functions:

Line Integral for scalar functions



Line integral for vector functions



$$\int_C f(x, y) ds = \int_a^b f(x(t), y(t)) |\vec{r}'| dt$$

$$\int_C \vec{F}(x, y) dr = \int_a^b \vec{F}(\vec{r}) \cdot \vec{r}' dt$$

Although it is harder to picture, these functions can have any number of dimensions. So the domain could be 3D for a scalar function, or for the vector case, both the domain and output vectors can be 3D. But the form for line integrals is the same, just add additional variables.

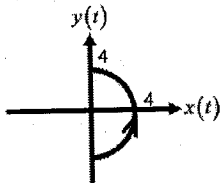
A scalar function line integral example...

Ex) Evaluate $\int_C xy^4 ds$ where C is the right half of circle $x^2 + y^2 = 16$

The integrand xy^4 is not a vector, therefore this is the scalar function case, so we use:

First, parametrize the curve:

$$\begin{aligned} x(t) &= 4 \cos t \\ y(t) &= 4 \sin t \end{aligned} \quad \text{where } -\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$$



$$\int_C f(x, y) ds = \int_a^b f(x(t), y(t)) |\vec{r}'| dt$$

Next, build the position vector and take it's derivative:

$$\vec{r} = \langle 4 \cos t, 4 \sin t \rangle \quad \vec{r}' = \langle -4 \sin t, 4 \cos t \rangle$$

Then, find the magnitude of the position function derivative:

$$|\vec{r}'| = \sqrt{(-4 \sin t)^2 + (4 \cos t)^2} = \sqrt{16 \sin^2 t + 16 \cos^2 t} = 4$$

Finally, build the line integral and evaluate:

$$\int_C xy^4 ds = \int_{-\pi/2}^{\pi/2} (4 \cos t)(4 \sin t)^4 4 dt = \frac{8192}{5}$$

A vector function line integral example...

Ex) Evaluate $\int_C \vec{F} \cdot d\vec{r}$ where $F(x, y) = \langle x + y, y - z, z^2 \rangle$
and C is given by $r(t) = \langle t^2, t^3, t^2 \rangle$ $0 \leq t \leq 1$

The integrand is clearly a vector, therefore this is the vector function case, so we use:

Here, the curve has already been parametrized, and we have the position vector, so we take it's derivative:

$$\vec{r}'(t) = \langle 2t, 3t^2, 2t \rangle$$

But before we can take the dot product, we need to evaluate the vector function at $r(t)$ by plugging the components of r into the x , y and z of F :

$$\vec{F}(r) = \langle (t^2) + (t^3), (t^3) - (t^2), (t^2)^2 \rangle = \langle t^2 + t^3, t^3 - t^2, t^4 \rangle$$

We then take the dot product: $\vec{F}(r) \cdot \vec{r}'(t) = \langle t^2 + t^3, t^3 - t^2, t^4 \rangle \cdot \langle 2t, 3t^2, 2t \rangle$
 $= 2t^3 + 2t^4 + 3t^5 - 3t^4 + 2t^5 = 5t^5 - t^4 + 2t^3$

Finally, build the line integral and evaluate:

$$\int_0^1 (5t^5 - t^4 + 2t^3) dt = \frac{17}{15}$$

Line Integrals with respect to x, y, z

Sometimes the curve C is difficult to parametrize, or the problem is defined using functions for x and y separately.

Scalar function case:

The curve could be a line segment in just the x direction or y direction, called **line integrals along C with respect to x and y :**

$$\int_C f(x, y) dx = \int_a^b f(x(t), y(t)) x'(t) dt \quad \int_C f(x, y) dy = \int_a^b f(x(t), y(t)) y'(t) dt$$

When these occur together, they are written as:

$$\int_C P(x, y) dx + Q(x, y) dy$$

Vector function case:

If the vector function could be written as $F = \langle P, Q, R \rangle$ then:

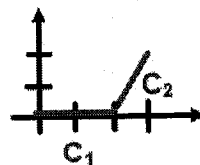
$$\int_C \vec{F} \cdot d\vec{r} = \int_C P dx + Q dy + R dz$$

To see why this might be easier, we'll consider an example...

Example...scalar function line integral wrt x,y

#7) Evaluate $\int_C xy \, dx + (x-y) \, dy$ where C consists of line segmnts from (0,0) to (2,0) and (2,0) to (3,2).

We're going to have to evaluate each curve separately, but we don't have to come up with some new parametrization of the curves with respect to t. We can instead just use x as the parameter:



$$\int_C^P xy \, dx + (x-y) \, dy = \int_{C_1} xy \, dx + (x-y) \, dy + \int_{C_2} xy \, dx + (x-y) \, dy$$

on C_1 :

$$x = x \text{ (the parameter)}$$

$$y = 0 \quad 0 \leq x \leq 2 \text{ and}$$

$$dx = dx$$

$$dy = 0$$

on C_2 :

$$x = x \text{ (the parameter)}$$

$$y = 2x - 4 \quad 2 \leq x \leq 3 \text{ and}$$

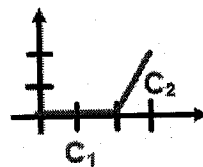
$$dx = dx$$

$$dy = 2dx$$

Substituting:

$$\begin{aligned} \int_C xy \, dx + (x-y) \, dy &= \int_0^2 (x)(0) \, dx + ((x)-(0))(0) + \int_2^3 (x)(2x-4) \, dx + ((x)-(2x-4))2dx \\ &= \int_0^2 0 \, dx + \int_2^3 (2x^2 - 4x + 2x - 4x + 8) \, dx \\ &= 0 + \int_2^3 (2x^2 - 6x + 8) \, dx \\ &= \left[\frac{2}{3}x^3 - 3x^2 + 8x \right]_2^3 = \frac{17}{3} \end{aligned}$$

Compare doing this with a parametrization using t:



$$\int_C^P xy \, dx + (x-y) \, dy = \int_{C_1} xy \, dx + (x-y) \, dy + \int_{C_2} xy \, dx + (x-y) \, dy$$

on C_1 :

$$r = (1-t)r_0 + t r_1$$

$$r = (1-t)\langle 0,0 \rangle + t \langle 2,0 \rangle$$

$$r = \langle 2t, 0 \rangle \quad 0 \leq t \leq 1 \text{ and}$$

$$x = 2t \quad dx = 2dt$$

$$y = 0 \quad dy = 0$$

on C_2 :

$$r = (1-t)r_0 + t r_1$$

$$r = (1-t)\langle 2,0 \rangle + t \langle 3,2 \rangle$$

$$r = \langle t+2, 2t \rangle \quad 0 \leq t \leq 1 \text{ and}$$

$$x = t+2 \quad dx = dt$$

$$y = 2t \quad dy = 2dt$$

Substituting:

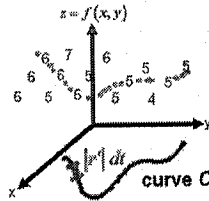
$$\begin{aligned} \int_C xy \, dx + (x-y) \, dy &= \int_0^1 (2t)(0) 2dt + ((2t)-(0))(0) + \int_0^1 (t+2)(2t) \, dt + ((t+2)-(2t)) 2dt \\ &= \int_0^1 0dt + \int_0^1 (2t^2 + 4t + 2t + 4 - 4t) \, dt \\ &= 0 + \int_0^1 (2t^2 + 2t + 4) \, dt = \left[\frac{2}{3}t^3 + t^2 + 4t \right]_0^1 = \frac{17}{3} \end{aligned}$$

so you can reparametrize with t, or just use x or y, whichever you feel is easier.

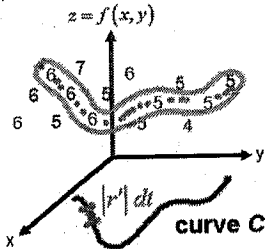
Physical interpretation of scalar line integrals

Line integral for scalar functions

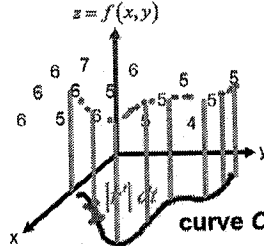
A scalar line integral is summing scalar numerical values over a path:



If the function values represents density of a physical material, then the line integral would be the total mass of an infinitely thin 'strip' along the path (not very realistic):

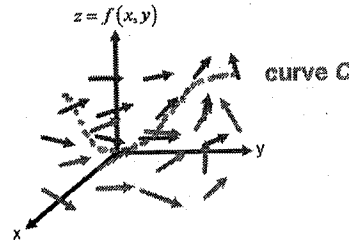


You could also think of the function value as the height of a surface, and then the line integral would represent the area on one side of a 'curtain' following the path down to the x-y plane (also not too useful):



A vector line integral is summing the dot product of a vector field with a direction, so it represents a summation of the amount of the field in the direction of the path along the path (much more useful than the scalar case).

In general, both the domain and vector field can have any number of dimensions.



For example, if the vector field is a force field such as gravity or fluid flow which exerts a force on particles passing through the fluid, then the vector line integral would represent the dot product of force times distance which is **work**. It represents a summation of the individual contributions of work of the amount of force in the direction of travel times that small distance, so the vector line integral represents the **total work done by the force field along the path**.

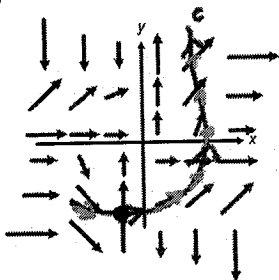


This is an extremely important application and occurs throughout the study of physics.

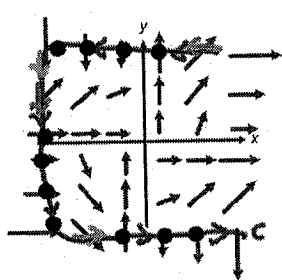
Intuitive understanding of Work/summation around a vector field

To get a better feel for vector line integrals, let's consider the 2D case of a force field and estimate what the vector line integral will be along different paths.

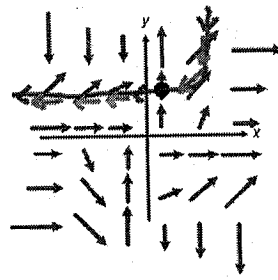
We need to estimate how much of the field is pointing in the direction of travel, and then add up little segments along the path (in blue) (green indicates the vector is in direction of travel, red against direction of travel, black dots are where field is perpendicular to travel).



For this path, the field is aligned with the direction of travel for most of the curve, so the line integral would be positive.



For this path, the field is perpendicular to the path for much of the curve (dot product = 0) and for the rest of the curve, there is about an even balance of aligned and not-aligned field, so the line integral would be approximately zero.



For this path, the field is almost always in the opposite direction of travel, so this line integral would be negative.

Examples

#10) Evaluate $\int_C xyz^2 ds$ where C is the line segment from $(-1, 5, 0)$ to $(1, 6, 4)$

$$\vec{r} = (1-t)\langle -1, 5, 0 \rangle + t\langle 1, 6, 4 \rangle = \langle 2t-1, t+5, 4t \rangle$$

$0 \leq t \leq 1$

$$\vec{r}'(t) = \langle 2, 1, 4 \rangle$$

$$|\vec{r}'(t)| = \sqrt{4+1+16} = \sqrt{21}$$

$$\begin{aligned} \int_C xyz^2 ds &= \int_0^1 (2t-1)(t+5)(4t)^2 \sqrt{21} dt \\ &= \int_0^1 (2t^2+9t-5) 16t^2 \sqrt{21} dt \\ &= 16\sqrt{21} \int_0^1 (2t^4+9t^3-5t^2) dt \\ &= 16\sqrt{21} \left[\frac{2}{5}t^5 + \frac{9}{4}t^4 - \frac{5}{3}t^3 \right]_0^1 \\ &= 16\sqrt{21} \left[\frac{2}{5} + \frac{9}{4} - \frac{5}{3} \right] \\ &= 16\sqrt{21} \frac{59}{60} = \boxed{\frac{236\sqrt{21}}{15}} \end{aligned}$$

#14) Evaluate $\int_C zdx + xdy + ydz$ where $C: x=t^2, y=t^3, z=t^2 \quad 0 \leq t \leq 1$

$$\vec{F} = \langle z, x, y \rangle$$

$$\vec{r} = \langle t^2, t^3, t^2 \rangle$$

$$\vec{r}' = \langle 2t, 3t^2, 2t \rangle$$

$$\vec{F}(\vec{r}) = \langle t^2, t^2, t^3 \rangle$$

$$\begin{aligned} \vec{F} \cdot \vec{r}' &= \langle t^2, t^2, t^3 \rangle \cdot \langle 2t, 3t^2, 2t \rangle \\ &= 2t^3 + 3t^4 + 2t^4 \end{aligned}$$

$$\int_0^1 (2t^3 + 5t^4) dt$$

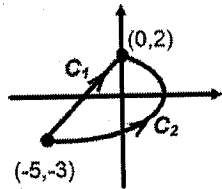
$$= \left[\frac{1}{2}t^4 + t^5 \right]_0^1 = \boxed{\frac{3}{2}}$$

16.3: The Fundamental Theorem for Line Integrals

Some fields are special...

Consider the following vector line integral: Evaluate $\int_C y^2 dx + x dy$ but let's evaluate it over two different paths between the same two endpoints, which we'll call curves C_1 and C_2 ...

C_1 : line segment
from $(-5, -3)$ to $(0, 2)$



C_2 : parabolic path $x = 4 - y^2$
from $(-5, -3)$ to $(0, 2)$

These two paths would have different parametrizations...

use standard line segment:

$$\begin{aligned}\vec{r} &= (1-t)\langle -5, -3 \rangle + t\langle 0, 2 \rangle \\ &= \langle 5t-5, 5t-3 \rangle \\ x &= 5t-5 \\ y &= 5t-3 \\ 0 &\leq t \leq 1\end{aligned}$$

just use y as the parameter:

$$\begin{aligned}\vec{r} &= \langle 4-t^2, t \rangle \\ x &= 4-t^2 \\ y &= t \\ -3 &\leq t \leq 2\end{aligned}$$

Now that we have r , evaluate the vector line integral along each curve...

C_1 : line segment
from $(-5, -3)$ to $(0, 2)$

$$\begin{aligned}\vec{r} &= \langle 5t-5, 5t-3 \rangle \\ \vec{r}' &= \langle 5, 5 \rangle \\ \vec{F} &= \langle y^2, x \rangle \\ \vec{F}(\vec{r}) &= \langle (5t-3)^2, (5t-5) \rangle \\ \vec{F} \cdot \vec{r}' &= \langle (5t-3)^2, (5t-5) \rangle \cdot \langle 5, 5 \rangle \\ &= 5(5t-3)^2 + 5(5t-5) \\ &= 125t^2 - 125t + 20\end{aligned}$$

$$\int_0^1 (125t^2 - 125t + 20) dt$$

$$= \left[\frac{125}{3}t^3 - \frac{125}{2}t^2 + 20t \right]_0^1$$

$$= \boxed{\frac{5}{6}}$$

In general, a vector line integral along different paths yields a different result - it is dependent upon the path taken through the field.

C_2 : parabolic path $x = 4 - y^2$
from $(-5, -3)$ to $(0, 2)$

$$\begin{aligned}\vec{r} &= \langle 4-t^2, t \rangle \\ \vec{r}' &= \langle -2t, 1 \rangle \\ \vec{F} &= \langle y^2, x \rangle \\ \vec{F}(\vec{r}) &= \langle (t)^2, (4-t^2) \rangle \\ \vec{F} \cdot \vec{r}' &= \langle (t)^2, (4-t^2) \rangle \cdot \langle -2t, 1 \rangle \\ &= -2t(t^2) + 1(4-t^2) \\ &= -2t^3 - t^2 + 4\end{aligned}$$

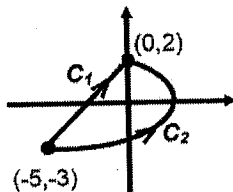
$$\int_{-3}^2 (-2t^3 - t^2 + 4) dt$$

$$= \left[-\frac{1}{2}t^4 - \frac{1}{3}t^3 + 4t \right]_{-3}^2$$

$$= \boxed{\frac{245}{6}}$$

Now let's evaluate along the same paths, but for this vector field: $\int_C (3 + 2xy) dx + (x^2 - 3y^2) dy$

C_1 : line segment
from $(-5, -3)$ to $(0, 2)$



C_2 : parabolic path $x = 4 - y^2$
from $(-5, -3)$ to $(0, 2)$

We have the same parametrizations as before...

use standard line segment:

$$\begin{aligned}\vec{r} &= (1-t)\langle -5, -3 \rangle + t\langle 0, 2 \rangle \\ &= \langle 5t-5, 5t-3 \rangle \\ x &= 5t-5 \\ y &= 5t-3 \\ 0 &\leq t \leq 1\end{aligned}$$

just use y as the parameter:

$$\begin{aligned}\vec{r} &= \langle 4-t^2, t \rangle \\ x &= 4-t^2 \\ y &= t \\ -3 &\leq t \leq 2\end{aligned}$$

...but the field is different, so we recompute the vector line integrals:

C_1 : line segment
from $(-5, -3)$ to $(0, 2)$

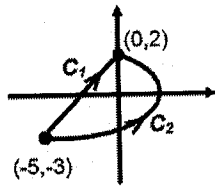
$$\begin{aligned}\vec{r} &= \langle 5t-5, 5t-3 \rangle \\ \vec{r}' &= \langle 5, 5 \rangle \\ \vec{F} &= \langle 3+2xy, x^2-3y^2 \rangle\end{aligned}$$

$$\begin{aligned}\vec{F}(\vec{r}) &= \langle 3+2(5t-5)(5t-3), (5t-5)^2-3(5t-3)^2 \rangle \\ &= \langle 50t^2-80t+33, -50t^2+40t-2 \rangle \\ \vec{F} \cdot \vec{r}' &= \langle 50t^2-80t+33, -50t^2+40t-2 \rangle \cdot \langle 5, 5 \rangle \\ &= 5(50t^2-80t+33) + 5(-50t^2+40t-2) \\ &= -200t+155\end{aligned}$$

$$\begin{aligned}\int_0^1 (-200t+155) dt \\ &= [-100t^2+155t]_0^1 \\ &= \boxed{55}\end{aligned}$$

This vector field is special...we get the same result integrating along different paths.

Vector fields like this are called conservative vector fields.



C_2 : parabolic path $x = 4 - y^2$
from $(-5, -3)$ to $(0, 2)$

$$\begin{aligned}\vec{r} &= \langle 4-t^2, t \rangle \\ \vec{r}' &= \langle -2t, 1 \rangle \\ \vec{F} &= \langle 3+2xy, x^2-3y^2 \rangle \\ \vec{F}(\vec{r}) &= \langle 3+2(4-t^2)(t), (4-t^2)^2-3(t)^2 \rangle \\ &= \langle -2t^3+8t+3, t^4-11t^2+16 \rangle \\ \vec{F} \cdot \vec{r}' &= \langle -2t^3+8t+3, t^4-11t^2+16 \rangle \cdot \langle -2t, 1 \rangle \\ &= -2t(-2t^3+8t+3) + 1(t^4-11t^2+16) \\ &= 5t^4-27t^2-6t+16\end{aligned}$$

$$\begin{aligned}\int_{-3}^2 (5t^4-27t^2-6t+16) dt \\ &= [t^5-9t^3-3t^2+16t]_{-3}^2 \\ &= \boxed{55}\end{aligned}$$

Conservative Vector Fields

Conservative Vector Fields have the property that vector line integrals computed between two points will have the same value regardless of path taken (as long as the endpoints are the same)...they are **independent of path**.

This field is conservative:

$$\vec{F} = \langle 3+2xy, x^2-3y^2 \rangle$$

This field is not conservative:

$$\vec{F} = \langle y^2, x \rangle$$

How can we tell if a vector field is conservative? The following is true for conservative vector fields and can be used to test if a vector field is conservative:

For $\vec{F} = \langle P, Q \rangle$, the vector field is conservative if

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

How do we determine if a field is conservative in higher dimensions? (We need to talk about that after covering a later section).

Checking the fields in the example:

$$\vec{F} = \langle 3 + 2xy, x^2 - 3y^2 \rangle$$

$$P = 3 + 2xy \quad Q = x^2 - 3y^2$$

$$\frac{\partial P}{\partial y} = 2x \quad \frac{\partial Q}{\partial x} = 2x$$

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

conservative

$$\vec{F} = \langle y^2, x \rangle$$

$$P = y^2 \quad Q = x$$

$$\frac{\partial P}{\partial y} = 2y \quad \frac{\partial Q}{\partial x} = 1$$

$$\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$$

not conservative

The Fundamental Theorem of Line Integrals

If a vector field is a conservative, not only will you get the same vector line integral regardless of path taken, but there is an alternative way to compute the vector line integral which is sometimes easier.

Remember, the Fundamental Theorem of Calculus (part 2)?

$$\int_a^b f(x) dx = F(b) - F(a)$$

We use this whenever we evaluate definite integrals, and it states that we can sum the effects of the function over the entire integral by plugging the endpoints into the antiderivative of the function. (We only need to evaluate at the endpoints.)

Something similar applies for vector line integrals, but only in the case of a vector line integral for a **conservative** vector field:

$$\int_c \vec{F} \cdot d\vec{r} = f(\text{endpoint}) - f(\text{startpoint})$$

This function, f , is called the **potential function** of the field, and is conceptually similar to the 'antiderivative' of the field.

Potential Function for a Conservative Vector Field

A theorem states that if a field is conservative:

$$\text{There exists a scalar function } f(x, y) \text{ such that: } \nabla f = \langle f_x, f_y \rangle = \vec{F}$$

...and this scalar function is the potential function which is used in the fundamental theorem of line integrals to compute the vector line integral.

