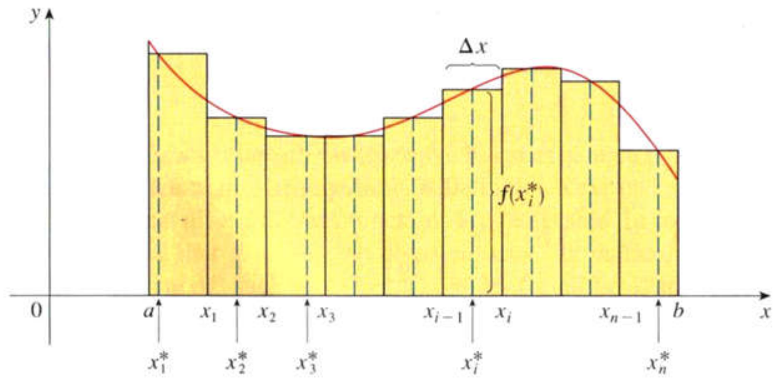
15.1: Double Integrals Over Rectangles

15.2: Iterated Integrals

Single-variable integral = area under curve, Riemann Sum

In single-variable calculus, we learned that integrals are objects which 'sum' things, and the first thing we considered was summing rectangles to find the area under a function curve:



When summing rectangles, this is called a Riemann Sum and gives the approximate area:

$$A \approx \sum_{i=1}^{n} f\left(x_i *\right)\Delta x$$

Taking the limit for a large number of rectangles, gives a smoothed version close enough to the curve to call the actual area:

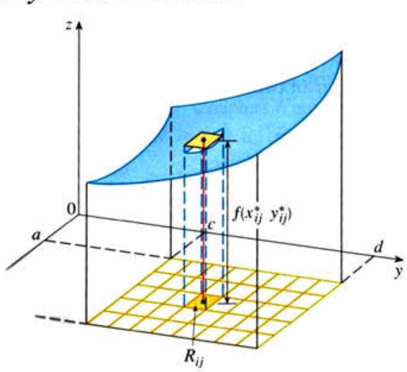
$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(x_i *\right)\Delta x$$

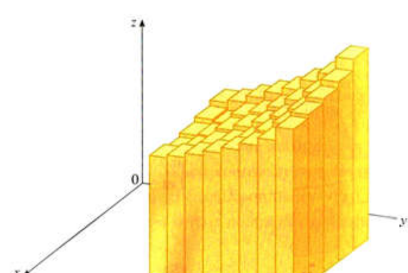
Then we called this smoothed sum the definite integral from *a* to *b*:

$$A = \int_a^b f\left(x\right) dx$$

Double Integrals Over Rectangles in the Domain

In multivariable calculus, the domain has more than one variable. First, we'll consider just a 2D domain. Now instead of rectangles of width Δx in the domain we have rectangular prisms with area $\Delta x \Delta y$ in the domain:





Summing these rectangular prisms is called a double Riemann Sum and gives approximate volume under the surface :

$$V \approx \sum_{i=i}^{n} \sum_{j=1}^{n} f\left(x_{ij}*, y_{ij}*\right)\Delta A$$

Taking the limit for a large number of prisms, gives a smoothed version close enough to the curve to call the actual volume:

$$V = \lim_{n\to\infty} \sum_{i=i}^{n} \sum_{j=1}^{n} f\left(x_{ij}*, y_{ij}*\right)\Delta A$$

Then we called this smoothed sum the definite **double integral over region *R*** in the domain:

$$V = \iint_R f\left(x, y\right) dA$$

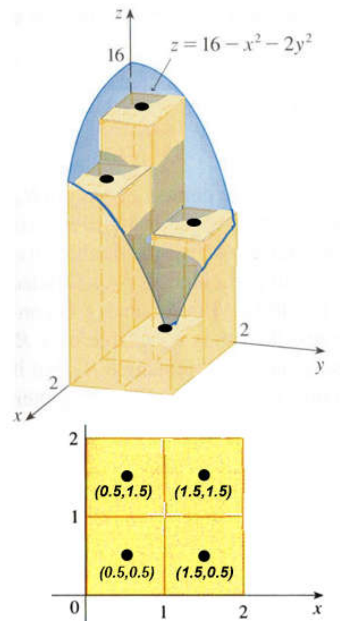
Double Riemann Sums - Midpoint Rule

When we computed Riemann Sums by hand in previous calculus classes, we had various ways to compute the rectangle areas: midpoint rule, trapezoidal rule, Simpson's rule) and these all have counterpoints for double-integrals. We'll just focus on Midpoint Rule, which says we choose the center of each rectangle *(x_{center}, y_{center})* and use this to find *f(x,y)* for the height of the prism:

Ex) Use the Midpoint Rule to approximate the value of the integral

$$\iint_R \left(16-x^2-2y^2\right) dA \quad where \quad R=\{(x,y)\,|\,0\le x\le 2, 0\le y\le 2\}$$

(x,y)	f(x,y)	*	ΔA	=	V



Average Value

We've seen average value before, and it works similarly for multivariable functions and is always a sum of something divided by the quantity of that thing:

discrete numbers

$$AV=\frac{3+4+5+8}{4}$$
$$=5$$

single variable functions

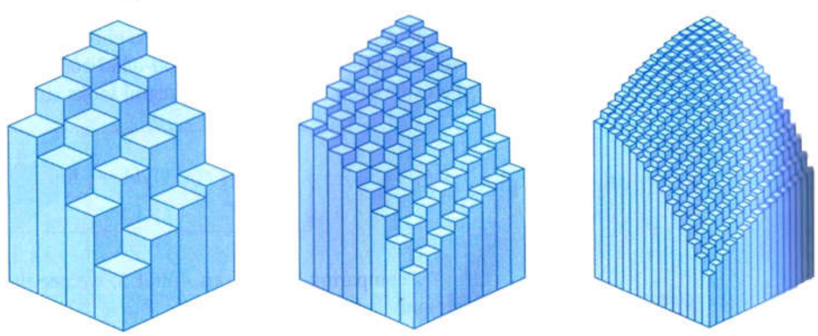
$$AV=\frac{1}{b-a}\int_a^b f(x)\,dx$$

multivariable functions

$$AV=\frac{1}{A(R)}\iint_R f(x,y)\,dA$$

Computing actual volume by computing the definite double integral

We can get a better approximation of the volume under a surface by finding a double Riemann Sum with more rectangles...



(a) $m=n=4$, $V\approx 41.5$ (b) $m=n=8$, $V\approx 44.875$ (c) $m=n=16$, $V\approx 46.469$

...but in single variable calculus we learned to evaluate the definite integral using the Fundamental Theorem of Calculus:

$$\int_a^b f(x)\,dx = F(b)-F(a)$$
$$where\ F(x)\ is\ the\ antiderivative\ of\ f(x)$$

So we need something like this for double integrals...

Computing the definite double integral using Fubini's Theorem

Fubini's Theorem says that you can compute integrals iteratively, that is, you can compute the inner integral first, and then use the result as the integrand to compute the outer integral:

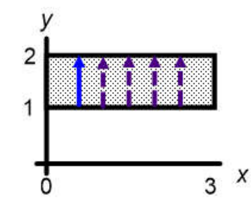
> *If f is continuous on the rectangle $R = \{(x,y) \mid a \le x \le b, c \le y \le d\}$, then*
>
> $$\iint_R f(x,y)\, dA = \int_a^b \int_c^d f(x,y)\, dy\, dx = \int_c^d \int_a^b f(x,y)\, dx\, dy$$

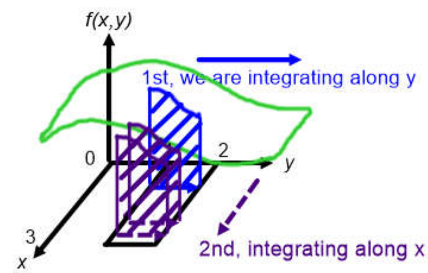
...and we can choose which variable to make the inner and which the outer variable of integration (whichever is more convenient).

This is easiest to understand by doing some examples...

Ex) $\int_0^3 \int_1^2 x^2 y\, dy\, dx$

note: this is how you determine which bounds go with which variable





Special case: f(x,y) can be factored into separate *x* and *y* factors

The last example is also a special case: where we are able to factor the *f(x,y)* into separate factors each containing only one of the variables. In cases like these:

> *If* $f(x,y) = g(x)\,h(y)$
>
> $$\iint_R f(x,y)\, dA = \int_c^d \int_a^b g(x)\,h(y)\, dx\, dy = \int_a^b g(x)\,dx \int_c^d h(y)\,dy$$

Ex) $\int_0^3 \int_1^2 x^2 y\, dy\, dx$

Another example...