Computing the Potential Function of a Conservative Vector Field

For our conservative field example: $\vec{F} = \langle 3 + 2xy, x^2 - 3y^2 \rangle$

We know a function, $f(x, y)$, exists such that $\nabla f = \langle f_x, f_y \rangle = \vec{F}$

Starting with this definition, we know: $f_x = 3 + 2xy$

$$f_y = x^2 - 3y^2$$

We start to find f by taking the antiderivative of f_x with respect to x :

$$\begin{aligned} f(x, y) &= \int f_x \, dx = \int (3 + 2xy) \, dx \\ &= 3x + x^2 y + C \end{aligned}$$

This integration constant, C , is a constant with respect to x , but could be a function of y , so we replace it with some currently unknown function of y :

$$f(x, y) = 3x + x^2 y + g(y)$$

To establish $g(y)$, now we take the derivative of both sides with respect to y :

$$\frac{d}{dy}[f(x, y)] = \frac{d}{dy}[3x + x^2 y + g(y)]$$

$$f_y = x^2 + g'(y)$$

But we know what f_y must equal from the field: $x^2 - 3y^2 = x^2 + g'(y)$

$$\text{So: } g'(y) = -3y^2$$

Now we can take the antiderivative of this with respect to y :

$$\int g'(y) \, dy = \int -3y^2 \, dy$$

$$g(y) = -y^3 + K$$

Substituting this into the earlier expression for $f(x, y)$ gives us the potential function:

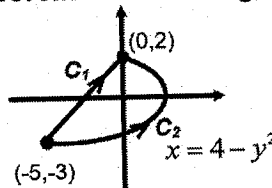
$$f(x, y) = 3x + x^2 y - y^3 + k$$

Evaluating a vector line integral using the Fundamental Theorem of Line Integrals

Evaluate $\int_C (3 + 2xy) \, dx + (x^2 - 3y^2) \, dy$ along the two paths:

First check to see if the field is conservative:

$$\begin{aligned} P &= 3 + 2xy & Q &= x^2 - 3y^2 & \frac{\partial P}{\partial y} &= \frac{\partial Q}{\partial x} & \text{so the field is} \\ \frac{\partial P}{\partial y} &= 2x & \frac{\partial Q}{\partial x} &= 2x & \frac{\partial P}{\partial x} &= \frac{\partial Q}{\partial y} & \text{conservative, and we} \\ & & & & & & \text{need only evaluate at} \\ & & & & & & \text{the endpoints.} \end{aligned}$$



Because the field is conservative we can find the potential function (shown previously):

$$f(x, y) = 3x + x^2 y - y^3 + k$$

Now we can use the fundamental theorem of line integrals to evaluate the vector line integral:

$$\int_C \vec{F} \cdot d\vec{r} = f(\text{endpoint}) - f(\text{startpoint})$$

$$\int_C (3 + 2xy) \, dx + (x^2 - 3y^2) \, dy = f(0, 2) - f(-5, -3)$$

$$= [3x + x^2 y - y^3]_{(-5, -3)}^{(0, 2)}$$

$$= (3(0) + (0)^2(2) - (2)^3) - (3(-5) + (-5)^2(-3) - (-3)^3)$$

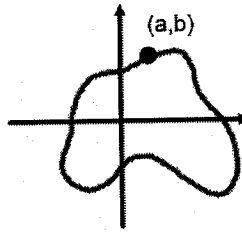
$$= -8 + 15 + 75 - 27$$

$$= \boxed{55} \quad (\text{along all paths between these two points})$$

Vector Line Integral for conservative field around a closed path is zero

If the starting and ending points for a vector line integral are the same, we say the path is a 'closed path'.

In that case, if the field is conservative, we can evaluate using the Fundamental Theorem of Line Integrals:



$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(\text{endpoint}) - f(\text{startpoint})$$

...but since the points are the same, **the result will always be zero around a closed path.**

Tree for evaluating line integrals

Is integrand a scalar or vector function?

scalar function $f(x, y)$

$$\int_C f(x, y) ds = \int_a^b f(\vec{r}) |\vec{r}'| dt$$

vector function $\vec{F}(x, y) = \langle P, Q \rangle$

Is the field conservative?

$$\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$$

no

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(r) \cdot \vec{r}' dt$$

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

yes

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(\vec{end}) - f(\vec{start})$$

where $\nabla f = \vec{F}$

Examples

#3) Determine whether or not \vec{F} is a conservative vector field. If it is, find a potential function f such that $\nabla f = \vec{F}$

conservative? $\vec{F}(x,y) = \langle 2x-3y, -3x+4y-8 \rangle$
 $\frac{\partial P}{\partial y} = -3$ $\frac{\partial Q}{\partial x} = -3$ = so **yes**

$f_x = 2x-3y$ $f_y = -3x+4y-8$

so $f(x,y) = x^2 - 3xy + 2y^2 - 8y + C$

$f(x,y) = \int (2x-3y) dx$ $-3x + g'(y) = -3x + 4y - 8$

$f(x,y) = x^2 - 3xy + g(y)$ $g'(y) = 4y - 8$

$f_y = -3x + g'(y)$ $g(y) = \int (4y-8) dy$
 $g(y) = 2y^2 - 8y + C$

#4) Determine whether or not \vec{F} is a conservative vector field. If it is, find a potential function f such that $\nabla f = \vec{F}$

$\vec{F}(x,y) = \langle e^x \cos y, e^x \sin y \rangle$
 conservative? $\frac{\partial P}{\partial y} = -e^x \sin y$ $\frac{\partial Q}{\partial x} = e^x \sin y$
 \neq so **no** (no potential function)

#12) Evaluate the integral $\int_C \vec{F} \cdot d\vec{r}$ $\vec{F}(x,y) = \langle x^2, y^2 \rangle$

where C is the arc of parabola $y = 2x^2$ from $(-1,2)$ to $(2,8)$

conservative? $\frac{\partial P}{\partial y} = 0$ $\frac{\partial Q}{\partial x} = 0$ = so yes

$f_x = x^2$ $f_y = y^2$
 $f(x,y) = \int x^2 dx$ $g'(y) = y^2$
 $f(x,y) = \frac{1}{3}x^3 + g(y)$ $g(y) = \int y^2 dy$
 $f_y = g'(y)$ $g(y) = \frac{1}{3}y^3 + C$

so $f(x,y) = \frac{1}{3}x^3 + \frac{1}{3}y^3 + C$

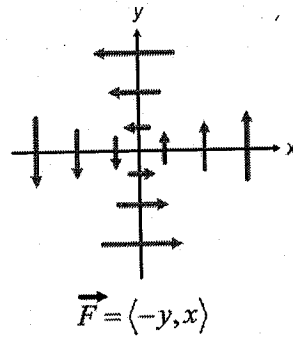
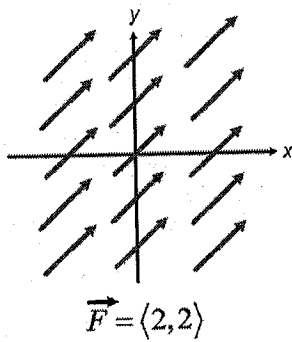
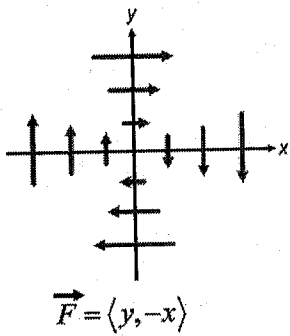
no $\int_C \vec{F} \cdot d\vec{r} = \left[\frac{1}{3}x^3 + \frac{1}{3}y^3 \right]_{(-1,2)}^{(2,8)}$

$\left[\frac{1}{3}(2)^3 + \frac{1}{3}(8)^3 \right] - \left[\frac{1}{3}(-1)^3 + \frac{1}{3}(2)^3 \right] = \boxed{171}$

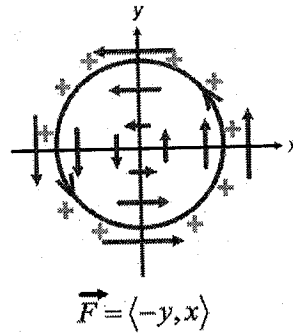
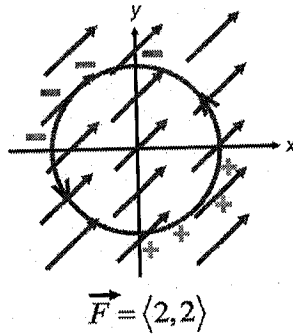
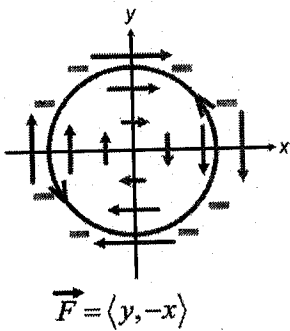
16.4: Green's Theorem

The concept of circulation

Consider these 2D vector fields:



If we computed vector line integrals around a counter-clockwise, closed, circular path...



... $\vec{F} \cdot \vec{r}'$ will be positive when the path direction and the field direction are aligned, negative when in opposite directions, and zero when perpendicular.

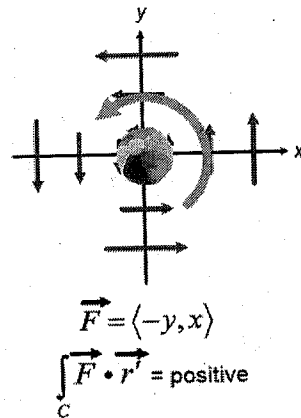
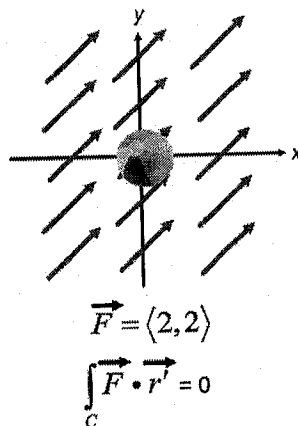
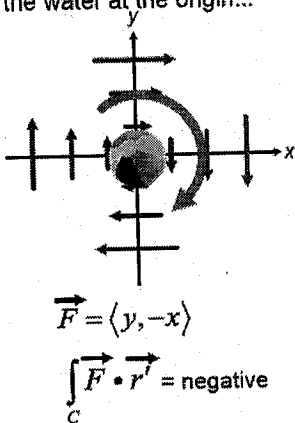
When these are summed over the entire path with an integral, the result will be:

$$\int_C \vec{F} \cdot \vec{r}' = \text{negative}$$

$$\int_C \vec{F} \cdot \vec{r}' = 0$$

$$\int_C \vec{F} \cdot \vec{r}' = \text{positive}$$

Now what if we imagine these fields represented current flow of a body of water and we placed a ball in the water at the origin...



...the ball would:

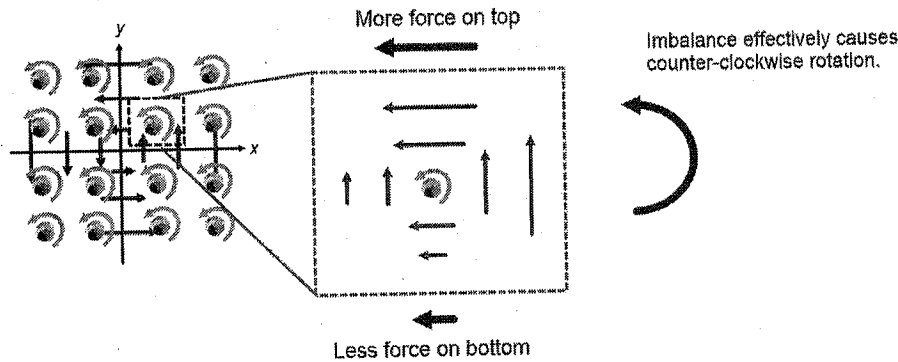
rotate clockwise
"negative circulation"

not rotate
"zero circulation"

rotate counter-clockwise
"positive circulation"

'Microscopic' circulation

If you had many really tiny beach balls, they would all rotate counter-clockwise in a positive circulation field, even if they were not at the origin.

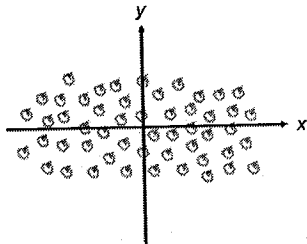


This is because there is an imbalance in the amount of force on the ball on opposite sides. Horizontally, there are higher forces above and lower forces below, so the effective motion is left on top, and 'less left' (effectively right) on the bottom.

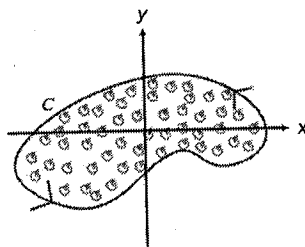
The same is true for the vertical forces and the imbalance also causes counter-clockwise rotation.

'Macroscopic' vs 'Microscopic' circulation

If you considered every point in a region within a vector field, every point would have a 'microscopic' rotation:



Then if you are taking a vector line integral for a closed path in this field, you can think of this as summing the contributions of all the 'microscopic' circulations that are within the region enclosed by the path...



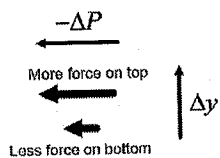
...the total could be considered the 'macroscopic' circulation and its value would be positive if the majority of microscopic circulations were positive, negative if the majority of microscopic circulations were negative, and zero if the microscopic circulations balanced out to zero.

A great website with even more detail and visualization tools:
https://mathinsight.org/greens_theorem_idea

An expression for circulation

For us to mathematically sum the individual microscopic circulations, we need a mathematical expression for circulation. Let's return to the idea that imbalance in force on opposite sides of a point causes rotation. If our field is defined to be: $\vec{F} = \langle P, Q \rangle$

These purple horizontal arrows for top and bottom would be changes in the field are really changes in the x-component of the field as we move in the y-direction...



...and would cause a positive circulation if this change was negative (the field was pointing in the negative x-direction):

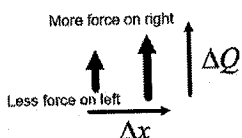
$$-\frac{\partial P}{\partial y}$$

Combining these gives an expression for the circulation at this point:

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

(Seem familiar? Our conservative field test: conservative fields have circulation = 0)

The vertical component of the field would work similarly, imbalance causing rotation, but now this would be changes in the y-component of the field as we move in the x-direction...



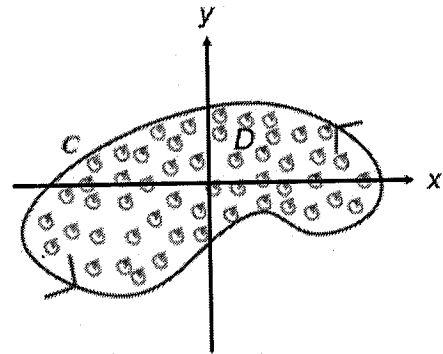
...and would cause a positive circulation if this change was positive (the field was pointing in the positive y-direction):

$$\frac{\partial Q}{\partial x}$$

Green's Theorem

Now we can define the structure and theorem for this section. **Green's Theorem** is a mathematical statement that the vector line integral in 2D is the sum of the individual microscopic circulations:

$$\oint_C P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

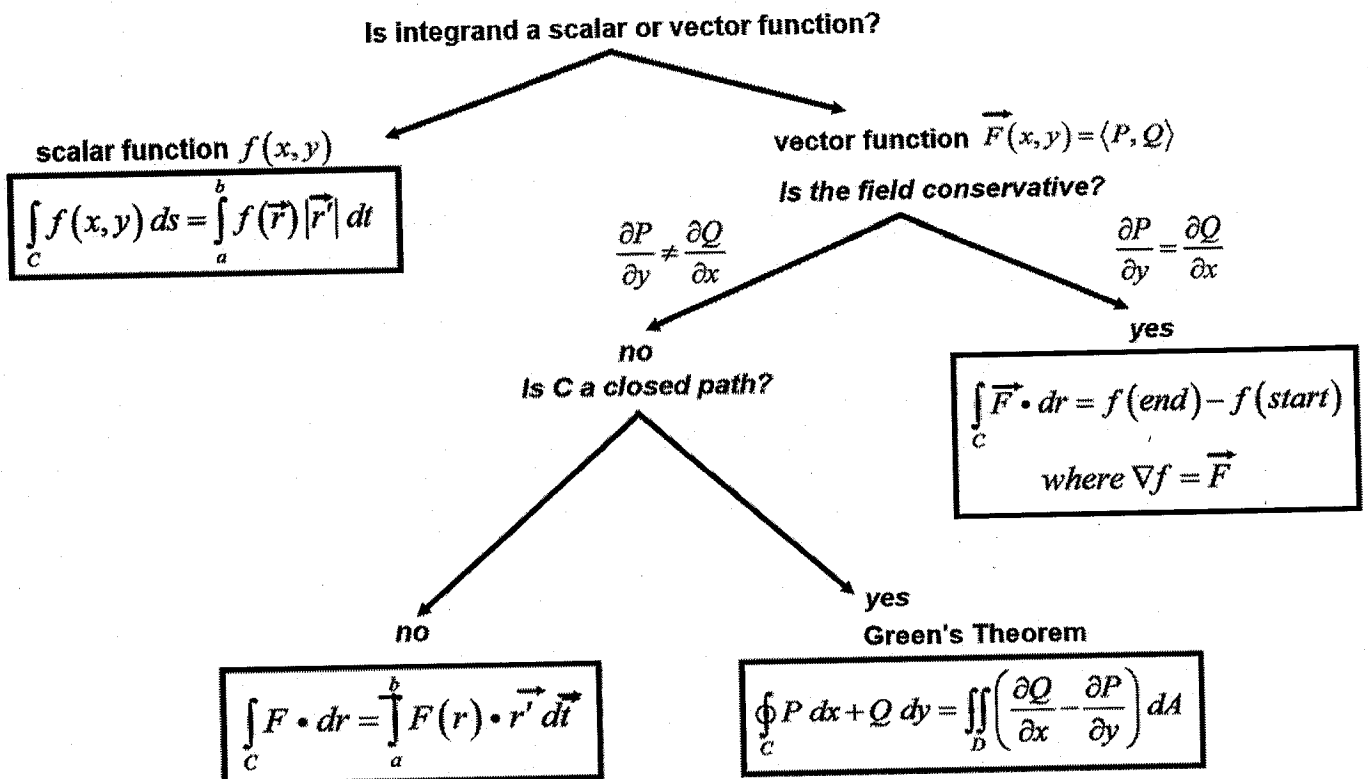


Green's Theorem only applies when the field and path are 2D and when the path is counter-clockwise (known as 'positive orientation'). This is why the line integral is written with a modified notation:

$$\oint_C P dx + Q dy \text{ or } \oint_C P dx + Q dy$$

(If the circulation is in the clockwise direction, you must multiply the result by -1 to account for this negative circulation.)

Tree for evaluating line integrals



Examples

#1) Evaluate the line integral (a) directly and (b) using Green's Theorem.

$$\oint_C (x-y) dx + (x+y) dy \quad C: \text{circle with center } (0,0), \text{ radius} = 2$$

(a) directly

conservative? $P = x - y$ $Q = x + y$

$$\frac{\partial P}{\partial y} = -1 \neq \frac{\partial Q}{\partial x} = 1 \quad \text{so not conservative, can't use potential function}$$

parametrize C: $x = 2 \cos t$ $y = 2 \sin t$ $0 \leq t \leq 2\pi$

$$r = \langle 2 \cos t, 2 \sin t \rangle \quad r' = \langle -2 \sin t, 2 \cos t \rangle$$

$$F(r) = \langle (2 \cos t) - (2 \sin t), (2 \cos t) + (2 \sin t) \rangle$$

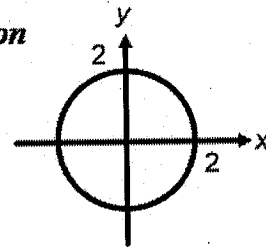
$$F \cdot r' = \langle 2 \cos t - 2 \sin t, 2 \cos t + 2 \sin t \rangle \cdot \langle -2 \sin t, 2 \cos t \rangle$$

$$= -2 \sin t (2 \cos t - 2 \sin t) + 2 \cos t (2 \cos t + 2 \sin t)$$

$$= -4 \sin t \cos t + 4 \sin^2 t + 4 \cos^2 t + 4 \sin t \cos t$$

$$= 4$$

$$\oint_C (x-y) dx + (x+y) dy = \int_0^{2\pi} F \cdot r' = \int_0^{2\pi} 4 = \boxed{8\pi}$$



(b) using Green's Theorem

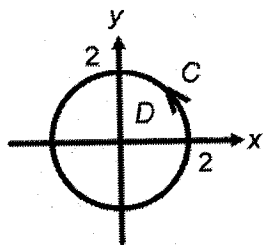
$$P = x - y \quad Q = x + y$$

$$\frac{\partial P}{\partial y} = -1 \quad \frac{\partial Q}{\partial x} = 1$$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = (1) - (-1) = 2$$

$$\oint_C (x-y) dx + (x+y) dy = \iint_D 2 dA = \int_0^{2\pi} \int_0^2 2 r dr d\theta$$

$$= 2 \int_0^{2\pi} d\theta \int_0^2 r dr = 2 [\theta]_0^{2\pi} \left[\frac{1}{2} r^2 \right]_0^2 = 2(2\pi)(2) = \boxed{8\pi}$$

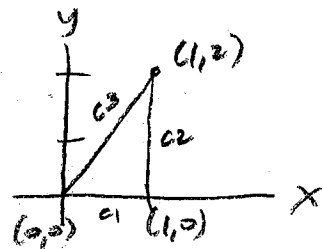


$$\oint_C P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

Examples

#3) Evaluate the line integral (a) directly and (b) using Green's Theorem.

$\oint_C (xy) dx + (x^2 y^3) dy$ C is the triangle with vertices (0,0), (1,0), and (1,2)



(a) direct

conservative? $\frac{\partial P}{\partial y} = x$ $\frac{\partial Q}{\partial x} = 2xy^3$ \neq

(c1) $\vec{r} = (1-t)\langle 0,0 \rangle + t\langle 1,0 \rangle$ $0 \leq t \leq 1$

$\vec{r} = \langle t, 0 \rangle$ $\vec{r}' = \langle 1, 0 \rangle$

$\vec{F}(\vec{r}) = \langle 0, 0 \rangle$ $\vec{F} \cdot \vec{r}' = \langle 0, 0 \rangle \cdot \langle 1, 0 \rangle = 0$

$\int_0^1 0 dt = 0$

(c2) $\vec{r} = (1-t)\langle 1,0 \rangle + t\langle 1,2 \rangle$ $0 \leq t \leq 1$

$\vec{r} = \langle 1, 2t \rangle$ $\vec{r}' = \langle 0, 2 \rangle$

$\vec{F}(\vec{r}) = \langle 2t, 8t^3 \rangle$ $\vec{F} \cdot \vec{r}' = \langle 2t, 8t^3 \rangle \cdot \langle 0, 2 \rangle = 16t^3$

$\int_0^1 16t^3 dt = [4t^4]_0^1 = 4$

(c3) use x as parameter

$\vec{r} = \langle t, 2t \rangle$ $\vec{r}' = \langle 1, 2 \rangle$ t : from 1 to 0

$\vec{F}(\vec{r}) = \langle 2t^2, 8t^5 \rangle$ $\vec{F} \cdot \vec{r}' = \langle 2t^2, 8t^5 \rangle \cdot \langle 1, 2 \rangle = 2t^2 + 16t^5$

$\int_1^0 (2t^2 + 16t^5) dt = \left[\frac{2}{3}t^3 + \frac{16}{6}t^6 \right]_1^0 = -\frac{10}{3}$

so $\oint_C = 0 + 4 + (-\frac{10}{3}) = \frac{2}{3}$

(b) using Green's

$\oint_C xy dx + x^2 y^3 dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \iint_D (2xy^3 - x) dA$



$= \int_0^1 \int_0^{2x} (2xy^3 - x) dy dx$

(dy) $\int_0^{2x} (2xy^3 - x) dy = \left[\frac{1}{2}xy^4 - xy \right]_0^{2x} = (8x^5 - 2x^2)$

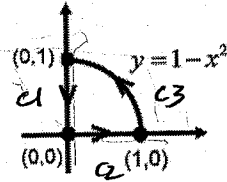
(dx) $\int_0^1 (8x^5 - 2x^2) dx = \left[\frac{8}{6}x^6 - \frac{2}{3}x^3 \right]_0^1 = \frac{2}{3}$

Examples

#4) Evaluate the line integral (a) directly and (b) using Green's Theorem.

$\oint_C (x) dx + (y) dy$ C consists of the line segments from $(0,1)$ to $(0,0)$ and from $(0,0)$ to $(1,0)$ and the parabola $y=1-x^2$ from $(1,0)$ to $(0,1)$

do this one last
 $\vec{F} = \langle x, y \rangle$



(a) direct

conservative? $\frac{\partial p}{\partial y} = 0$ $\frac{\partial q}{\partial x} = 0 \Rightarrow$ yes

Since this is a closed path, $\oint_C = 0$
but if we didn't check first and proceeded...

(1) $\vec{r} = (1-t)\langle 0,1 \rangle + t\langle 0,0 \rangle$ $0 \leq t \leq 1$ $\vec{r}' = \langle 0, -1 \rangle$

$\vec{F}(\vec{r}) = \langle 0, 1-t \rangle$ $\vec{F} \cdot \vec{r}' = \langle 0, 1-t \rangle \cdot \langle 0, -1 \rangle = t-1$ $\int_0^1 (t-1) dt = \left[\frac{1}{2}t^2 - t \right]_0^1 = \left[-\frac{1}{2} \right]$

(2) $\vec{r} = (1-t)\langle 0,0 \rangle + t\langle 1,0 \rangle = \langle t, 0 \rangle$ $0 \leq t \leq 1$, $\vec{r}' = \langle 1, 0 \rangle$
 $\vec{F}(\vec{r}) = \langle t, 0 \rangle$ $\vec{F} \cdot \vec{r}' = \langle t, 0 \rangle \cdot \langle 1, 0 \rangle = t$ $\int_0^1 t dt = \left[\frac{1}{2}t^2 \right]_0^1 = \left[\frac{1}{2} \right]$

(3) use $x=t$: $\vec{r} = \langle t, 1-t^2 \rangle$ $(1,0) \rightarrow (0,1)$ $\vec{r}' = \langle 1, -2t \rangle$
 $\vec{F}(\vec{r}) = \langle t, 1-t^2 \rangle$ $\vec{F} \cdot \vec{r}' = \langle t, 1-t^2 \rangle \cdot \langle 1, -2t \rangle = t - 2t + 2t^3$

total: $\frac{1}{2} + \frac{1}{2} + (-1) = 0$ \checkmark $\int_0^1 (t - 2t + 2t^3) dt = \left[\frac{1}{2}t^2 - t^2 + \frac{1}{2}t^4 \right]_0^1 = 0$

(b) use Green's Theorem

$$\oint_C x dx + y dy = \iint_D \left(\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y} \right) dA = \iint_D (0 - 0) dA = 0$$

Examples

#8) Evaluate the line integral using Green's Theorem along the given positively oriented curve.

$$\oint_C (xe^{-2x}) dx + (x^4 + 2x^2y^2) dy \quad C \text{ is the boundary of the region between the circles}$$

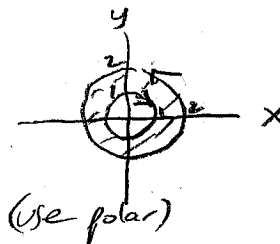
$$x^2 + y^2 = 1 \text{ and } x^2 + y^2 = 4$$

conservative? $\frac{\partial Q}{\partial x} = 4x^3 + 4xy^2$ $\frac{\partial P}{\partial y} = 0$ not conservative

$$\oint_C (xe^{-2x}) dx + (x^4 + 2x^2y^2) dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

$$= \iint_D 4x^3 + 4xy^2 dA$$

use polar $x = r \cos \theta$
 $y = r \sin \theta$



set up

$$= \int_0^{2\pi} \int_1^2 4(r \cos \theta)^3 + 4(r \cos \theta)(r \sin \theta)^2 r dr d\theta$$

$$= \int_0^{2\pi} \int_1^2 (4r^3 \cos^3 \theta + 4r^3 \cos \theta \sin^2 \theta) r dr d\theta$$

$$= \int_0^{2\pi} (\cos^3 \theta + \cos \theta \sin^2 \theta) d\theta \int_1^2 r^4 dr$$

$$= \int_0^{2\pi} \cos \theta (\cos^2 \theta + \sin^2 \theta) d\theta \int_1^2 r^4 dr$$

$$= \int_0^{2\pi} \cos \theta d\theta \int_1^2 r^4 dr$$

$$= (\sin \theta)_0^{2\pi} \left[\frac{1}{5} r^5 \right]_1^2$$

$$= (\sin(2\pi) - \sin(0)) \left(\frac{1}{5} (2)^5 - \frac{1}{5} (1)^5 \right)$$

$$= (0 - 0) \left(\frac{32}{5} \right) = \boxed{0}$$

not conservative, but line integral = 0
along this path.

Examples

#12) Evaluate the line integral using Green's Theorem.

$\vec{F}(x,y) = (y^2 \cos x, x^2 + 2y \sin x)$ C is the triangle from (0,0) to (2,6) to (2,0) to (0,0)

conservative? $\frac{\partial Q}{\partial x} = 2x + 2y \cos x$ $\frac{\partial P}{\partial y} = 2y \cos x$ (not conservative)

$$\oint (y^2 \cos x) dx + (x^2 + 2y \sin x) dy$$

$$= \iint \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

$$= \iint (2x + 2y \cos x - 2y \cos x) dA$$

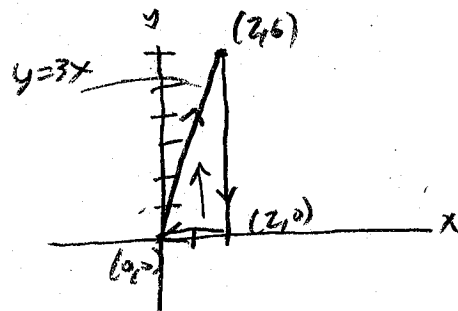
$$= \int_0^2 \int_0^{3x} 2x dy dx$$

$$\textcircled{dy} \int_0^{3x} 2x dy = 2x \left[y \right]_0^{3x} = 2x(3x) = 6x^2$$

$$\textcircled{dx} \int_0^2 6x^2 dx = \left[2x^3 \right]_0^2 = 16$$

but this is a clockwise path
(negative orientation)

so answer is $\boxed{-16}$



16.5: Curl and Divergence

Definition and computation of Curl and Divergence

Recall the use of the symbol 'del' to define the gradient vector for a scalar, multivariable function f ...

$$\nabla f = \langle f_x, f_y, f_z \rangle = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$

...and the del symbol can be thought of as a vector of partial derivative operators:

$$\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$$

Now that we are working with vector functions, what happens if we take the dot product or cross product of del with the vector function? We get what is called **divergence** and **curl**:

Divergence

(divergence - dot product)

$$\operatorname{div} F = \nabla \cdot \vec{F}$$

$$= \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle \cdot \langle P, Q, R \rangle$$

$$= \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

(output is a scalar at each point,
a scalar field)

Curl

(curl - cross product)

$$\operatorname{curl} F = \nabla \times \vec{F}$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

$$= \left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle$$

(output is a vector at each point,
a vector field)

Ex) Find the divergence and curl of the vector field $\vec{F} = \langle xe^{yz}, ye^{xz}, ze^{xy} \rangle$

$$\operatorname{div} \vec{F} = \nabla \cdot \vec{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle xe^{yz}, ye^{xz}, ze^{xy} \rangle$$

$$= \frac{\partial}{\partial x} [xe^{yz}] + \frac{\partial}{\partial y} [ye^{xz}] + \frac{\partial}{\partial z} [ze^{xy}]$$

$$= \boxed{e^{yz} + e^{xz} + e^{xy}}$$

$$\operatorname{curl} \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xe^{yz} & ye^{xz} & ze^{xy} \end{vmatrix}$$

'other direction'
because this
one is negative

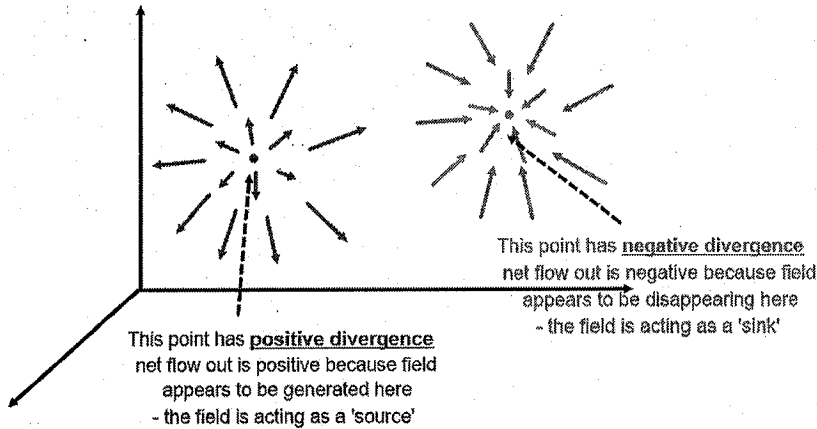
$$= \left\langle \left(\frac{\partial}{\partial y} [ze^{xy}] - \frac{\partial}{\partial z} [ye^{xz}] \right), \left(\frac{\partial}{\partial z} [xe^{yz}] - \frac{\partial}{\partial x} [ze^{xy}] \right), \left(\frac{\partial}{\partial x} [ye^{xz}] - \frac{\partial}{\partial y} [xe^{yz}] \right) \right\rangle$$

$$= \boxed{\langle (xze^{xy} - yze^{xz}), (xye^{yz} - yze^{xy}), (yze^{xz} - xze^{yz}) \rangle}$$

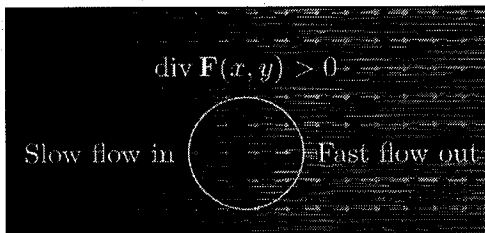
Meaning of Divergence

$$\text{div } F = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

The divergence of a vector field at a point is a measure of the net flow out of the field:



Note: every point in a field has a divergence value, and its value is a little more subtle, in reality, than just 'flows in' or 'flows out'. For example, if you had a field like this where the field is flowing to the right, but with smaller magnitude on the left and larger magnitude on the right...

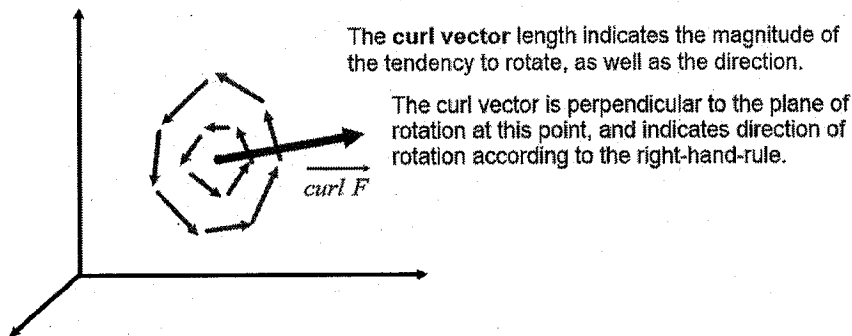


...than this point would also have a positive divergence, because in its region 'less' is flowing in than is flowing out due to the imbalance in magnitudes. This region is behaving as if it is 'sourcing' field because more appears to be leaving than entering.

Meaning of Curl

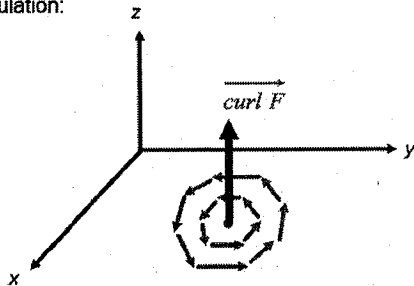
$$\text{curl } F = \left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle$$

The curl of a vector field at a point is a measure of the tendency for a vector field to rotate:



Curl is like a 3D version of circulation

If the rotation at a point is in the x-y plane, then the curl vector would only have a z-component, and would equal our expression for circulation, so you can think of curl as the 3D equivalent of circulation:



$$\text{curl } F = \nabla \times F$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & 0 \end{vmatrix}$$

$$= \left\langle 0, 0, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle$$

Examples

#2) Evaluate the divergence and curl of $\vec{F}(x, y, z) = \langle x^2yz, xy^2z, xyz^2 \rangle$

$$\begin{aligned}\operatorname{div} \vec{F} &= \nabla \cdot \vec{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle x^2yz, xy^2z, xyz^2 \rangle \\ &= 2xyz + 2xyz + 2xyz = \boxed{6xyz}\end{aligned}$$

$$\begin{aligned}\operatorname{curl} \vec{F} &= \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2yz & xy^2z & xyz^2 \end{vmatrix} \\ &= \left\langle \left(\frac{\partial}{\partial y}[xyz^2] - \frac{\partial}{\partial z}[xy^2z] \right), \left(\frac{\partial}{\partial z}[x^2yz] - \frac{\partial}{\partial x}[xyz^2] \right), \left(\frac{\partial}{\partial x}[xy^2z] - \frac{\partial}{\partial y}[x^2yz] \right) \right\rangle \\ &= \boxed{\langle (xz^2 - xy^2), (x^2z - yz^2), (y^2z - x^2z) \rangle}\end{aligned}$$

If a vector field is conservative, curl = 0

It can be shown that if a field is conservative (and therefore there exists a potential function), then the curl will be zero.

Technically, however, the converse is not necessarily true, just because a field has a zero curl does not guarantee that the field is conservative. But checking curl = 0 can be used to show conclusively that a field is **not** conservative.

In practice, when curl = 0 this is a strong indication that the field is conservative, so we should attempt to find a potential function. If we are able to, then this **does** indicate that the field is conservative.

When a field has curl = 0, it does not have a tendency to rotate, so we say the field is **irrotational**.

What does divergence = 0 means?

If you imagine that a field represents something like a physical medium (such as a fluid or a gas), then $\operatorname{div} \vec{F}$ represents the net rate of change with respect to time of the mass of fluid (or gas) flowing from a point, per unit volume.

If the divergence is zero, that means there is no net flow out of the field at that point...any 'fluid' that is flowing in is all flowing out, which means there is no 'compression' of the field (accumulation of fluid in that spot).

Therefore, when a field has divergence = 0, it does not compress, and is called **incompressible**.

Examples

Ex) Determine whether or not the vector function is conservative.

If it is conservative, find potential function f such that $\vec{F} = \nabla f$

$$F(x, y, z) = \langle x, y, z \rangle$$

In 2D, we would have checked $\frac{\partial^2}{\partial x^2} \stackrel{?}{=} \frac{\partial^2}{\partial y^2}$

here in 3D, if field is conservative, $\text{curl } \vec{F} = 0$ but we can't use that as a test because it is a necessary but not sufficient condition.

Instead, we must try to find a potential function, $f(x, y, z)$ such that $\nabla f = \vec{F}$

$$\vec{F} = \langle x, y, z \rangle$$

$$\text{so } f_x = x \quad f_y = y \quad f_z = z$$

$$f(x, y, z) = \int x dx$$

$$f(x, y, z) = \frac{1}{2}x^2 + g(y, z)$$

$$f_y = g'(y, z) \rightarrow g'(y, z) = y$$

$$g(y, z) = \int y dy$$

$$g(y, z) = \frac{1}{2}y^2 + h(z)$$

$$\text{so } f(x, y, z) = \frac{1}{2}x^2 + \frac{1}{2}y^2 + h(z)$$

$$\text{and } f_z = h'(z) \rightarrow h'(z) = z$$

$$h(z) = \int z dz$$

$$h(z) = \frac{1}{2}z^2 + C$$

$$\text{so } f(x, y, z) = \frac{1}{2}x^2 + \frac{1}{2}y^2 + \frac{1}{2}z^2 + C$$

there is a potential function such that $\nabla f = \vec{F}$

so \vec{F} is conservative

Vector form of Green's Theorem

From the previous section, for 2D, Green's Theorem says the line integral of a vector over a closed, counter-clockwise, path can be evaluated by integrating the expression for circulator over the enclosed area:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

If we wrote this using curl...

$$= \iint_D \left(\left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle \cdot \langle 0, 0, 1 \rangle \right) dA$$

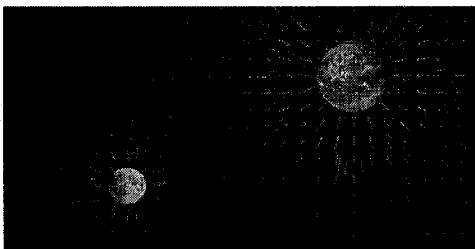
...we get the vector form of Green's Theorem:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D (\text{curl } \mathbf{F} \cdot \mathbf{k}) dA$$

Videos about the concepts of divergence and curl

The YouTube channel 3Blue1Brown is run by Grant Sanderson, and excellent math/science video producer who did most of the Calc III videos for the Khan Academy website. His separate channel has some really excellent videos, with very good animations and explanations, including this one about divergence and curl:

<https://www.youtube.com/watch?v=rB83DpBJQsE>



Div of curl?

Taking the curl of a vector field (at a point) produces a vector, so the result for all points is also a vector field. You could then take the divergence of this curl of the vector field.

But the result can be shown to be (theorem 11 in the textbook):

If $\vec{F} = \langle P, Q, R \rangle$ is a vector field on \mathbb{R}^3 and $P, Q,$ and R have continuous second-order partial derivatives, then

$$\text{div curl } \vec{F} = 0$$

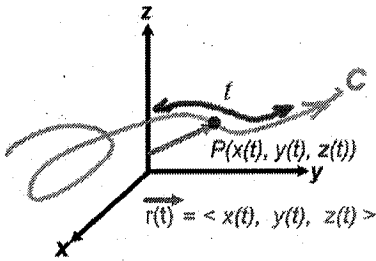
divergence of gradient $\nabla \cdot (\nabla \mathbf{F}) = \nabla \cdot \left\langle \frac{\partial P}{\partial x_1}, \frac{\partial Q}{\partial y_1}, \frac{\partial R}{\partial z_1} \right\rangle$
 Laplacian operator $\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 Q}{\partial y^2} + \frac{\partial^2 R}{\partial z^2}$

16.6: Parametric Surfaces and their Areas

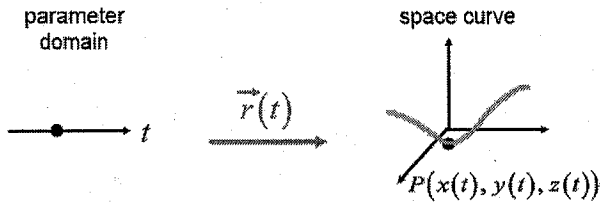
Parametric Surfaces are similar to Space Curves

We've encountered the idea that you can use a parameter, t , to 'move along' a space curve...

Space Curve



We only need one dimension in the 'parameter domain' to move along this space curve to produce all points on the space curve...



...and you can think of the position vector, r , as something that converts t values in the parameter domain to points on the space curve (like a function).