$$\int_0^1\int_0^3 e^{x+3y}\ dx\ dy$$

can be made special case

$$\int_0^1\int_0^3 e^{x+3y}\ dx\ dy$$

$$\int_0^1\int_0^3 e^{x}e^{3y}\ dx\ dy$$

$$\int_0^1 e^{3y}\ dy\int_0^3 e^{x}\ dx$$

$$\left[\frac{e^{3y}}{3}\right]_0^1\left[e^x\right]_0^3$$

$$\left(\frac{e^3}{3}-\frac{1}{3}\right)\left(e^3-1\right)$$

$$\frac{e^3}{3}e^3-\frac{e^3}{3}-\frac{1}{3}e^3+\frac{1}{3}$$

$$\frac{1}{3}e^6-\frac{2}{3}e^3+\frac{1}{3}$$

if you don't see the special case, it is more difficult

$$\int_0^1\int_0^3 e^{x+3y}\ dx\ dy$$

$$\int_0^1\left[\int_0^3 e^{x+3y}\ dx\right]dy$$

$$u=x+3y$$

$$\frac{du}{dx}=1 \qquad du=dx$$

$$\left[\int_0^3 e^{x+3y}\ dx\right]=\int_{3y}^{3+3y} e^u\ du=\left[e^u\right]_{3y}^{3+3y}=e^{3+3y}-e^{3y}$$

$$\int_0^1\left(e^{3+3y}-e^{3y}\right)dy=\int_0^1 e^{3+3y}dy-\int_0^1 e^{3y}dy$$

$$u=3+3y$$

$$\frac{du}{dy}=3 \qquad du=3dy$$

$$\int_3^6 e^u\,\frac{1}{3}du-\int_0^1 e^{3y}dy$$

$$\left[\frac{1}{3}e^u\right]_3^6-\left[\frac{e^3}{3}\right]_0^1$$

$$\left(\frac{e^6}{3}-\frac{1}{3}e^3\right)-\left(\frac{e^3}{3}-\frac{e^0}{3}\right)$$

$$\frac{1}{3}e^6-\frac{2}{3}e^3+\frac{1}{3}$$

More examples...

$$\int_0^1\int_1^2 \left(4x^3 - 9x^2y^2\right)\,dy\,dx$$ **(can't be factored for the special case)**

inner: when working with y, x is treated as a constant...

$$\int_1^2 \left(4x^3 - 9x^2y^2\right)dy$$

$$\left[4x^3y - 3x^2y^3\right]_1^2$$

$$\left(4x^3(2) - 3x^2(2)^3\right) - \left(4x^3(1) - 3x^2(1)^3\right)$$

$$8x^3 - 24x^2 - 4x^3 + 3x^2$$

$$4x^3 - 21x^2$$ the result is typically a function of (contains) the outer variable

outer: now integrate with respect to x...

$$\int_0^1 \left(4x^3 - 21x^2\right)dx$$

$$\left[x^4 - 7x^3\right]_0^1$$

$$\left((1)^4 - 7(1)^3\right) - \left((0)^4 - 7(0)^3\right) = -6$$

Try these...

$$\int_0^2\int_0^{\pi/2} x\sin y\,dy\,dx$$

$$\int_1^4\int_1^2 \left(\frac{x}{y} + \frac{y}{x}\right)dy\,dx$$

More examples...

$$\iint_R \frac{xy^2}{x^2+1}\,dA \quad R=\{(x,y)\mid 0\le x\le 1,-3\le y\le 3\}$$

$$\int_0^1\int_{-3}^3 \frac{xy^2}{x^2+1}\,dy\,dx$$

$$\int_0^1 \frac{x}{x^2+1}\,dx \quad * \quad \int_{-3}^3 y^2\,dy$$

$$u=x^2+1 \quad * \quad \left[\frac{y^3}{3}\right]_{-3}^3$$

$$du=2xdx \quad * \quad \frac{(3)^3}{3}-\frac{(-3)^3}{3}$$

$$\int_1^2 \frac{1}{u}\frac{1}{2}du \quad * \quad 18$$

$$\left[\frac{1}{2}\ln|u|\right]_1^2 \quad * \quad 18$$

$$\frac{1}{2}\ln|2|-\frac{1}{2}\ln|1| \quad * \quad 18$$

$$9\ln 2$$

$$\int_0^2\int_0^\pi r\sin^2\theta\,d\theta\,dr$$

$$\int_0^2 r\,dr \quad * \quad \int_0^\pi \sin^2\theta\,d\theta$$

$$\left[\frac{r^2}{2}\right]_0^2 \quad * \quad \frac{1}{2}\int_0^\pi (1-\cos 2\theta)\,d\theta \quad \textit{use identity}: \sin^2\theta=\frac{1-\cos 2\theta}{2}$$