Parametric equations:

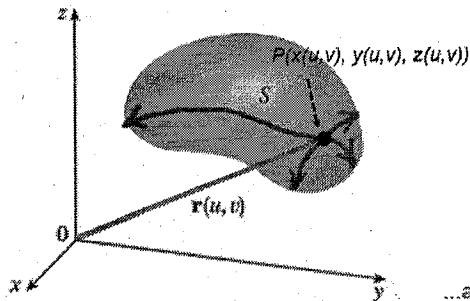
$$x = x(t)$$

$$y = y(t)$$

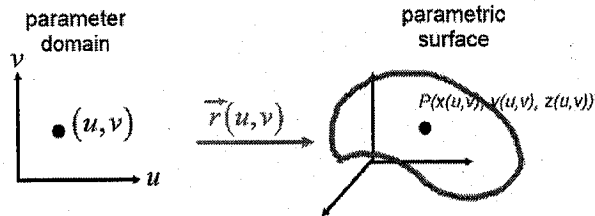
$$z = z(t)$$

If we need to represent a generic 2D surface in 3D space, to cover all points on this surface, we will need a 2D parameter domain, so instead of parameter, t , we now define parameters u and v ...

Parametric Surface



We need two dimensions in the 'parameter domain' to move across this surface to produce all points on the surface...



...and you can think of the position vector, r , as something that converts points (u, v) in the parameter domain to points on the parametric surface (like a function).

Parametric equations:

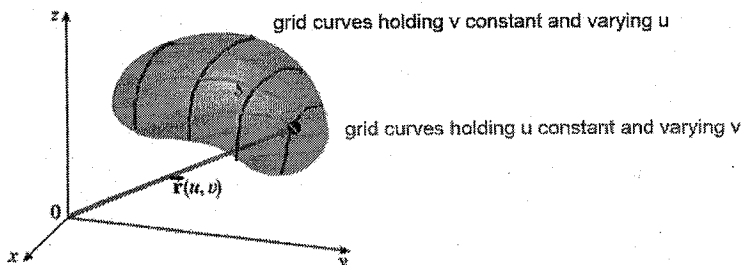
$$x = x(u, v)$$

$$y = y(u, v)$$

$$z = z(u, v)$$

Grid Curves

If we hold one of the parameters constant and vary the other one, this will trace out a curve along the parametric surface, which is called a 'grid curve'.



Examples

#1) Determine whether point P lies on the given plane.

$$\vec{r}(u,v) = \langle 2u+3v, 1+5u-v, 2+u+v \rangle$$

add last 2 rows:

$$\begin{array}{r} 5u-v=9 \\ + (u+v=2) \\ \hline 6u=11 \end{array}$$

$$u = \frac{11}{6}$$

backsubstitute:

$$u+v=2$$

$$\left(\frac{11}{6}\right) + v = 2$$

$$v = 2 - \frac{11}{6} = \frac{1}{6}$$

$$u = \frac{11}{6}$$

$$v = \frac{1}{6}$$

now, check in last equation:

$$u = \frac{11}{6}, v = \frac{1}{6}$$

$$2u + 3v = 7$$

$$2\left(\frac{11}{6}\right) + 3\left(\frac{1}{6}\right) \stackrel{?}{=} 7$$

$$\frac{25}{6} \neq 7$$

so P is not on the plane

$$P(7,10,4) \rightarrow \begin{cases} 2u+3v=7 \\ 1+5u-v=10 \\ 2+u+v=4 \end{cases} = \begin{cases} 2u+3v=7 \\ 5u-v=9 \\ u+v=2 \end{cases}$$

Parametrizing Parametric Surfaces

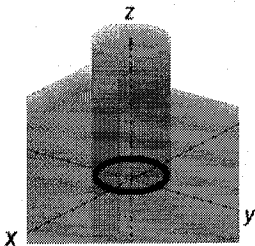
Parametrization of a parametric surface means finding two parameters which span the domain and then a set of functions of these parameters to produce the coordinate values for each dimension in the output space where the surface resides.

Because there are different possible choices for the parameters, parametrizations of parametric surfaces are not unique.

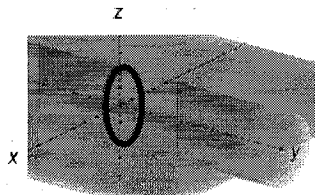
Let's consider some common surfaces and their parametrizations...

Parametrizing Cylinders

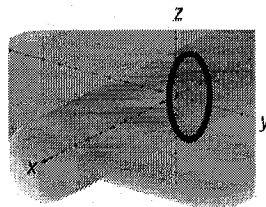
$$x^2 + y^2 = a^2$$



$$x^2 + z^2 = a^2$$



$$y^2 + z^2 = a^2$$



For the plane with circular grid curves, we can use polar parameters, a, θ . The last dimension is not a parameter, but is used when specifying the point in 3D space...

Possible parametrizations: $\vec{r}(z, \theta)$

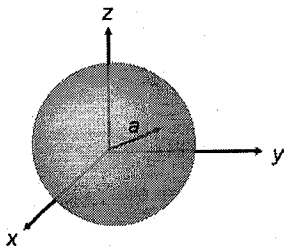
$$\begin{aligned} x &= a \cos \theta \\ y &= a \sin \theta \\ z &= z \end{aligned}$$

these could be reversed - just depends upon which axis you want to define as the reference for θ

$$\begin{aligned} x &= a \cos \theta \\ y &= y \\ z &= a \sin \theta \end{aligned}$$

$$\begin{aligned} x &= x \\ y &= a \cos \theta \\ z &= a \sin \theta \end{aligned}$$

Parametrizing Spheres



$$x^2 + y^2 + z^2 = a^2$$

For spheres, we can just use polar coordinates, and use the θ, ϕ angles as the parameters, with the radius being a constant on the spherical surface:

$$\vec{r}(\phi, \theta)$$

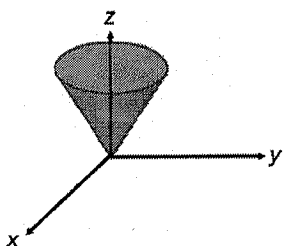
$$x = a \sin \phi \cos \theta$$

$$y = a \sin \phi \sin \theta$$

$$z = a \cos \phi$$

Parametrizing Cones

$$z = \sqrt{x^2 + y^2}$$



For the cone, there are two ways we could go...

...we could use the fact that the grid curves are circles and use polar coordinates as the parameters:

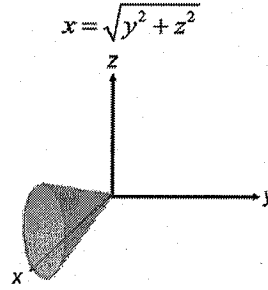
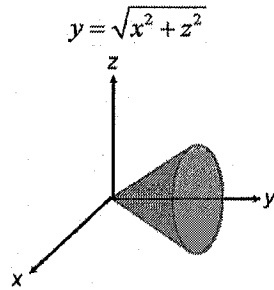
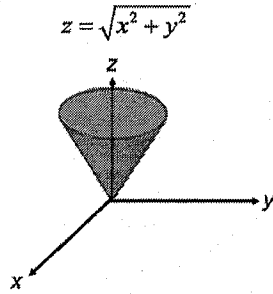
$$\vec{r}(r, \theta) \quad \begin{aligned} x &= r \cos \theta \\ y &= r \sin \theta \\ z &= r \end{aligned}$$

...or we could take advantage of the fact that the surface is expressed with z as a function of x and y and just use x, y as the parameters:

$$\vec{r}(x, y) \quad \begin{aligned} x &= x \\ y &= y \\ z &= \sqrt{x^2 + y^2} \end{aligned}$$

This illustrates that there are often multiple possible parametrizations. You just need any two parameter variables and a set of equations such that if you vary the two parameters over all possible combinations of their values, the parametric equations map out points that cover the entire parametric surface.

Parametrizing Cones



Possible parametrizations:

$$\begin{array}{ll} \vec{r}(r, \theta) & \vec{r}(x, y) \\ x = r \cos \theta & x = x \\ y = r \sin \theta & y = y \\ z = r & z = \sqrt{x^2 + y^2} \end{array}$$

$$\begin{array}{ll} \vec{r}(r, \theta) & \vec{r}(x, y) \\ x = r \cos \theta & x = x \\ y = r & y = \sqrt{x^2 + z^2} \\ z = r \sin \theta & z = z \end{array}$$

$$\begin{array}{ll} \vec{r}(r, \theta) & \vec{r}(x, y) \\ x = r & x = \sqrt{y^2 + z^2} \\ y = r \cos \theta & y = y \\ z = r \sin \theta & z = z \end{array}$$

Parametrizing General Surfaces

For 98% of surfaces, it is possible to express the surface as $z = f(x, y)$

and in these cases, the simplest way to parametrize is to just use x, y as the two parameters and make z a function of these:

$$\begin{array}{l} x = x \\ \vec{r}(x, y) \quad y = y \\ z = f(x, y) \end{array}$$

Parametrizing a Plane

Once we have the equation of a plane in the form $ax + by + cz = d$

we can choose x, y as the parameters and solve for z in terms of these to parametrize:

$$\begin{array}{l} x = x \\ \vec{r}(x, y) \quad y = y \\ z = \frac{d - ax - by}{c} \end{array}$$

Examples

#19) Find a parametric representation for the surface of the plane that passes through $(1, 2, -3)$ and contains the vectors $\langle 1, 1, -1 \rangle$ and $\langle 1, -1, 1 \rangle$

(method different from textbook solution)

Normal to plane $\vec{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{vmatrix}$

$$= \langle (1-1), (-1-1), (-1-1) \rangle$$

$$= \langle 0, -2, -2 \rangle$$

plane: $0x - 2y - 2z = \langle 0, -2, -2 \rangle \cdot \langle 1, 2, -3 \rangle$

$$-2y - 2z = 0 - 4 + 6$$

$$-2y - 2z = 2$$

$$-y - z = 1$$

$$\text{so } z = -y - 1$$

parameterization: $\begin{array}{l} x = x \\ y = y \\ z = -y - 1 \end{array}$

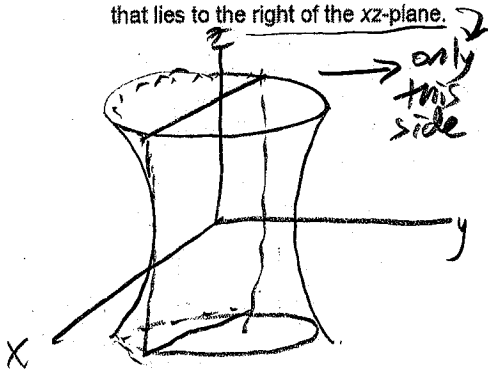
more commonly written: $\vec{r}(x, y) = \langle x, y, -y - 1 \rangle$

range of parameters $\begin{cases} -\infty < x < \infty \\ -\infty < y < \infty \end{cases}$
should also be given

Examples

#21) Find a parametric representation for the surface of the part of the hyperboloid $x^2 + y^2 - z^2 = 1$

that lies to the right of the xz -plane.



y is varying
so use x, z
as the
parameters

$$x = x$$

$$z = z$$

$$y^2 = z^2 - x^2 + 1$$

$$y = \pm \sqrt{z^2 - x^2 + 1}$$

$$y = \sqrt{z^2 - x^2 + 1}$$

use + (positive
 y only)

$$\text{so } \vec{r}(x, z) = \langle x, \sqrt{z^2 - x^2 + 1}, z \rangle$$

Restricting the parameter domain

The parameters may be restricted to only a portion of the parameter domain and the result is that only a portion of the parametric surface is produced.

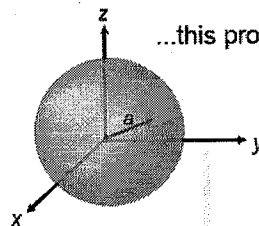
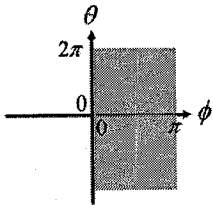
Consider the sphere: $x^2 + y^2 + z^2 = a^2$

that has been parametrized as $\vec{r}(\phi, \theta) = \langle a \sin \phi \cos \theta, a \sin \phi \sin \theta, a \cos \phi \rangle$

If we allow the parameters to vary as...

$$0 \leq \phi \leq \pi$$

$$0 \leq \theta \leq 2\pi$$

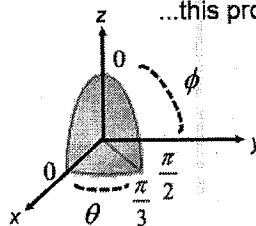
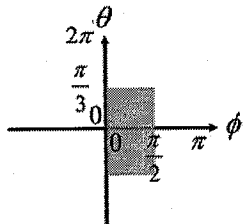


...this produces the entire parametric surface.

But if we restrict the parameters to...

$$0 \leq \phi \leq \frac{\pi}{2}$$

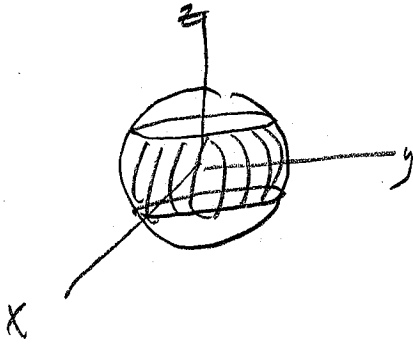
$$0 \leq \theta \leq \frac{\pi}{3}$$



...this produces only a portion of the surface.

NOTE: parametrization isn't complete unless the limitations on the parameters are also specified.

#24) Find a parametric representation for the surface of the part of the sphere $x^2 + y^2 + z^2 = 16$ that lies between the planes $z = -2$ and $z = 2$



when $z=2$: $x^2 + y^2 + (2)^2 = 16$
 $x^2 + y^2 = 12$

z is varying, make x, y the parameters
 — or — could use spherical (better)

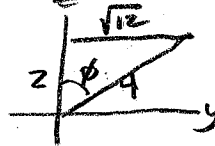
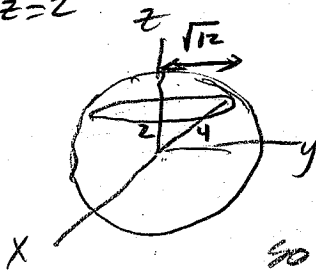
$x = \rho \sin \phi \cos \theta$
 $y = \rho \sin \phi \sin \theta$
 $z = \rho \cos \phi$

so $\vec{r}(\phi, \theta) = \langle 4 \sin \phi \cos \theta, 4 \sin \phi \sin \theta, 4 \cos \phi \rangle$

but we need limits...

projecting onto $y-z$ plane:

intersection when $z=2$
 $\rightarrow x^2 + y^2 = 12$
 $r^2 = 12$
 $r = \sqrt{12}$



$\sin \phi = \frac{\sqrt{12}}{4} = \frac{2\sqrt{3}}{4} = \frac{\sqrt{3}}{2}$

so $\phi = \pi/3$

so limits for parameters are:

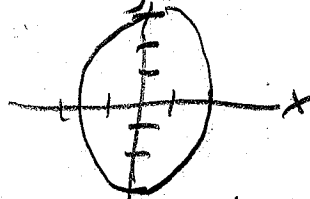
$\frac{\pi}{3} \leq \phi \leq \frac{2\pi}{3}, \quad 0 \leq \theta \leq 2\pi$

#4) Identify the surface with the given vector equation $\vec{r}(u, v) = \langle 2 \sin u, 3 \cos u, v \rangle \quad 0 \leq v \leq 2$

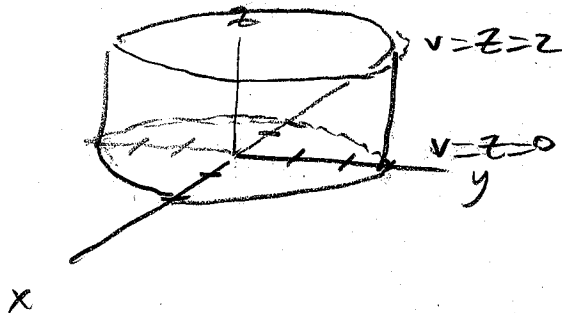
ignoring z $\vec{r}(u, v) = \langle 2 \sin u, 3 \cos u \rangle$

limit for v
 (no limit given for u)

is a parametrization for an ellipse:



so $\vec{r}(u, v) = \langle 2 \sin u, 3 \cos u, v \rangle$ extrudes this up in z direction:
 into an elliptical cylinder:



Tangent Planes

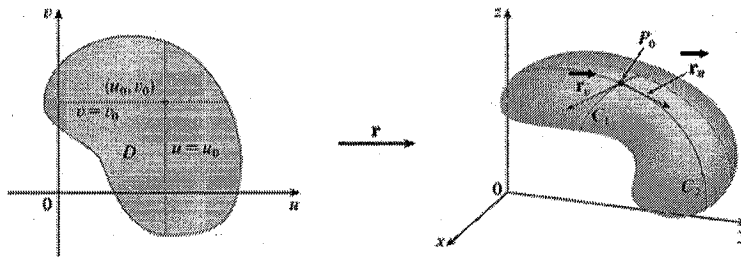
If you hold parameter v constant and traverse across u in the parameter domain, this traces curve C_1 on the surface. The tangent to this curve would be \vec{r}' a vector whose components are found by taking the partial derivatives in the $x, y,$ and z directions, but as we move in the u direction, so we call this vector \vec{r}_u and define it as follows:

$$\vec{r}_u = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle$$

Similarly, if we hold parameter u constant and vary v in the parameter domain, this traces curve C_2 on the surface which has tangent vector \vec{r}_v defined as:

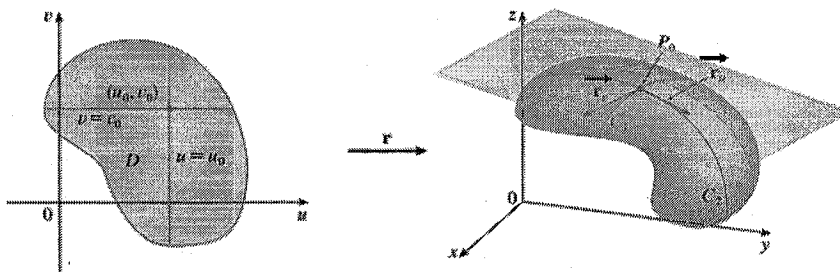
$$\vec{r}_v = \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right\rangle$$

(Although these don't have the 'prime' notation, we understand these are derivatives due to their subscripts)



Tangent Planes

These two tangent vectors are both tangent to the surface at point P_0 so we can use these vectors to define a tangent plane...



...and we can find the normal vector for this plane by taking the cross product of these two tangent vectors:

Normal vector which defines tangent plane to a parametric surface is:

$$\vec{n} = \vec{r}_u \times \vec{r}_v$$

where:

$$\vec{r}_u = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle \quad \vec{r}_v = \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right\rangle$$

Examples

#34) Find an equation of the tangent plane to the given parametric surface as the specified point.

$$x = u^2, y = v^2, z = uv; \quad u=1, v=1$$

$$\vec{r}(u,v) = \langle u^2, v^2, uv \rangle$$

$$\vec{r}_u = \langle 2u, 0, v \rangle \quad \vec{r}_v = \langle 0, 2v, u \rangle$$

at this point $u=1, v=1$, so

$$\vec{r}_u = \langle 2, 0, 1 \rangle \quad \vec{r}_v = \langle 0, 2, 1 \rangle$$

$$\vec{n} = \vec{r}_u \times \vec{r}_v = \begin{vmatrix} + & - & + \\ 2 & 0 & 1 \\ 0 & 2 & 1 \end{vmatrix} = \langle 0-2, -(2-0), 4-0 \rangle$$

$$\vec{n} = \langle -2, -2, 4 \rangle \text{ or } 2 \langle -1, -1, 2 \rangle$$

use this for \vec{n} for the plane

point on surface and plane: when $u=v=1$

$$x = u^2 = (1)^2 = 1$$

$$y = v^2 = (1)^2 = 1$$

$$z = uv = (1)(1) = 1$$

$$\text{so } \vec{r}_0 = \langle 1, 1, 1 \rangle$$

$$\text{w/ } \vec{n} = \langle -1, -1, 2 \rangle$$

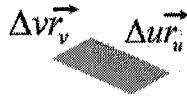
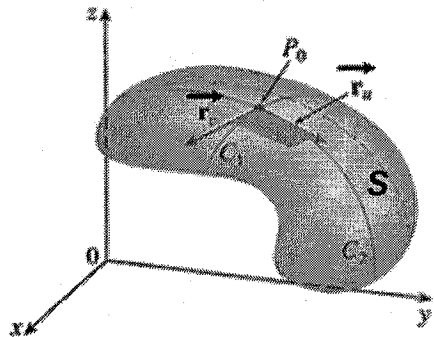
$$\text{tangent plane: } ax + by + cz = \vec{n} \cdot \vec{r}_0$$

$$\begin{aligned} (-1)x + (-1)y + (2)z &= \langle -1, -1, 2 \rangle \cdot \langle 1, 1, 1 \rangle \\ &= (-1)(1) + (-1)(1) + (2)(1) \\ &= -1 - 1 + 2 \\ &= 0 \end{aligned}$$

$$\text{tangent plane: } \boxed{-x - y + 2z = 0}$$

Area of a Surface

Further, if we go a small amount in the parameter domain, Δu we travel $\Delta u \vec{r}_u$ in the direction of this tangent vector, and similarly travel $\Delta v \vec{r}_v$ along the other tangent vector.



The area of this infinitesimally small rectangle can be found by taking the magnitude of the cross-product (from earlier):

$$\begin{aligned} \text{area} &= |(\Delta u \vec{r}_u) \times (\Delta v \vec{r}_v)| \\ &= |\vec{r}_u \times \vec{r}_v| \Delta u \Delta v \end{aligned}$$

In the limit as the size of this rectangle approaches zero, the rectangle is a good approximation for a small portion of the area of the surface, so to find the total surface area, we use a 2D integral to sum these areas:

Area of Surface: $A(S) = \iint_D |\vec{r}_u \times \vec{r}_v| dA$
 (D is the region in the parameter domain)

Area of a Surface: special case where $z = f(x, y)$

In the special case where the surface represents a function where $z = f(x, y)$ the parametrization of the surface would be: $\vec{r}(x, y) \quad x = x \quad y = y \quad z = f(x, y)$

...and the tangent vectors would be: $\vec{r}_x = \left\langle \frac{\partial x}{\partial x}, \frac{\partial y}{\partial x}, \frac{\partial z}{\partial x} \right\rangle = \left\langle \frac{\partial x}{\partial x}, \frac{\partial y}{\partial x}, \frac{\partial f(x, y)}{\partial x} \right\rangle = \left\langle 1, 0, \frac{\partial f}{\partial x} \right\rangle$

$$\vec{r}_y = \left\langle \frac{\partial x}{\partial y}, \frac{\partial y}{\partial y}, \frac{\partial z}{\partial y} \right\rangle = \left\langle \frac{\partial x}{\partial y}, \frac{\partial y}{\partial y}, \frac{\partial f(x, y)}{\partial y} \right\rangle = \left\langle 0, 1, \frac{\partial f}{\partial y} \right\rangle$$

...and the cross-product would be:

$$\vec{r}_x \times \vec{r}_y = \begin{vmatrix} i & j & k \\ 1 & 0 & \frac{\partial f}{\partial x} \\ 0 & 1 & \frac{\partial f}{\partial y} \end{vmatrix} = \left\langle -\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1 \right\rangle$$

...so the area would be:

$$\iint_D \left| \left\langle -\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1 \right\rangle \right| dA = \iint_D \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2} dA$$

...which can be written...

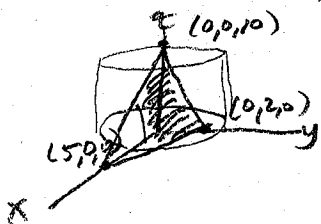
$$A(S) = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} dA$$

...the formula used in Calc1/Calc2 for surface area.

Examples

#38) Find the area of the part of the surface of the plane $2x + 5y + z = 10$

that lies inside the cylinder $x^2 + y^2 = 9$

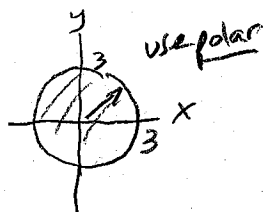


$$\vec{r}(x,y) = \langle x, y, 10 - 2x - 5y \rangle$$

$$\vec{r}_x = \langle 1, 0, -2 \rangle \quad \vec{r}_y = \langle 0, 1, -5 \rangle$$

$$\vec{r}_x \times \vec{r}_y = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -2 \\ 0 & 1 & -5 \end{vmatrix} = \langle 2, 5, 1 \rangle$$

$$|\vec{r}_x \times \vec{r}_y| = \sqrt{2^2 + 5^2 + 1^2} = \sqrt{30}$$



$$A(S) = \iint_D \sqrt{30} dA = \int_0^{2\pi} \int_0^3 \sqrt{30} r dr d\theta$$

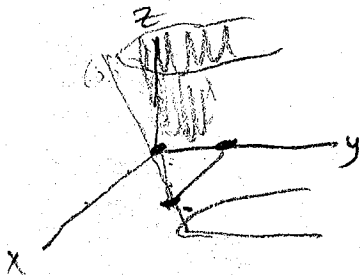
$$= \sqrt{30} \int_0^{2\pi} d\theta \int_0^3 r dr$$

$$= \sqrt{30} [2\pi] \left[\frac{1}{2} r^2 \right]_0^3$$

$$= \sqrt{30} (2\pi) \left(\frac{9}{2} \right) = \boxed{9\pi\sqrt{30}}$$

#42) Find the area of the part of the surface $z = 1 + 3x + 2y^2$

that lies above the triangle with vertices $(0,0)$, $(0,1)$, $(2,1)$

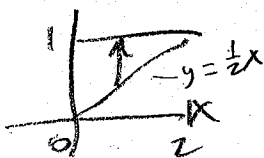


$$\vec{r}(x,y) = \langle x, y, 1 + 3x + 2y^2 \rangle$$

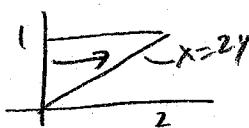
$$\vec{r}_x = \langle 1, 0, 3 \rangle \quad \vec{r}_y = \langle 0, 1, 4y \rangle$$

$$\vec{r}_x \times \vec{r}_y = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 3 \\ 0 & 1 & 4y \end{vmatrix} = \langle -3, -4y, 1 \rangle$$

$$|\vec{r}_x \times \vec{r}_y| = \sqrt{9 + 16y^2 + 1} = \sqrt{10 + 16y^2}$$



$$A(S) = \iint_D \sqrt{10 + 16y^2} dA = \int_0^2 \int_{1/2 x}^1 \sqrt{10 + 16y^2} dy dx$$



$$A(S) = \int_0^1 \int_0^{2y} \sqrt{10 + 16y^2} dx dy \quad (\text{this one is easier})$$

$$\text{(dx)} \int_0^{2y} (10 + 16y^2)^{1/2} dx = (10 + 16y^2)^{1/2} [x]_0^{2y} = (10 + 16y^2)^{1/2} (2y)$$

$$\text{(dy)} \int_0^1 (10 + 16y^2)^{1/2} 2y dy \quad \begin{array}{l} u = 10 + 16y^2 \\ du = 32y dy \\ \frac{1}{32} du = y dy \end{array}$$

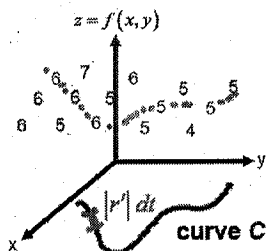
$$\frac{2}{32} \int_{10}^{26} u^{1/2} du = \frac{1}{16} \frac{2}{3} [u^{3/2}]_{10}^{26} = \boxed{\frac{1}{24} [26^{3/2} - 10^{3/2}]}$$

16.7: Surface Integrals

Surface Integrals of Scalar Functions

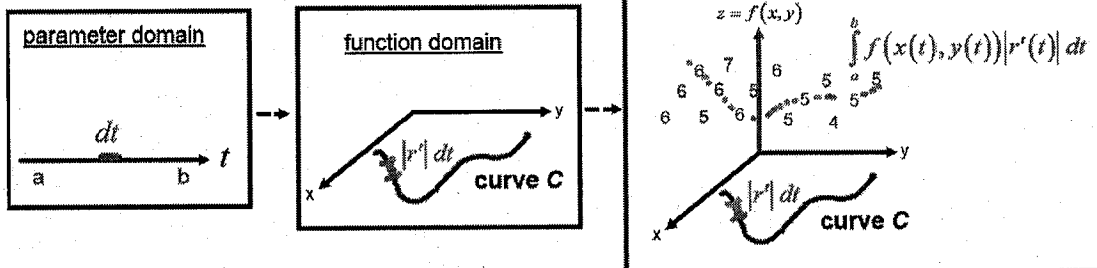
In 16.2 we defined line integrals for scalar functions, initially for a 2D scalar function...

Line Integral for scalar functions

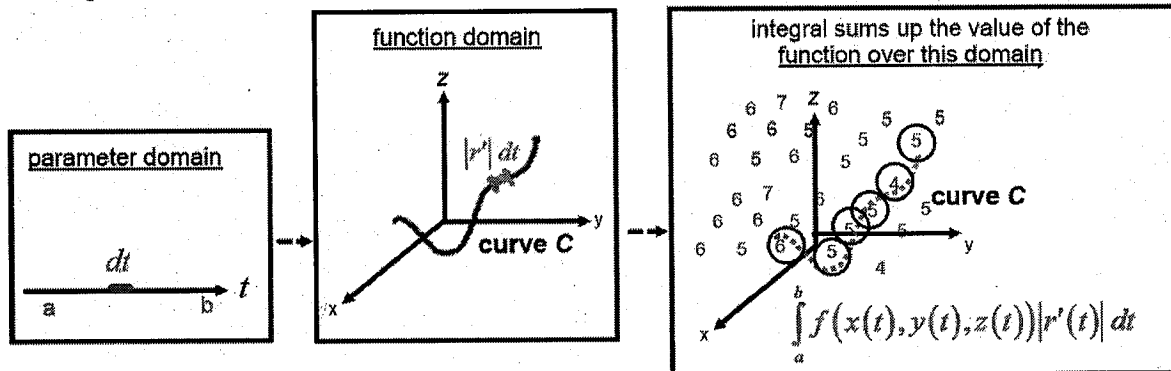


$$\int_C f(x, y) ds = \int_a^b f(x(t), y(t)) |r'| dt$$

...which you can think of this way...



Although it is a little harder to picture, we said in 16.2 this can be extended to higher dimensions...

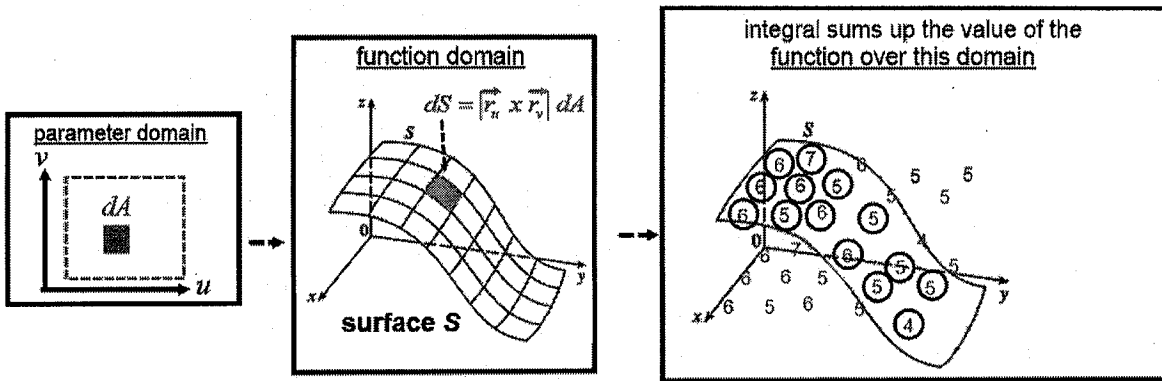


$$\int_C f(x, y, z) ds = \int_a^b f(x(t), y(t), z(t)) |r'(t)| dt$$

A line integral over a curve is summing the scalar numerical values of a multivariable scalar function over some 1-D curved path through the domain of the function.

The integrand is the function value at a point multiplied by the magnitude of the derivative of the position vector which is the non-normalized tangent vector in the direction of t moving along the curve.

In this chapter, we now extend this concept to a 2D parameter domain, which means we define a 2D surface in the function domain, and therefore we are now summing the scalar function value over a 2D surface:



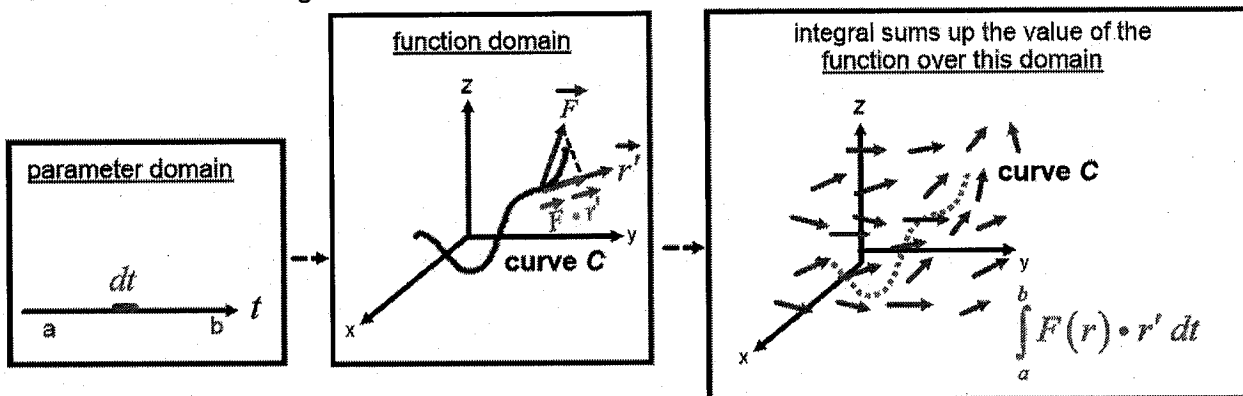
The area of the 'patch' of surface covered by a small area dA in the parameter domain would be approximated by $dS = |\vec{r}_u \times \vec{r}_v| dA$, a small portion of the tangent plane at that point.

Summing the values of the function over this surface gives the surface integral of the scalar function:

$$\iint_S f(x, y, z) dS = \iint_D f(\vec{r}(u, v)) |\vec{r}_u \times \vec{r}_v| dA$$

Surface Integrals of Vector Functions

We can define something similar for vector functions.

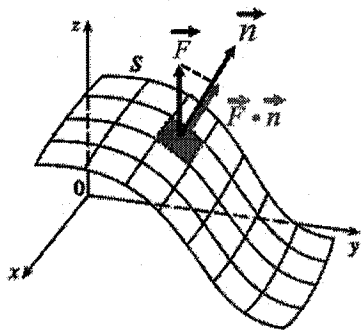


$$\int_C \vec{F}(x, y, z) dr = \int_a^b \vec{F}(r) \cdot \vec{r}' dt$$

Earlier, we said for a vector function, a line integral over a curve is now summing the amount of the vector field pointing in the direction of the curve at each point over some 1-D curved path through the domain of the function.

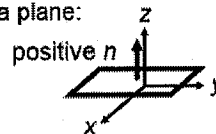
The integrand is the dot product of the field vector with the direction vector (to get the component of the field pointing along the path).

For the 2D surface version, we use a 2D domain region, but now we are summing components of the vector field instead of numerical values of a scalar field. But what is the higher dimension equivalent of taking the component of the field in the direction of the curve?



We still take the dot product to get the component of the field in a direction, but now the direction is the normal vector for the surface, n .

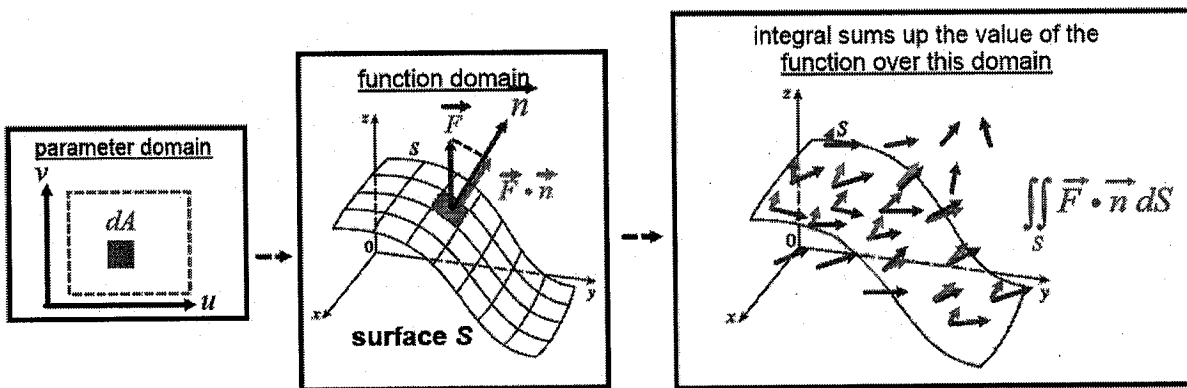
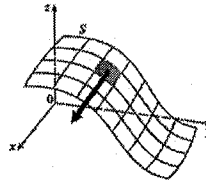
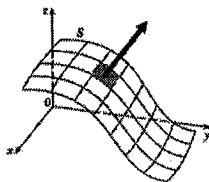
This presents a small complication...there are really two normal vectors for a surface, one on each side. We define n as the vector perpendicular to the tangent plane and point 'outward' if you viewed the surface as a sphere: or upward for a plane:



In fact, for a curved surface we define positive and negative orientations as follows:

positive orientation for this surface

negative orientation for this surface



Summing the values of the components of field that are normal to the surface (in the positive orientation direction) the over this surface gives the surface integral of the vector function:

$$\iint_S F \cdot dS = \iint_S \vec{F} \cdot \vec{n} dS \quad \text{where } n = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$$

surface integral of the vector function is also known as the 'flux of F across S'

Using the fact that $dS = |\vec{r}_x \times \vec{r}_y| dA$

$$\iint_S F \cdot dS = \iint_S \left[F(r(u,v)) \cdot \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|} \right] dS = \iint_S \left[F(r(u,v)) \cdot \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|} \right] |\vec{r}_u \times \vec{r}_v| dA$$

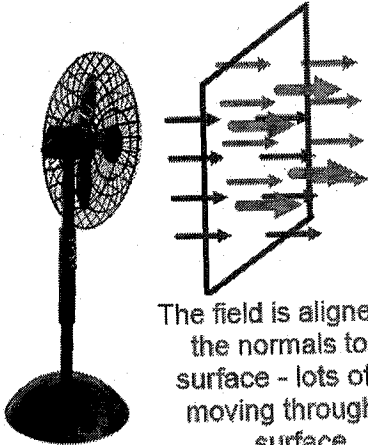
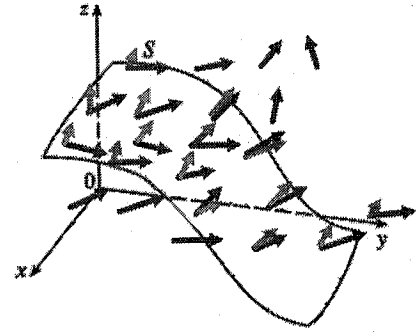
$$\iint_S F \cdot dS = \iint_S \vec{F} \cdot (\vec{r}_u \times \vec{r}_v) dA \quad \text{is a form that is easier to use in computation}$$

Physical Interpretation of Flux

You can think of flux (surface integral of a vector function) as meaning how much of the field goes 'through' the surface.

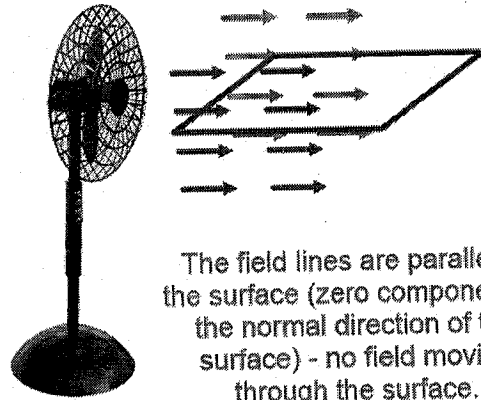
Depending upon the orientation of the surface, you could have maximum flux or minimum (maybe even no flux).

Imagine a fan blowing air and the flow of air being represented by a field:



The field is aligned with the normals to the surface - lots of field moving through the surface.

Surface with maximum flux

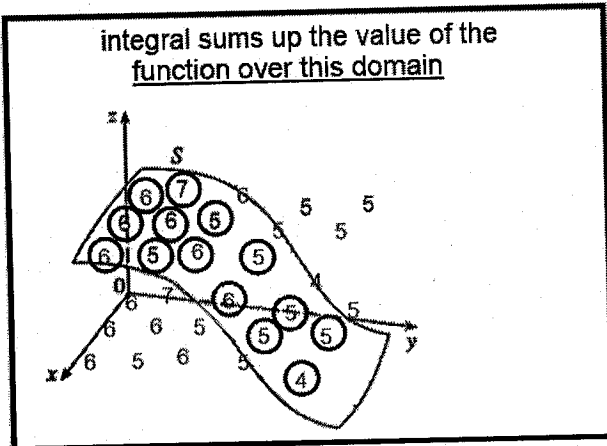


The field lines are parallel to the surface (zero component in the normal direction of the surface) - no field moving through the surface.

Surface with minimum flux

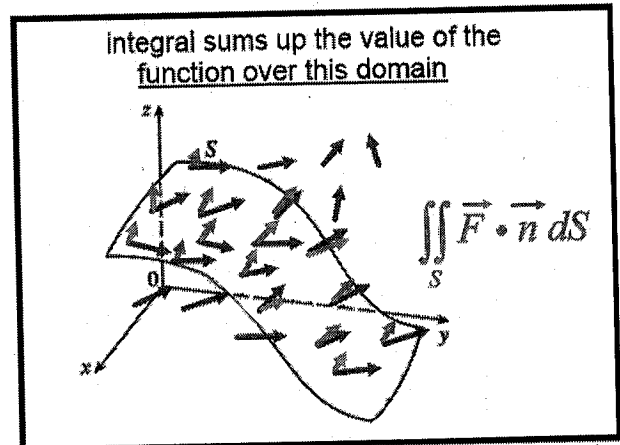
Summary (so far)

Surface integral of the scalar function:



$$\iint_S f(x, y, z) dS = \iint_D f(\vec{r}(u, v)) |\vec{r}_u \times \vec{r}_v| dA$$

Surface integral of the vector function:



$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \vec{n} dS \quad \text{where } \vec{n} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$$

- or -

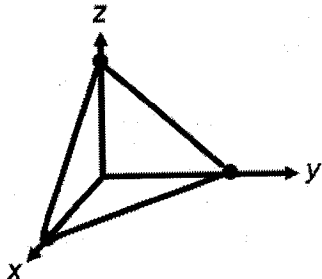
$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot (\vec{r}_u \times \vec{r}_v) dA$$

Example

#7) Evaluate the surface integral $\int_S yz \, dS$

$$\iint_S f(x,y,z) \, dS = \iint_D f(\vec{r}(u,v)) |\vec{r}_u \times \vec{r}_v| \, dA$$

where S is the part of the plane $x + y + z = 1$ that lies in the first octant.