$$2-0 \quad * \quad \frac{1}{2}[\theta]_0^\pi-\frac{1}{2}\int_0^\pi \cos 2\theta\,d\theta$$

$$2 \quad * \quad \frac{1}{2}\pi \quad - \quad u=2\theta,\ du=2d\theta$$

$$2 \quad * \quad \frac{1}{2}\pi \quad -\frac{1}{4}\int_0^{2\pi}\cos u\,d\theta$$

$$2 \quad * \quad \frac{1}{2}\pi \quad -\frac{1}{4}[-\sin u]_0^{2\pi}$$

$$2 \quad * \quad \frac{1}{2}\pi \quad -\frac{1}{4}(0-0)$$

$$\pi$$

More examples...

$$\iint_R x\sin(x+y)\,dA \quad R=\left\{(x,y)\,|\,0\le x\le\frac{\pi}{6},\,0\le y\le\frac{\pi}{3}\right\}$$

$$\int_0^{\pi/3}\int_0^{\pi/6} x\sin(x+y)\,dx\,dy$$

$$inner:\quad \int_0^{\pi/6} x\sin(x+y)\,dx$$

$$u=x \qquad dv=\sin(x+y)dx$$

$$\frac{du}{dx}=1 \qquad \int 1\,dv=\int\sin(x+y)dx$$

$$du=dx \qquad v=-\cos(x+y)$$

$$\int_0^{\pi/6} x\sin(x+y)\,dx=-x\cos(x+y)-\int_0^{\pi/6}-\cos(x+y)\,dx$$

$$=\left[-x\cos(x+y)+\sin(x+y)\right]_0^{\pi/6}$$

$$=-\frac{\pi}{6}\cos\left(\frac{\pi}{6}+y\right)+(0)\cos\left(\frac{\pi}{6}+y\right)+\sin\left(\frac{\pi}{6}+y\right)-\sin(0+y)$$

$$outer:\quad \int_0^{\pi/3}\left(-\frac{\pi}{6}\cos\left(\frac{\pi}{6}+y\right)+\sin\left(\frac{\pi}{6}+y\right)-\sin(y)\right)dy$$

$$=\left[-\frac{\pi}{6}\sin\left(\frac{\pi}{6}+y\right)-\cos\left(\frac{\pi}{6}+y\right)+\cos(y)\right]_0^{\pi/3}$$

$$=\left(-\frac{\pi}{6}\sin\left(\frac{\pi}{6}+\left(\frac{\pi}{3}\right)\right)-\cos\left(\frac{\pi}{6}+\left(\frac{\pi}{3}\right)\right)+\cos\left(\frac{\pi}{3}\right)\right)-\left(-\frac{\pi}{6}\sin\left(\frac{\pi}{6}+(0)\right)-\cos\left(\frac{\pi}{6}+(0)\right)+\cos(0)\right)$$

$$=-\frac{\pi}{6}\sin\left(\frac{\pi}{2}\right)-\cos\left(\frac{\pi}{2}\right)+\cos\left(\frac{\pi}{3}\right)+\frac{\pi}{6}\sin\left(\frac{\pi}{6}\right)+\cos\left(\frac{\pi}{6}\right)-\cos(0)$$

$$=-\frac{\pi}{6}[1]-[0]+\left[\frac{1}{2}\right]+\frac{\pi}{6}\left[\frac{1}{2}\right]+\left[\frac{\sqrt{3}}{2}\right]-1$$

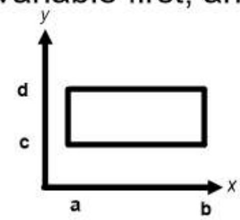
$$=\frac{\sqrt{3}-1}{2}-\frac{\pi}{12}$$