Parametrize the surface using x, y :

$$\vec{r}(x,y,z) = \langle x, y, 1-x-y \rangle$$

Derivatives on the tangent plane:

$$\vec{r}_x = \langle 1, 0, -1 \rangle \quad \vec{r}_y = \langle 0, 1, -1 \rangle$$

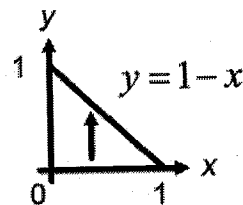
The surface normal vector:

Magnitude:

$$\vec{r}_x \times \vec{r}_y = \begin{vmatrix} i & j & k \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} = \langle 1, 1, 1 \rangle \quad |\vec{r}_x \times \vec{r}_y| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$$

The surface integral:

$$\iint_D f(\vec{r}(x,y)) |\vec{r}_x \times \vec{r}_y| \, dA = \int_0^{1-x} \int_0^{1-x-y} yz \sqrt{3} \, dy \, dx$$



$$= \int_0^{1-x} \int_0^{1-x-y} y(1-x-y) \sqrt{3} \, dy \, dx$$

$$= \sqrt{3} \int_0^{1-x} \int_0^{1-x-y} (y - xy - y^2) \, dy \, dx$$

$$= \sqrt{3} \int_0^{1-x} \left[\frac{1}{2} y^2 - \frac{1}{2} xy^2 - \frac{1}{3} y^3 \right]_0^{1-x-y} \, dx$$

$$= \sqrt{3} \int_0^1 \left[\frac{1}{2} (1-x)^2 - \frac{1}{2} x (1-x)^2 - \frac{1}{3} (1-x)^3 - 0 + 0 + 0 \right] \, dx$$

$$= \sqrt{3} \int_0^1 \left(-\frac{1}{6} x^3 + \frac{1}{2} x^2 - \frac{1}{2} x + \frac{1}{6} \right) \, dx$$

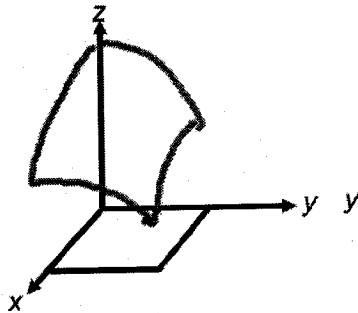
$$= \sqrt{3} \left[-\frac{1}{24} x^4 + \frac{1}{6} x^3 - \frac{1}{4} x^2 + \frac{1}{6} x \right]_0^1 = \boxed{\frac{\sqrt{3}}{24}}$$

Example

#19) Evaluate the surface integral for the given vector field and the oriented surface

$F(x, y, z) = \langle xy, yz, zx \rangle$ S is the part of the paraboloid $z = 4 - x^2 - y^2$ that lies above the square $0 \leq x \leq 1$, $0 \leq y \leq 1$ and has upward orientation.

$$\iint_S F \cdot dS = \iint_S F \cdot (r_u \times r_v) dA$$



Parametrize the surface using x, y :

$$\vec{r}(x, y, z) = \langle x, y, 4 - x^2 - y^2 \rangle$$

Derivatives on the tangent plane:

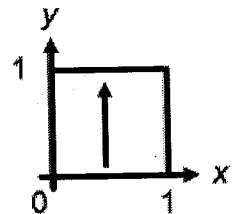
$$\vec{r}_x = \langle 1, 0, -2x \rangle \quad \vec{r}_y = \langle 0, 1, -2y \rangle$$

The surface normal vector:

$$\vec{r}_x \times \vec{r}_y = \begin{vmatrix} i & j & k \\ 1 & 0 & -2x \\ 0 & 1 & -2y \end{vmatrix} = \langle 2x, 2y, 1 \rangle$$

The surface integral:

$$\begin{aligned} \iint_S F \cdot dS &= \iint_S F \cdot (r_u \times r_v) dA \\ &= \iint_S \langle xy, yz, zx \rangle \cdot \langle 2x, 2y, 1 \rangle dA \\ &= \iint_S (2x^2y + 2y^2z + zx) dA \\ &= \int_0^1 \int_0^1 (2x^2y + 2y^2(4 - x^2 - y^2) + (4 - x^2 - y^2)x) dy dx \\ &= \int_0^1 \int_0^1 (2x^2y + 8y^2 - 2x^2y^2 - 2y^4 + 4x - x^3 - xy^2) dy dx \\ &= \int_0^1 \left[x^2y^2 + \frac{8}{3}y^3 - \frac{2}{3}x^2y^3 - \frac{2}{5}y^5 + 4xy - x^3y - \frac{1}{3}xy^3 \right]_0^1 dx \\ &= \int_0^1 \left(x^2 + \frac{8}{3} - \frac{2}{3}x^2 - \frac{2}{5} + 4x - x^3 - \frac{1}{3}x \right) dx \\ &= \left[\frac{1}{3}x^3 + \frac{8}{3}x - \frac{2}{9}x^3 - \frac{2}{5}x + 2x^2 - \frac{1}{4}x^4 - \frac{1}{6}x^2 \right]_0^1 = \frac{713}{180} \end{aligned}$$



Simpler forms if you can parametrize as $\langle x, y, g(x,y) \rangle$

These first two examples are actually somewhat special: it was possible to parametrize the surface by expressing one variable in terms of the other two, for example $z = g(x,y)$. When this is true, you can use a simpler version of the surface integral formulas (shown here for the case where x and y are the parameters, but could also work for yz , or xz as parameters).

Surface integral of the scalar function:

$$\iint_S f(x,y,z) dS = \iint_D f(\vec{r}(u,v)) |\vec{r}_u \times \vec{r}_v| dA$$



simplifies to

$$\text{if } r(x,y) = \langle x, y, g(x,y) \rangle$$

then

$$\iint_S f(x,y,z) dS = \iint_D f(x,y,g(x,y)) \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1} dA$$

Surface integral of the vector function:

$$\iint_S F \cdot dS = \iint_S \vec{F} \cdot \vec{n} dS \text{ where } n = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$$

- or -

$$\iint_S F \cdot dS = \iint_S \vec{F} \cdot (\vec{r}_u \times \vec{r}_v) dA$$



$$\text{if } r(x,y) = \langle x, y, g(x,y) \rangle$$

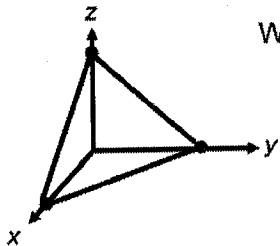
$$\text{and } F = \langle P, Q, R \rangle$$

$$\text{then } \iint_S F \cdot dS = \iint_S \left(-P \frac{\partial g}{\partial x} - Q \frac{\partial g}{\partial y} + R \right) dA$$

Example (rework using the simpler formula)

#7) Evaluate the surface integral $\int_S yz \, dS$

where S is the part of the plane $x + y + z = 1$ that lies in the first octant.

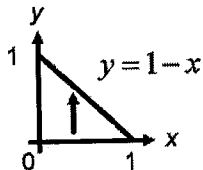


We would have the same parametrization...

$$\vec{r}(x, y, z) = \langle x, y, 1 - x - y \rangle$$

...but because we just used x, y for the parameters, we can use this formula for the surface integral:

$$\iint_S f(x, y, z) \, dS = \iint_D f(x, y, g(x, y)) \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1} \, dA$$



$$\frac{\partial z}{\partial x} = -1, \quad \frac{\partial z}{\partial y} = -1$$

$$\iint_S f(x, y, z) \, dS = \iint_D y(1-x-y) \sqrt{(-1)^2 + (-1)^2 + 1} \, dA$$

(a little simpler because the derivatives aren't vectors and no cross product)

$$= \sqrt{3} \int_0^1 \int_0^{1-x} (y - xy - y^2) \, dy \, dx$$

$$= \sqrt{3} \int_0^1 \left[\frac{1}{2}y^2 - \frac{1}{2}xy^2 - \frac{1}{3}y^3 \right]_0^{1-x} \, dx$$

$$= \sqrt{3} \int_0^1 \left(\frac{1}{2}(1-x)^2 - \frac{1}{2}x(1-x)^2 - \frac{1}{3}(1-x)^3 \right) \, dx$$

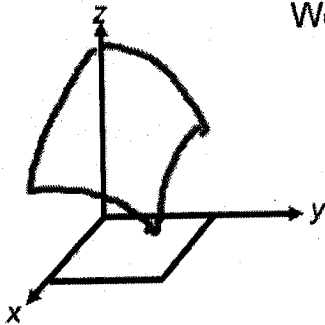
$$= \sqrt{3} \int_0^1 \left(-\frac{1}{6}x^3 + \frac{1}{2}x^2 - \frac{1}{2}x + \frac{1}{6} \right) \, dx$$

$$= \sqrt{3} \left[-\frac{1}{24}x^4 + \frac{1}{6}x^3 - \frac{1}{4}x^2 + \frac{1}{6}x \right]_0^1 = \boxed{\frac{\sqrt{3}}{24}}$$

Example

#19) Evaluate the surface integral for the given vector field and the oriented surface

$F(x, y, z) = \langle xy, yz, zx \rangle$ S is the part of the paraboloid $z = 4 - x^2 - y^2$ that lies above the square $0 \leq x \leq 1$, $0 \leq y \leq 1$ and has upward orientation.

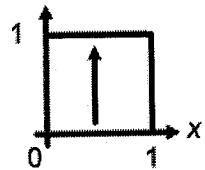


We would have the same parametrization...

$$\vec{r}(x, y, z) = \langle x, y, 4 - x^2 - y^2 \rangle$$

...but because we just used x, y for the parameters, we can use this formula for the surface integral:

$$\iint_S F \cdot dS = \iint_S \left(-P \frac{\partial g}{\partial x} - Q \frac{\partial g}{\partial y} + R \right) dA$$



$$P = xy, \quad Q = yz, \quad R = zx \quad g(x, y) = 4 - x^2 - y^2$$

$$\frac{\partial g}{\partial x} = -2x, \quad \frac{\partial g}{\partial y} = -2y$$

$$\iint_S F \cdot dS = \iint_S \left(-P \frac{\partial g}{\partial x} - Q \frac{\partial g}{\partial y} + R \right) dA$$

$$= \iint_S \left(-(xy)(-2x) - (yz)(-2y) + zx \right) dA$$

$$= \int_0^1 \int_0^1 \left(-(xy)(-2x) - (y(4 - x^2 - y^2))(-2y) + (4 - x^2 - y^2)x \right) dy dx$$

$$= \int_0^1 \int_0^1 \left(2x^2y + 8y^2 - 2x^2y^2 + 2y^4 + 4x - x^3 - xy^2 \right) dy dx$$

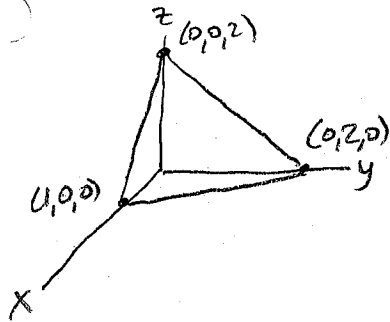
$$= \int_0^1 \left[x^2y^2 + \frac{8}{3}y^3 - \frac{2}{3}x^2y^3 + \frac{2}{5}y^5 + 4xy - x^3y - \frac{1}{3}xy^3 \right]_0^1 dx$$

$$= \int_0^1 \left[x^2 + \frac{8}{3} - \frac{2}{3}x^2 + \frac{2}{5} + 4x - x^3 - \frac{1}{3}x \right] dx$$

$$= \left[\frac{1}{3}x^3 + \frac{8}{3}x - \frac{2}{9}x^3 + \frac{2}{5}x + 2x^2 - \frac{1}{4}x^4 - \frac{1}{6}x^2 \right]_0^1 = \boxed{\frac{713}{180}}$$

Example (setup only, check answers with key and technology)

#6) Evaluate the surface integral $\iint_S xy \, dS$ S is the triangular region with vertices (1,0,0), (0,2,0), and (0,0,2)



First, need equation for the plane:

2 vectors in plane: $\langle 1, 0, -2 \rangle$ $\langle 0, 2, -2 \rangle$

$$\vec{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -2 \\ 0 & 2 & -2 \end{vmatrix} = \langle 4, 2, 2 \rangle = 2 \langle 2, 1, 1 \rangle$$

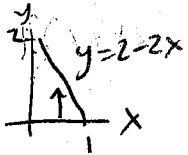
$n = \langle 2, 1, 1 \rangle$

$\vec{r}_0 = \langle 1, 0, 0 \rangle$ (pick a point)

plane: $2x + y + z = \langle 2, 1, 1 \rangle \cdot \langle 1, 0, 0 \rangle = 2$ $2x + y + z = 2$

so $z = 2 - 2x - y$

parametrization: $\vec{r}(x,y) = \langle x, y, 2-2x-y \rangle$



$$\iint_S xy \, dS = \iint_D xy \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1} \, dA$$

$\frac{\partial z}{\partial x} = -2$ $\frac{\partial z}{\partial y} = -1$

$$= \int_0^1 \int_0^{2-2x} xy \sqrt{(-2)^2 + (-1)^2 + 1} \, dy \, dx$$

$$= \sqrt{6} \int_0^1 \int_0^{2-2x} xy \, dy \, dx \quad \text{setup complete}$$

$$= \left[\frac{\sqrt{6}}{6} \right] \quad (\text{use technology})$$

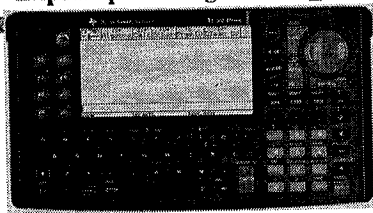
or if you want: $\int_0^1 \int_0^{2-2x} xy \, dy = \left[\frac{1}{2} xy^2 \right]_0^{2-2x} = \frac{1}{2} x(2-2x)^2 = 2(x - 2x^2 + x^3)$

$$\int_0^1 2\sqrt{6} (x - 2x^2 + x^3) \, dx = 2\sqrt{6} \left[\frac{1}{2} x^2 - \frac{2}{3} x^3 + \frac{1}{4} x^4 \right]_0^1$$

$$= 2\sqrt{6} \left(\frac{1}{12} \right) = \left[\frac{\sqrt{6}}{6} \right]$$

Online tech for integrals (Ti-92 emulator)

https://tiplanet.org/emu68k_fork/



- 'F3' to access calc menu
- '2' to integrate
- 'F3' and '2' again for 2nd integral
- enter inside integral using this form: integrand, (letter of variable), (lower limit), (upper limit) then close parenthesis.
- then enter a comma, and repeat for outer integral.

Example: $\int_0^1 \int_0^{2-2x} \sqrt{6} xy \, dy \, dx$

Here is the calculator text string:

`1/12*sqrt(6)`

the inner integral

Ti-89 is the same except you have to select 'home' first

Example (setup only, check answers with key and technology)

#9) Evaluate the surface integral $\iint_S yz \, dS$ S is the surface with parametric equations:

$$x = u^2, \quad y = u \sin v, \quad z = u \cos v$$

$$0 \leq u \leq 1, \quad 0 \leq v \leq \frac{\pi}{2}$$

$$\vec{F}(u,v) = \langle u^2, u \sin v, u \cos v \rangle$$

$$\text{use } \iint_D f(\vec{r}) |\vec{r}_u \times \vec{r}_v| \, dA$$

$$\vec{r}_u = \langle 2u, \sin v, \cos v \rangle \quad \vec{r}_v = \langle 0, u \cos v, -u \sin v \rangle$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \uparrow & \uparrow & \uparrow \\ 2u & \sin v & \cos v \\ 0 & u \cos v & -u \sin v \end{vmatrix} = \langle -u \sin^2 v - u \cos^2 v, 2u^2 \sin v, 2u^2 \cos v \rangle$$

$$= \langle -u(\sin^2 v + \cos^2 v), 2u^2 \sin v, 2u^2 \cos v \rangle$$

$$= \langle -u, 2u^2 \sin v, 2u^2 \cos v \rangle$$

$$|\vec{r}_u \times \vec{r}_v| = \sqrt{(-u)^2 + (2u^2 \sin v)^2 + (2u^2 \cos v)^2} = \sqrt{u^2 + 4u^4 \sin^2 v + 4u^4 \cos^2 v}$$

$$= \sqrt{u^2 + 4u^4(\sin^2 v + \cos^2 v)} = \sqrt{u^2 + 4u^4} = \sqrt{u^2} \sqrt{1 + 4u^2} = u \sqrt{1 + 4u^2}$$

$$\int_0^{\frac{\pi}{2}} \int_0^1 (u \sin v)(u \cos v) u \sqrt{1 + 4u^2} \, du \, dv \quad \text{setup complete}$$

use technology to check

or by hand:

$$\int_0^{\frac{\pi}{2}} \sin v \cos v \, dv \int_0^1 u^3 \sqrt{1 + 4u^2} \, du$$

$$w = \cos v \\ dw = -\sin v \, dv$$

$$-\int_1^0 w \, dw$$

$$-\left[\frac{1}{2} w^2 \right]_1^0 \\ \left(\frac{1}{2} \right)$$

$$w = 1 + 4u^2 \quad 4u^2 = w - 1 \\ dv = 8u \, du \quad u^2 = \frac{1}{4}(w - 1) \\ \frac{1}{8} dw = u \, du$$

$$\int_1^5 w^{1/2} \frac{1}{4}(w - 1) \frac{1}{8} dw$$

$$\frac{1}{32} \int_1^5 (w^{3/2} - w^{1/2}) \, dw$$

$$\frac{1}{32} \left[\frac{2}{5} w^{5/2} - \frac{2}{3} w^{3/2} \right]_1^5$$

$$\frac{1}{32} \left[\frac{2}{5} 25\sqrt{5} - \frac{2}{5} 5\sqrt{5} - \frac{2}{3} + \frac{2}{3} \right]$$

$$\frac{1}{32} \left[\frac{30\sqrt{5}}{3} - \frac{10\sqrt{5}}{3} + \frac{4}{15} \right]$$

$$\left(\frac{1}{2} \right) \left(\frac{1}{32} \left[\frac{20\sqrt{5}}{3} + \frac{4}{15} \right] \right)$$

$$\boxed{\frac{5}{48} \sqrt{5} + \frac{1}{240}}$$

Example (setup only, check answers with key and technology)

#26) Evaluate the surface integral $\iint_S \vec{F} \cdot d\vec{S}$ for the given vector field F and the oriented

surface S . In other words, find the flux of F across S . $F(x, y, z) = \langle xy, 4x^2, yz \rangle$

S is the surface $z = xe^y$ $0 \leq x \leq 1$, $0 \leq y \leq 1$ with upward orientation.

use $\vec{r}(x, y) = \langle xy, xe^y \rangle$

$\vec{r}_x = \langle 1, 0, e^y \rangle$ $\vec{r}_y = \langle 0, 1, xe^y \rangle$

$\vec{r}_x \times \vec{r}_y = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & e^y \\ 0 & 1 & xe^y \end{vmatrix} = \langle -e^y, -xe^y, 1 \rangle$

use $\iint_S \vec{F} \cdot \vec{n} \, dA$

$\vec{F} \cdot \vec{n} = \langle xy, 4x^2, yz \rangle \cdot \langle -e^y, -xe^y, 1 \rangle$

$= -xye^y - 4x^3e^y + yz$

← substitute ze^y

$\iint_S \vec{F} \cdot d\vec{S} = \int_0^1 \int_0^1 (-xye^y - 4x^3e^y + yz) \, dy \, dx$

$= \int_0^1 \int_0^1 (-xye^y - 4x^3e^y + \underbrace{y(xe^y)}_{\text{cancels}}) \, dy \, dx$

$= \int_0^1 \int_0^1 -4x^3e^y \, dy \, dx$ set y complete

use technology

(or by hand)

(dy) $-4x^3 \int_0^1 e^y \, dy = -4x^3 [e^y]_0^1 = -4x^3 (e-1)$

(dx) $\int_0^1 -4x^3(e-1) \, dx = -(e-1) [x^4]_0^1 = -(e-1)$

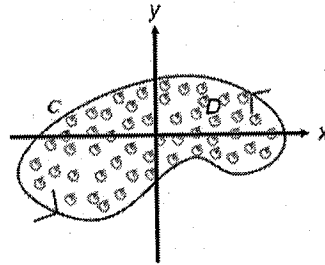
$1-e$

16.8: Stokes' Theorem

Stokes' Theorem extends Green's Theorem for higher dimensions

Green's Theorem stated that the vector line integral in 2D is the sum of the individual microscopic circulations:

$$\oint_C P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$



*DD. Planar
not on
last example*

In general, to evaluate a line integral along a path, we had to use the more general...

$$\int_C \vec{F}(x,y) dr = \int_a^b \vec{F}(r) \cdot \vec{r}' dt$$

...and evaluate along the path (sometimes, multiple sections of path)

But if we had a closed-path, then we could use Green's Theorem that allows us to integrate the expression for circulation over the area in the domain representing the bounded region. Even though this was a double-integral, it was often easier than computing the line integral along the path.

We then defined curl of a field...

...and noted that curl represent the circulation concept extended to higher dimensions.

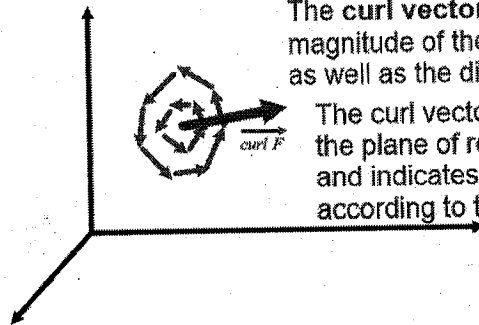
Curl
(curl - cross product)
 $curl F = \nabla \times F$

$$= \begin{vmatrix} i & j & k \\ \partial & \partial & \partial \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

$$= \left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle$$

(output is a vector at each point, a vector field)

The curl of a vector field at a point is a measure of the tendency for a vector field to rotate:



The curl vector length indicates the magnitude of the tendency to rotate, as well as the direction.

The curl vector is perpendicular to the plane of rotation at this point, and indicates direction of rotation according to the right-hand-rule.

If we put these two ideas together and use curl to represent circulation in something like Green's Theorem we get **Stokes' Theorem**:

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S curl \vec{F} \cdot \vec{n} dS = \iint_S curl \vec{F} \cdot d\vec{S}$$

Since $\int_C \vec{F} \cdot d\vec{r} = \int_C \vec{F} \cdot \vec{T} ds$

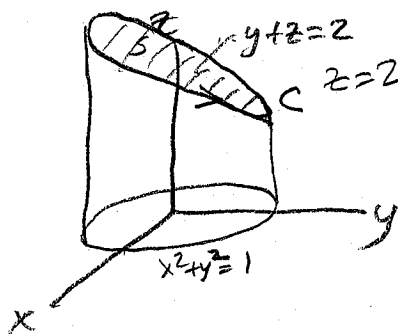
Stokes' Theorem says that the line integral around the boundary curve of S of the tangential component of \vec{F} is equal to the surface integral of the normal component of the curl of \vec{F} .

As with Green's Theorem, even though this requires evaluating a double-integral, this is often easier than directly evaluating the single integral along the path.

An example

Here is a first example for us to try: Evaluate $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = \langle -y^2, x, z^2 \rangle$ and C is the curve of intersection of the plane $z = 2 - y$ and the cylinder $x^2 + y^2 = 1$ with C oriented counter-clockwise when viewed from above.

by Stokes' Theorem: $\int_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl } \vec{F} \cdot \vec{n} \, dS$



$$\text{curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^2 & x & z^2 \end{vmatrix} = \langle 0, 0, 1+2y \rangle$$

for the surface, S , we can use the elliptical disk surface

parameterize: $\vec{r} = \langle x, y, 2-y \rangle$

we need a normal $\vec{n} = \vec{r}_x \times \vec{r}_y$

$$\vec{r}_x = \langle 1, 0, 0 \rangle \quad \vec{r}_y = \langle 0, 1, -1 \rangle$$

$$\vec{n} = \vec{r}_x \times \vec{r}_y = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{vmatrix} = \langle 0, 1, 1 \rangle$$

$$\text{so } \text{curl } \vec{F} \cdot \vec{n} = \langle 0, 0, 1+2y \rangle \cdot \langle 0, 1, 1 \rangle = 1+2y$$

$$\text{and } \int_C \vec{F} \cdot d\vec{r} = \iint_D (1+2y) \, dA = \text{for circular area in the parameter domain we can use polar}$$

$$= \int_0^{2\pi} \int_0^1 (1+2(rs)\sin\theta) r \, dr \, d\theta$$

$$x = r \cos\theta$$

$$y = r \sin\theta$$

$$= \int_0^{2\pi} \int_0^1 (r - 2r^2 \sin\theta) \, dr \, d\theta$$

$$\text{(dr)} \int_0^1 (r - 2r^2 \sin\theta) \, dr = \left[\frac{1}{2} r^2 - \frac{2}{3} r^3 \sin\theta \right]_0^1 = \frac{1}{2} - \frac{2}{3} \sin\theta$$

$$\text{(d}\theta\text{)} \int_0^{2\pi} \left(\frac{1}{2} - \frac{2}{3} \sin\theta \right) \, d\theta = \left[\frac{1}{2} \theta - \frac{1}{3} \sin^2\theta \right]_0^{2\pi}$$

$$= \left(\frac{1}{2}(2\pi) - \frac{1}{3} \sin^2(2\pi) \right) - \left(\frac{1}{2}(0) - \frac{1}{3} \sin^2(0) \right)$$

$$= \boxed{\pi}$$

What would be required to work the example directly

(This is much harder, don't use this method - but to see what this is equivalent to...)

First, we'd have to parametrize the curve with a single parameter, t . Start by finding an equation representing the curve in 3D by finding the intersection of the ellipse and cylinder:

$$y+z=2 \quad x^2+y^2=1$$

$$y=2-z \text{ substitute into } x^2+(2-z)^2=1$$

Since the 2nd term is squared, we could reverse what it inside, to get the equation of the elliptical curve in 3D:

$$x^2+(2-z)^2=1 \Rightarrow x^2+(z-2)^2=1$$

This is the expression of a circle not centered at the origin. We can use parametrization for a circle, but to handle the offset center, let's define a new variable w , then complete the parametrization:

$$w=z-2$$

$$\text{circle: } x^2+(w)^2=1$$

$$x=r \cos \theta = (1) \cos \theta = \cos \theta$$

$$w=r \sin \theta = (1) \sin \theta = \sin \theta$$

To return from w to z , we can substitute, then solve for x , y , and z to get the final parametrization of the curve:

$$x = \cos \theta$$

$$w = z - 2 = \sin \theta$$

$$z = 2 + \sin \theta$$

$$y = 2 - z = 2 - (2 + \sin \theta) = -\sin \theta$$

$$r(\theta) = \langle \cos \theta, -\sin \theta, 2 + \sin \theta \rangle \quad 0 \leq \theta \leq 2\pi$$

Now we would compute the line integral directly using $\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(r) \cdot \vec{r}' dt$ (our t is θ)

$$r(\theta) = \langle \cos \theta, -\sin \theta, 2 + \sin \theta \rangle$$

$$r' = \langle -\sin \theta, -\cos \theta, \cos \theta \rangle$$

$$F(r) = \langle -(-\sin \theta)^2, (\cos \theta), (2 + \sin \theta)^2 \rangle$$

$$= \langle -\sin^2 \theta, \cos \theta, 4 + 2\sin \theta + \sin^2 \theta \rangle$$

$$F \cdot r' = \langle -\sin^2 \theta, \cos \theta, 4 + 2\sin \theta + \sin^2 \theta \rangle \cdot \langle -\sin \theta, -\cos \theta, \cos \theta \rangle$$

$$= \sin^3 \theta - \cos^2 \theta + 4\cos \theta + 2\sin \theta \cos \theta + \sin^2 \theta \cos \theta$$

$$\int_C F \cdot dr = \int_0^{2\pi} (\sin^3 \theta - \cos^2 \theta + 4\cos \theta + 2\sin \theta \cos \theta + \sin^2 \theta \cos \theta) d\theta$$

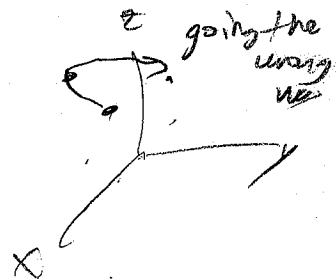
$$= \int_0^{2\pi} \sin^3 \theta d\theta - \int_0^{2\pi} \cos^2 \theta d\theta + \int_0^{2\pi} 4\cos \theta d\theta + \int_0^{2\pi} 2\sin \theta \cos \theta d\theta + \int_0^{2\pi} \sin^2 \theta \cos \theta d\theta$$

$$= \left[-\frac{1}{3}(2 + \sin^2 \theta) \cos \theta \right]_0^{2\pi} - \left[\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta \right]_0^{2\pi} + 4[\sin \theta]_0^{2\pi} + 2 \int_0^0 u du + \int_0^0 u^2 du$$

$$= 0 - \pi + 0 + 0 + 0 = -\pi$$

$$r = \langle \cos \theta, -\sin \theta, 2 + \sin \theta \rangle$$

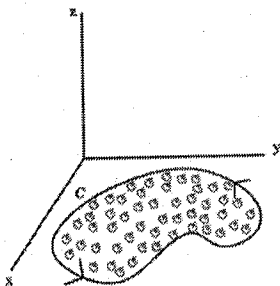
θ	r
0	$\langle 1, 0, 2 \rangle$
$\frac{\pi}{2}$	$\langle 0, -1, 3 \rangle$
π	$\langle -1, 0, 2 \rangle$



Computing a line integral using a surface

Both Green's Theorem and Stokes' Theorem allow us to compute a line integral around a closed path by instead integrating 'circulation' over a surface:

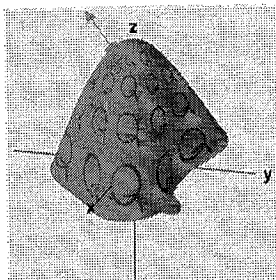
Green's Theorem



$$\oint_C P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

Green's Theorem stays in the x-y plane

Stokes' Theorem



$$\int_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl } \vec{F} \cdot \vec{n} dS = \iint_S \text{curl } \vec{F} \cdot d\vec{S}$$

Stokes' Theorem allows the path and surface to be 3-dimensional

...but both allow us to compute a 'path integral' using a surface.

Computing a surface integral of curl using a line integral or normal vector

Stokes' Theorem can also be used in reverse...to compute the surface integral of curl F by instead computing a line integral.

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl } \vec{F} \cdot \vec{n} dS = \iint_S \text{curl } \vec{F} \cdot d\vec{S}$$

easiest

When we do this, remember that:

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) dt$$

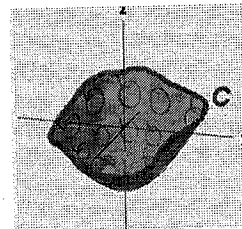
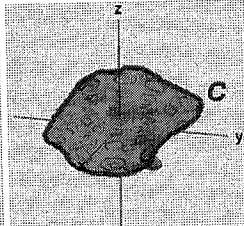
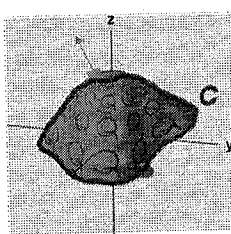
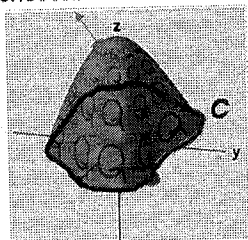
In some cases, the line integral would be more difficult, but for the surface we can easily compute a normal vector, n. Then we would use this form to compute:

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl } \vec{F} \cdot \vec{n} dS = \iint_S \text{curl } \vec{F} \cdot d\vec{S}$$

easiest

Line integral value is independent of which surface is used

Since there is a single correct value for the line integral (path integral), that means that integrating over any surface which has this path as the boundary will yield the same result - independent of which surface is chosen:



$$\int_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl } \vec{F} \cdot \vec{n} dS = \iint_S \text{curl } \vec{F} \cdot d\vec{S}$$

Evaluating over any of these surfaces is a valid way to find the line integral value because they all share the same curve, C.

Sometimes, we can take advantage of this to pick a particular surface which will be easiest to express or integrate. For example, if the curve happens to be in a plane, we can just pick the region enclosed in that plane to integrate over.

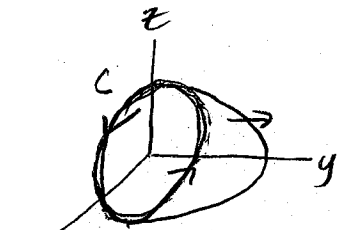
Example

#15) Verify that Stokes' Theorem is true for $F = \langle y, z, x \rangle$

if S is the hemisphere $x^2 + y^2 + z^2 = 1, y \geq 0$

oriented in the direction of the positive y -axis.

- Calculate the line integral directly over the boundary curve.



x note: polar defines θ this way

(at

so this curve is the negative of the result

$$\oint_C \vec{F} \cdot d\vec{r}$$

$$C: x^2 + z^2 = 1, y = 0$$

use polar!

$$x = r \cos \theta = \cos \theta$$

$$z = r \sin \theta = \sin \theta$$

so parametrization of curve is:

$$\vec{r} = \langle \cos \theta, 0, \sin \theta \rangle \quad 0 \leq \theta \leq 2\pi$$

$$\vec{r}' = \langle -\sin \theta, 0, \cos \theta \rangle$$

$$\vec{F}(\vec{r}) = \langle 0, \sin \theta, \cos \theta \rangle$$

$$\text{so } \vec{F} \cdot \vec{r}' = \langle 0, \sin \theta, \cos \theta \rangle \cdot \langle -\sin \theta, 0, \cos \theta \rangle = \cos^2 \theta$$

$$\text{then } \oint_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \vec{F}(\vec{r}) \cdot \vec{r}' d\theta = \int_0^{2\pi} \cos^2 \theta d\theta$$

(by table)

$$= \left[\frac{1}{2} \theta + \frac{1}{4} \sin 2\theta \right]_0^{2\pi}$$

$$= \frac{1}{2}(2\pi) + \frac{1}{4} \sin 4\pi - \frac{1}{2}(0) - \frac{1}{4} \sin 0$$

$$= \pi$$

but because curve direction that matches our orientation (right hand rule) is opposite of θ definition,

the actual answer is $\boxed{-\pi}$

Example

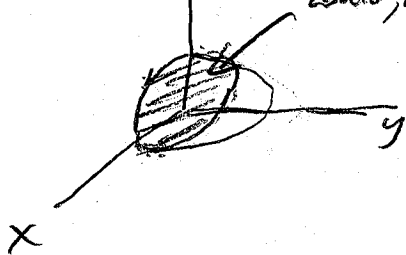
#15) Verify that Stokes' Theorem is true for $F = \langle y, z, x \rangle$

if S is the hemisphere $x^2 + y^2 + z^2 = 1, y \geq 0$

oriented in the direction of the positive y -axis.

Then I want us to also do the surface integral over another, easier, surface to see that it matches.

z



could, instead, use this surface since it has the same boundary curve,

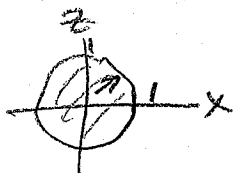
$\text{curl } \vec{F}$ is still $\langle -1, -1, -1 \rangle$

but now, very easy to find normal \vec{n}

it is just $\vec{n} = \langle 0, 0, 1 \rangle$ (direction \rightarrow match positive y)

$$\text{so } \text{curl } \vec{F} \cdot \vec{n} = \langle -1, -1, -1 \rangle \cdot \langle 0, 0, 1 \rangle = 0 + 0 - 1 = -1$$

and we can evaluate surface in x - z plane: (with polar)



$$\iint_S \text{curl } \vec{F} \cdot \vec{n} \, ds = \int_0^{2\pi} \int_0^1 (-1) r \, dr \, d\theta$$

$$\textcircled{1} - \int_0^1 r \, dr = -\left[\frac{1}{2}r^2\right]_0^1 = -\frac{1}{2}$$

$$\textcircled{2} \int_0^{2\pi} \left(-\frac{1}{2}\right) d\theta = -\frac{1}{2}(2\pi) = \boxed{-\pi}$$

much easier than using the hemisphere surface as defined in the problem.

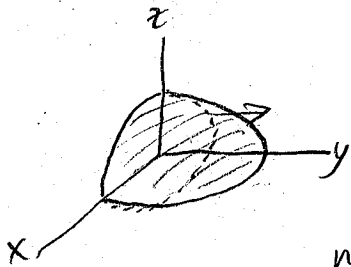
Example

#15) Verify that Stokes' Theorem is true for $F = \langle y, z, x \rangle$

if S is the hemisphere $x^2 + y^2 + z^2 = 1, y \geq 0$

oriented in the direction of the positive y -axis.

- Calculate the surface integral of curl over the surface as defined.



$$\iint_S \text{curl } \vec{F} \cdot d\vec{S} = \iint_S \text{curl } \vec{F} \cdot \vec{n} \, ds$$

$$\text{curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix} = \langle -1, -1, -1 \rangle$$

need to parametrize this surface... let's use spherical w/ $r=1$

$$\vec{r} = \langle \sin\phi \cos\theta, \sin\phi \sin\theta, \cos\phi \rangle$$

$$\text{then } \vec{r}_\theta = \langle \cos\phi \cos\theta, \cos\phi \sin\theta, -\sin\phi \rangle$$

$$\vec{r}_\phi = \langle -\sin\phi \sin\theta, \sin\phi \cos\theta, 0 \rangle$$

$$\left(\begin{array}{l} \frac{\pi}{2} \leq \phi \leq \pi \\ 0 \leq \theta \leq 2\pi \end{array} \right)$$

$$\vec{n} = \vec{r}_\theta \times \vec{r}_\phi = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos\phi \cos\theta & \cos\phi \sin\theta & -\sin\phi \\ -\sin\phi \sin\theta & \sin\phi \cos\theta & 0 \end{vmatrix}$$

$$= \langle \sin^2\phi \cos\theta, \sin^2\phi \sin\theta, \cos^2\theta \sin\phi \cos\phi + \sin^2\theta \sin\phi \cos\phi \rangle$$

$$= \langle \sin^2\phi \cos\theta, \sin^2\phi \sin\theta, \sin\phi \cos\phi \rangle$$

$$\text{then } \text{curl } \vec{F} \cdot \vec{n} = \langle -1, -1, -1 \rangle \cdot \langle \sin^2\phi \cos\theta, \sin^2\phi \sin\theta, \sin\phi \cos\phi \rangle$$

$$= -\sin^2\phi \cos\theta - \sin^2\phi \sin\theta - \sin\phi \cos\phi$$

$$\text{so } \iint_S \text{curl } \vec{F} \cdot \vec{n} \, ds = \int_0^{2\pi} \int_{\frac{\pi}{2}}^{\pi} (-\sin^2\phi \cos\theta - \sin^2\phi \sin\theta - \sin\phi \cos\phi) \, d\phi \, d\theta$$

$$\int_{\frac{\pi}{2}}^{\pi} \sin\phi \cos\phi \, d\phi$$

$$u = \sin\phi$$

$$du = \cos\phi \, d\phi$$

$$\int_{-1}^0 u \, du$$

do first

$$= -\cos\theta \left[\frac{1}{2}\phi - \frac{1}{4}\sin 2\phi \right]_{\frac{\pi}{2}}^{\pi} - \sin\theta \left[\frac{1}{2}\phi - \frac{1}{4}\sin 2\phi \right]_{\frac{\pi}{2}}^{\pi} - \left[\frac{1}{2}u \right]_{\frac{\pi}{2}}^{\pi}$$

$$= -\cos\theta \left[\frac{1}{2}(\pi) - \frac{1}{4}\sin(\pi) \right] - \frac{1}{2}(-\frac{\pi}{2}) + \frac{1}{4}\sin(\pi)$$

$$= -\cos\theta \left(\frac{\pi}{2} \right)$$

$$- \sin\theta \left(\frac{\pi}{2} \right) - 0$$

$$= \frac{\pi}{2} [\cos\theta - \sin\theta]$$

then do

$$\int_0^{2\pi} (\cos\theta - \sin\theta) \, d\theta = -\frac{\pi}{2} [\sin\theta + \cos\theta]_0^{2\pi}$$

$$= -\frac{\pi}{2} [\sin 2\pi + \cos 2\pi - \sin 0 + \cos 0] = -\frac{\pi}{2} (2)$$

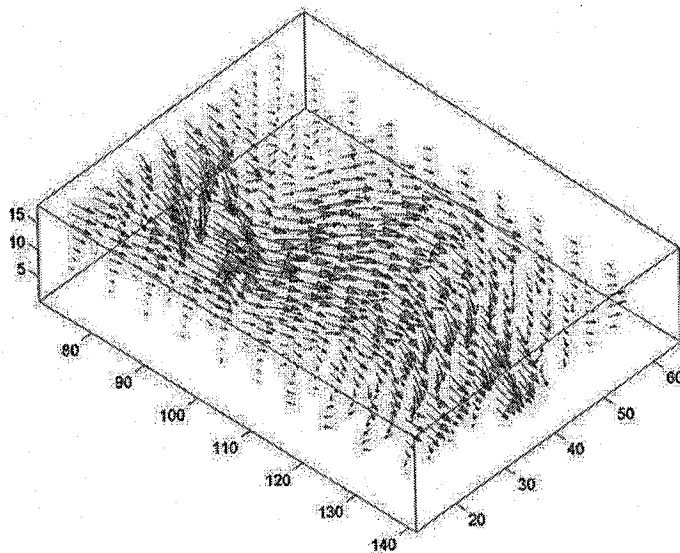
$$= \boxed{-\pi}$$

verifying Stokes' Theorem

16.9: The Divergence Theorem

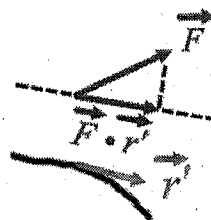
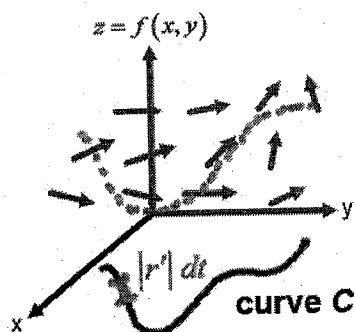
Let's review vector calculus...

In 16.1, we defined **vector fields** - the idea that at any point in the domain we could define a vector with magnitude and direction. Easiest to image as a flow of fluid and the vector being a force imparted by the fluid at the point in a particular direction.



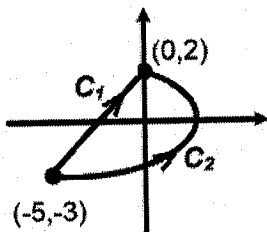
In 16.2, we defined **line (path) integrals** - that you could move along a path and the line integral would sum up the contributions of the little 'forces' from the vector field along the direction of the path.