15.3: Double Integrals Over General Regions in the domain

When region in domain is not a rectangle

When the area in the domain is a rectangle, we integrate either variable first, and the limits of integration are constants:

$$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

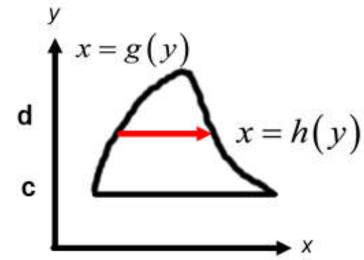


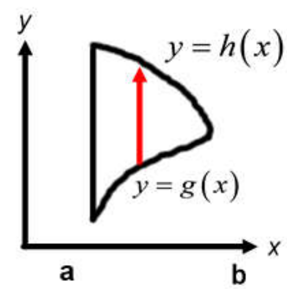
If the region is not rectangular, the edges are expressed as functions of one of the variables in terms of the other:

$$\iint_R f(x,y)\,dA = \int_c^d \int_{g(y)}^{h(y)} f(x,y)\,dx\,dy$$

$$\iint_R f(x,y)\,dA = \int_a^b \int_{g(x)}^{h(x)} f(x,y)\,dy\,dx$$

(Rectangular area is just a special case)



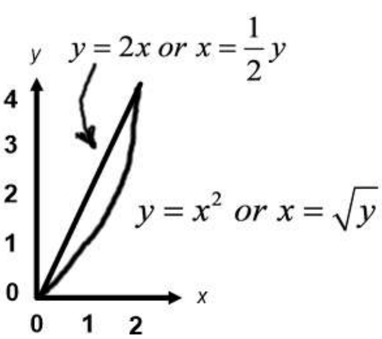


An example

Ex) $\iint_R \left(x^2 + y^2\right) dA$ $where\ D = \left\{(x,y) \mid 0 \le x \le 2,\ x^2 \le y \le 2x\right\}$

Determine intersections and graph first:

$$y = 2x \quad y = x^2$$
$$x^2 = 2x$$
$$x^2 - 2x = 0$$
$$x(x-2) = 0$$
$$x = 0,\ x = 2$$
$$y = 0,\ y = 4$$

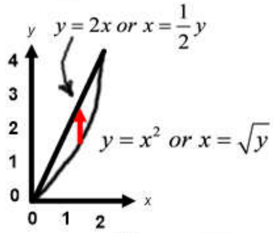


An example

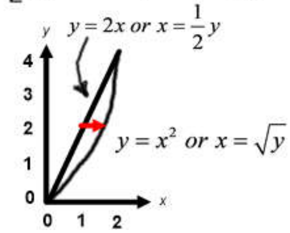
Ex) $\iint\limits_R \left(x^2+y^2\right) dA \;\; where\; D=\left\{(x,y)\,|\,0\le x\le 2,\, x^2\le y\le 2x\right\}$

We can choose either variable to integrate over first...

$$\begin{aligned}\iint\limits_R \left(x^2+y^2\right) dA &= \int_0^2\int_{x^2}^{2x}\left(x^2+y^2\right) dy\,dx\\ &=\int_0^2\left[x^2y+\frac{y^3}{3}\right]_{x^2}^{2x} dx\\ &=\int_0^2 2x^3+\frac{8}{3}x^3-\left(x^4+\frac{1}{3}x^6\right)dx\\ &=\int_0^2\frac{14}{3}x^3-x^4-\frac{1}{3}x^6\;dx\\ &=\left[\frac{14}{12}x^4-\frac{1}{5}x^5-\frac{1}{21}x^7\right]_0^2=\frac{216}{35}\end{aligned}$$

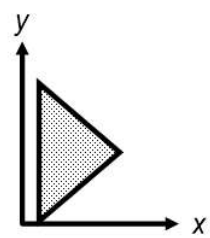


$$\begin{aligned}\iint\limits_R \left(x^2+y^2\right) dA &= \int_0^4\int_{y/2}^{\sqrt{y}}\left(x^2+y^2\right) dx\,dy\\ &=\int_0^4\left[\frac{1}{3}x^3+xy^2\right]_{y/2}^{\sqrt{y}} dy\\ &=\int_0^4\frac{1}{3}y^{3/2}+y^{5/2}-\left(\frac{1}{24}y^3+\frac{1}{2}y^3\right)dy\\ &=\int_0^4\frac{1}{3}y^{3/2}+y^{5/2}-\frac{13}{24}y^3\;dy\\ &=\left[\frac{2}{15}y^{5/2}+\frac{2}{7}y^{7/2}-\frac{13}{96}y^4\right]_0^4=\frac{216}{35}\end{aligned}$$

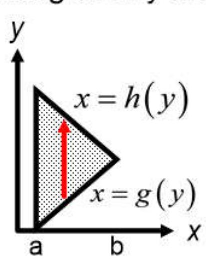


Sometimes one direction is a far better choice

If you have a region like this...