Line Integral along curve C



$$\int_C \vec{F}(x, y) dr = \int_a^b \vec{F}(r) \cdot \vec{r}' dt$$

In 16.3, we learned that the value you get from a line integral generally depends upon the path taken, but for special fields, called conservative vector fields, the path is independent and depends only upon the starting and ending points.



For $\vec{F} = \langle P, Q \rangle$, the vector field is conservative if

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

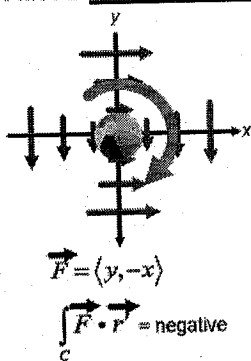
For conservative vector fields, there exists a scalar function $f(x, y)$ called the potential function such that:

$$\nabla f = \langle f_x, f_y \rangle = \vec{F}$$

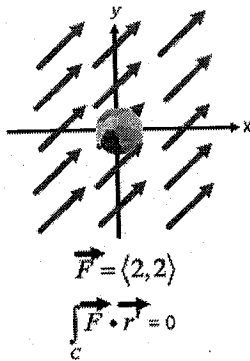
Then, by the Fundamental Theorem of Line Integrals:

$$\int_C \vec{F} \cdot d\vec{r} = f(\text{endpoint}) - f(\text{startpoint})$$

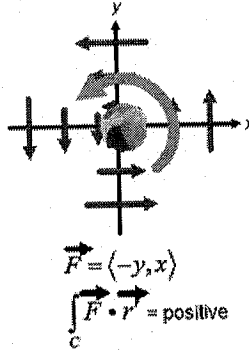
In 16.4, we defined **circulation** as the tendency of a vector field to cause rotation.



...the ball would:
 rotate clockwise
 "negative circulation"

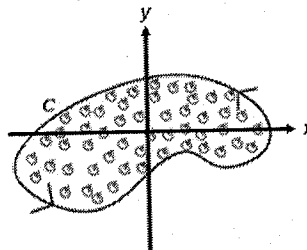


not rotate
 "zero circulation"



rotate counter-clockwise
 "positive circulation"

...and said that the a line integral around a closed path is like a 'macrocirculation' which is equal to the sum of all the 'microcirculations' enclosed by the path, which produced **Green's Theorem**



$$\oint_C P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

This gives us a choice: evaluate a line integral (often over multiple segments) or integrate the circulation over the 2D area enclosed (often easier).

In 16.5, we defined **divergence** and **curl**:

Divergence
(divergence - dot product)

$$\text{div } F = \nabla \cdot \vec{F}$$

$$= \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle \cdot \langle P, Q, R \rangle$$

$$= \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

(output is a scalar at each point, a scalar field)

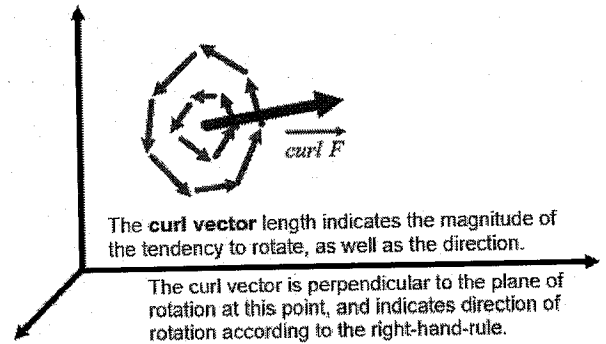
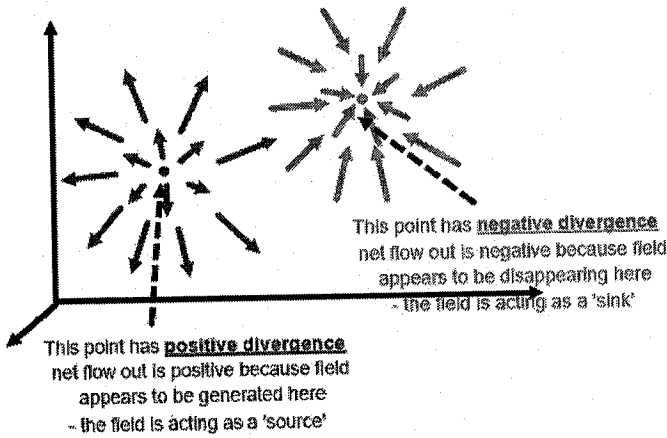
Curl
(curl - cross product)

$$\text{curl } F = \nabla \times \vec{F}$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

$$= \left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle$$

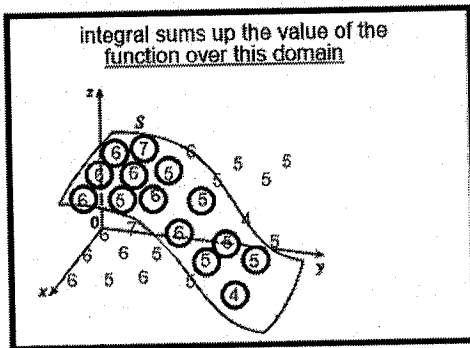
(output is a vector at each point, a vector field)



In 16.6, we defined **parametric surfaces**, the idea that just as we can parametrize curves using a parameter t to traverse the curve, we can use two parameters u, v to traverse a 2D surface.

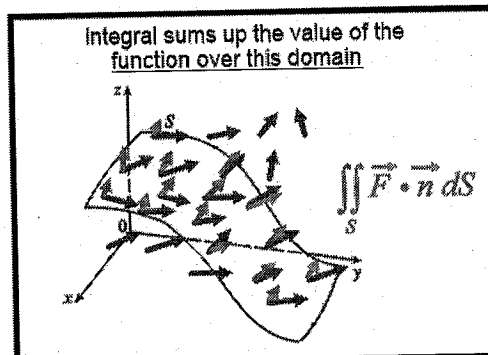
In 16.7, we defined **surface integrals**, which use integrals to sum something up over the entire 2D surface (usually existing in a 3D space). We did this for both scalar functions and vector functions:

Surface integral of the scalar function:



$$\iint_S f(x, y, z) dS = \iint_D f(\vec{r}(u, v)) |\vec{r}_u \times \vec{r}_v| dA$$

Surface integral of the vector function:



$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \vec{n} dS \quad \text{where } \vec{n} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$$

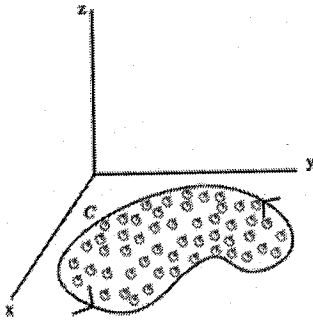
- or -

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot (\vec{r}_u \times \vec{r}_v) dA$$

In the vector field case, this is called a **flux integral** which represents the sum of the 'flow' of the amount of the field which is normal to the surface, through the surface, summed over the entire surface.

In 16.8, we saw that Stokes' Theorem generalizes Green's Theorem to higher dimensions:

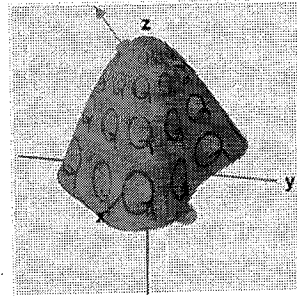
Green's Theorem



$$\oint_C P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

Green's Theorem stays in the x-y plane

Stokes' Theorem



$$\int_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl } \vec{F} \cdot \vec{n} dS = \iint_S \text{curl } \vec{F} \cdot d\vec{S}$$

Stokes' Theorem allows the path and surface to be 3-dimensional

But these both give us a choice:

- 1) evaluate a line integral (often over multiple segments)
 - 2) evaluate a 2D surface integral of the circulation (defined by the curl of F) over the 2D area enclosed
- ...whichever is easier.

16.9 - The Divergence Theorem

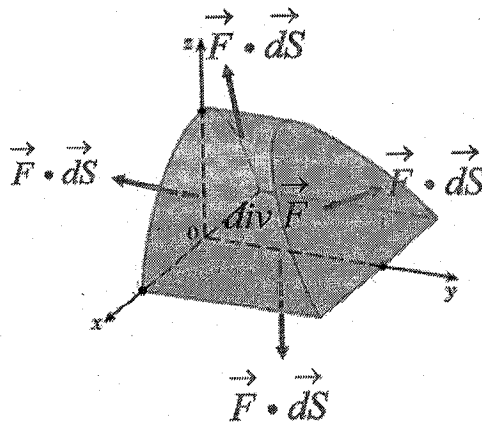
For our last CalcIII topic, we define the Divergence Theorem (aka Gauss' Theorem) which

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \text{div } \vec{F} dV$$

The sum of fluxes (outward defined as positive) across all surfaces of a closed surface

equals

The divergence of the vector field summed over the enclosed volume



For our last CalcIII topic, we define the **Divergence Theorem (aka Gauss' Theorem)** which

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \operatorname{div} \vec{F} dV$$

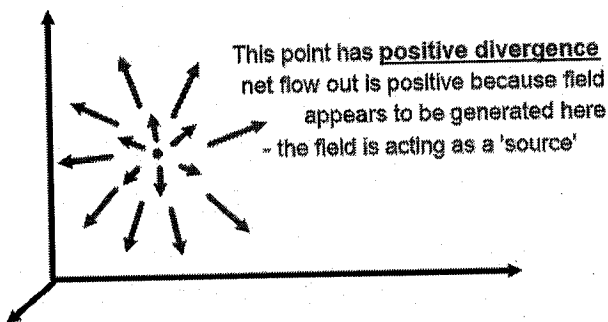
Similar to Green's and Stokes' Theorems, the Divergence Theorem gives us a choice...

- 1) evaluate surface integrals for all the surfaces of a closed surface
- 2) evaluate a 3D volume integral of the divergence of F

...whichever is easier.

Geometric interpretation of the Divergence Theorem

The divergence of a vector field at a point is a measure of the field's net 'source' or 'sink' flow at that point:



$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \operatorname{div} \vec{F} dV$$

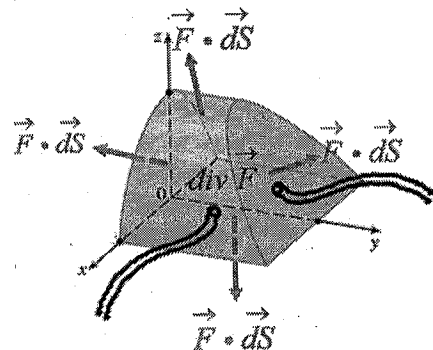
The sum of fluxes (outward defined as positive) across all surfaces of a closed surface equals

The divergence of the vector field summed over the enclosed volume

...means the sum of all outward 'source' flows from the points inside the volume, equals the total outward flow of field across the surfaces leaving the volume.

Imagine is the volume was a tank which could hold a liquid, but which allowed some liquid to escape through the porous sides of the tank.

If there were hoses that allowed fluid to enter the tank (sources) then the total amount of fluid expanding in the tank from the sources (the divergence) summed over the volume must equal the total amount of liquid that leaves the tank through the surfaces.

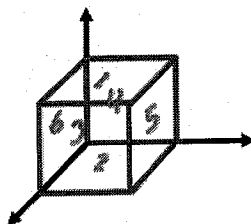


Ex #1. Verify that the Divergence Theorem is true for $\vec{F} = \langle 3x, xy, 2xz \rangle$
 if E is the cube bounded by the planes $x=0, x=1, y=0, y=1, z=0,$ and $z=1$.

$$\boxed{\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \operatorname{div} \vec{F} dV}$$

The difficult side $\iint_S \vec{F} \cdot d\vec{S}$

requires computing surface integrals for 6 sides:



always 'outward'

side 1: $\iint_S \vec{F} \cdot d\vec{S} = \iint_S \langle 3x, xy, 2xz \rangle \cdot \langle 0, 0, 1 \rangle dA$
 $= \int_0^1 \int_0^1 2xz dy dx = \int_0^1 \int_0^1 2x(1) dy dx$
 $= \int_0^1 2x [y]_0^1 dx = \int_0^1 2x dx$
 $= [x^2]_0^1 = \boxed{1}$

side 2: $\iint_S \vec{F} \cdot d\vec{S} = \iint_S \langle 3x, xy, 2xz \rangle \cdot \langle 0, 0, -1 \rangle dA$
 $= \int_0^1 \int_0^1 -2xz dy dx = \int_0^1 \int_0^1 -2x(0) dy dx$
 $= \int_0^1 \int_0^1 (0) dy dx = \boxed{0}$

side 3: $\iint_S \vec{F} \cdot d\vec{S} = \iint_S \langle 3x, xy, 2xz \rangle \cdot \langle 1, 0, 0 \rangle dA$
 $= \int_0^1 \int_0^1 3x dy dz = \int_0^1 \int_0^1 3(1) dy dz$
 $= \int_0^1 [3y]_0^1 dz = \int_0^1 3 dz$
 $= [3z]_0^1 = \boxed{3}$

side 4: $\iint_S \vec{F} \cdot d\vec{S} = \iint_S \langle 3x, xy, 2xz \rangle \cdot \langle -1, 0, 0 \rangle dA$
 $= \int_0^1 \int_0^1 -3x dy dz = \int_0^1 \int_0^1 -3(0) dy dz$
 $= \int_0^1 \int_0^1 0 dy dz = \boxed{0}$

side 5: $\iint_S \vec{F} \cdot d\vec{S} = \iint_S \langle 3x, xy, 2xz \rangle \cdot \langle 0, 1, 0 \rangle dA$
 $= \int_0^1 \int_0^1 xy dx dz = \int_0^1 \int_0^1 x(1) dx dz$
 $= \int_0^1 \left[\frac{1}{2} x^2 \right]_0^1 dz = \int_0^1 \frac{1}{2} dz$
 $= \left[\frac{1}{2} z \right]_0^1 = \boxed{\frac{1}{2}}$

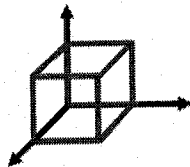
side 6: $\iint_S \vec{F} \cdot d\vec{S} = \iint_S \langle 3x, xy, 2xz \rangle \cdot \langle 0, -1, 0 \rangle dA$
 $= \int_0^1 \int_0^1 -xy dx dz = \int_0^1 \int_0^1 x(0) dx dz$
 $= \int_0^1 \int_0^1 0 dx dz = \boxed{0}$

Total outward flow = $1 + 0 + 3 + 0 + 1/2 + 0 = \boxed{9/2}$

Ex #1. Verify that the Divergence Theorem is true for $F = \langle 3x, xy, 2xz \rangle$
 if E is the cube bounded by the planes $x=0, x=1, y=0, y=1, z=0,$ and $z=1$.

$$\boxed{\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \operatorname{div} \vec{F} dV}$$

The easier side $\iiint_E \operatorname{div} \vec{F} dV$



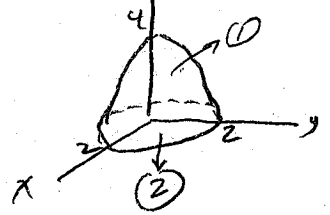
requires computing one triple integral:

$$\begin{aligned} \iiint_E \operatorname{div} F dV &= \int_0^1 \int_0^1 \int_0^1 \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle 3x, xy, 2xz \rangle dx dy dz \\ &= \int_0^1 \int_0^1 \int_0^1 \left(\frac{\partial}{\partial x}[3x] + \frac{\partial}{\partial y}[xy] + \frac{\partial}{\partial z}[2xz] \right) dx dy dz \\ &= \int_0^1 \int_0^1 \int_0^1 (3 + x + 2x) dx dy dz = \int_0^1 \int_0^1 \int_0^1 (3 + 3x) dx dy dz \\ &= \int_0^1 \int_0^1 \left[3x + \frac{3}{2}x^2 \right]_0^1 dy dz = \int_0^1 \int_0^1 \frac{9}{2} dy dz \\ &= \int_0^1 \left[\frac{9}{2}y \right]_0^1 dz = \int_0^1 \frac{9}{2} dz \\ &= \left[\frac{9}{2}z \right]_0^1 = \boxed{\frac{9}{2}} \end{aligned}$$

Ex #2. Verify that the Divergence Theorem is true for $F = \langle x^2, xy, z \rangle$

E is the solid bounded by the paraboloid $z = 4 - x^2 - y^2$ and the xy-plane

$$z = 4 - r^2$$



difficult side $\iint \vec{F} \cdot \vec{n} \, dS$

2 sides, 1 side:

Surface: $\vec{r}(x,y) = \langle x, y, 4 - x^2 - y^2 \rangle$

$$\vec{n}_1 = \vec{r}_x \times \vec{r}_y$$

$$\vec{r}_x = \langle 1, 0, -2x \rangle \quad \vec{r}_y = \langle 0, 1, -2y \rangle$$

$$\vec{n}_1 = \vec{r}_x \times \vec{r}_y = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -2x \\ 0 & 1 & -2y \end{vmatrix}$$

$$= \langle (0+2x), -(-2y-0), (1-0) \rangle$$

$$= \langle 2x, 2y, 1 \rangle$$

so $\vec{F} \cdot \vec{n}_1 = \langle x^2, xy, z \rangle \cdot \langle 2x, 2y, 1 \rangle$

$$= (x^2)(2x) + (xy)(2y) + (z)(1)$$

$$= 2x^3 + 2xy^2 + z$$

$$= 2x^3 + 2xy^2 + (4 - x^2 - y^2)$$

$$= 2x(x^2 + y^2) + 4 - (x^2 + y^2)$$

$$\iint (2x(x^2 + y^2) + 4 - (x^2 + y^2)) \, dA$$

use polar: $x = r \cos \theta$
 $y = r \sin \theta$

$$\int_0^{2\pi} \int_0^2 [2(r \cos \theta)r^2 + 4 - r^2] r \, dr \, d\theta$$

$$\int_0^{2\pi} \int_0^2 (2r^4 \cos \theta + 4r - r^3) \, dr \, d\theta$$

$$\int_0^{2\pi} \left[\frac{2}{5} r^5 \cos \theta + 2r^2 - \frac{1}{4} r^4 \right]_0^2 \, d\theta$$

$$= \frac{64}{5} \cos \theta + 4$$

$$\int_0^{2\pi} \left(\frac{64}{5} \cos \theta + 4 \right) \, d\theta$$

$$= \left[\frac{64}{5} \sin \theta + 4\theta \right]_0^{2\pi}$$

$$(0 + 8\pi) - (0 + 0) = \boxed{8\pi}$$

side 2: $\vec{n}_2 = \langle 0, 0, -1 \rangle$

$$\vec{F} \cdot \vec{n}_2 = \langle x^2, xy, z \rangle \cdot \langle 0, 0, -1 \rangle = -z$$

$$\iint (-z) \, dA \quad \text{but on xy plane } z=0$$

$$\iint (0) \, dA = \boxed{0}$$

$$\text{total} = 8\pi + 0 = \boxed{8\pi}$$

easier side $\iiint \text{div } \vec{F} \, dV$

$$\text{div } \vec{F} = \nabla \cdot \vec{F} = \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(xy) + \frac{\partial}{\partial z}(z)$$

$$= 2x + x + 1$$

$$= 3x + 1$$

$$\iiint (3x+1) \, dV \quad \text{use cylindrical:}$$

$$x = r \cos \theta$$

$$y = r \sin \theta$$

$$z = z$$

$$\int_0^{2\pi} \int_0^2 \int_0^{4-r^2} (3(r \cos \theta) + 1) r \, dz \, dr \, d\theta$$

$$\int_0^{2\pi} \int_0^2 (3r^2 \cos \theta + r) \, dr \, d\theta$$

$$\left[3r^2 \cos \theta + r \right]_0^{4-r^2} = (3r^2 \cos \theta + r)(4-r^2)$$

$$= 12r^2 \cos \theta - 3r^4 \cos \theta + 4r - r^3$$

$$\int_0^{2\pi} (12r^2 \cos \theta - 3r^4 \cos \theta + 4r - r^3) \, dr$$

$$\left[4r^3 \cos \theta - \frac{3}{5} r^5 \cos \theta + 2r^2 - \frac{1}{4} r^4 \right]_0^2$$

$$32 \cos \theta - \frac{96}{5} \cos \theta + 8 - 4$$

$$= \frac{64}{5} \cos \theta + 4$$

$$\int_0^{2\pi} \left(\frac{64}{5} \cos \theta + 4 \right) \, d\theta$$

$$\left[\frac{64}{5} \sin \theta + 4\theta \right]_0^{2\pi}$$

$$(0 + 8\pi) - (0 + 0)$$

$$= \boxed{8\pi} \quad \checkmark$$

A really interesting table in our textbook: Green's, Stokes', Divergence Theorem are higher dimensional versions of the Fundamental Theorem of Calculus. In each case we have an integral of a "derivative" over a region on the left side and the right side involves values of the original function on the boundary of the region:

Fundamental Theorem of Calculus

$$\int_a^b F'(x) dx = F(b) - F(a)$$



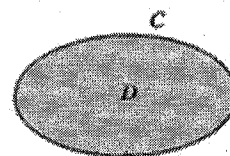
Fundamental Theorem for Line Integrals

$$\int_C \nabla f \cdot dr = f(r(b)) - f(r(a))$$



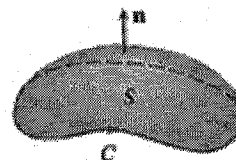
Green's Theorem

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \int_C P dx + Q dy$$



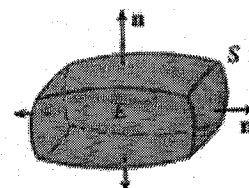
Stokes' Theorem

$$\iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r}$$



Divergence Theorem

$$\iiint_E \text{div } \mathbf{F} dV = \iint_S \mathbf{F} \cdot d\mathbf{S}$$



Homework notes: Start with #5, and do problems 1 and 3 last (they are the most work, but they are important for the test).