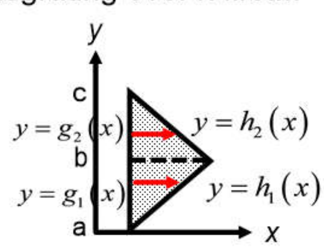
Integrating over y first...



...means you only need one integral:

$$\iint\limits_R f(x,y)\,dA=\int_a^b\int_{g(x)}^{h(x)} f(x,y)\,dy\,dx$$

Integrating over x first...

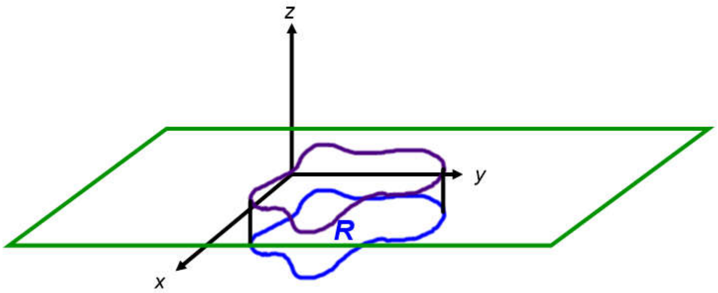


...means you need two integrals:

$$\iint\limits_R f(x,y)\,dA=\int_a^b\int_{g_1(y)}^{h_1(y)} f(x,y)\,dx\,dy+\int_b^c\int_{g_2(y)}^{h_x(y)} f(x,y)\,dx\,dy$$

You can use a volume structure to calculate the area in the domain

If you make the integrand = 1...



$$\iint_R f(x,y)\,dA = \iint_R 1\,dA = V = R\,(1) = \textit{numerical value of area } R$$

$$V\,u^3 = R\,u^2$$

Examples...

Ex) $\displaystyle\int_0^4 \int_0^{\sqrt{y}} xy^2\;dx\;dy$

Ex) $\displaystyle\int_0^8 \int_{\sqrt[3]{y}}^{2} e^{x^4}\;dx\;dy$

Examples...

Ex) $\iint\limits_R \frac{y}{x^5+1}\, dA \quad where\ R=\left\{(x,y)\,|\,0\le x\le 1,\ 0\le y\le x^2\right\}$

Examples...

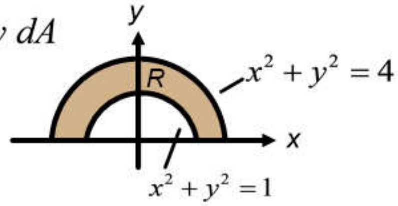
Ex) $\iint_R x\cos y\, dA$ *where R is bounded by* $y=0,\ y=x^2,\ x=1$

Ex) sketch the solid whose volume is given by the iterated integral: $\int_0^1 \int_0^{1-x} (1-x-y)\, dy\, dx$

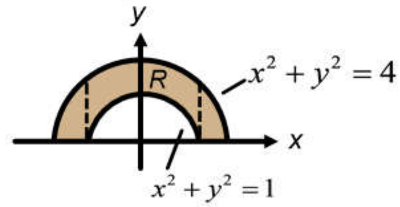
15.4: Double Integrals in Polar Coordinates

Polar coordinates is superior when the domain region is circular

This problem... $\iint_R 3y\,dA$



...is fairly difficult using our current methods:



$$\iint_R 3y\,dA=\int_{-2}^{-1}\int_{0}^{\sqrt{4-x^2}}3y\,dy\,dx+\int_{-1}^{1}\int_{\sqrt{1-x^2}}^{\sqrt{4-x^2}}3y\,dy\,dx+\int_{1}^{2}\int_{0}^{\sqrt{4-x^2}}3y\,dy\,dx$$

$$=\int_{-2}^{-1}\left[\frac{3}{2}y^2\right]_0^{\sqrt{4-x^2}}dx+\int_{-1}^{1}\left[\frac{3}{2}y^2\right]_{\sqrt{1-x^2}}^{\sqrt{4-x^2}}dx+\int_{1}^{2}\left[\frac{3}{2}y^2\right]_0^{\sqrt{4-x^2}}dx$$

$$=\int_{-2}^{-1}\frac{3}{2}\left(4-x^2\right)dx+\int_{-1}^{1}\frac{3}{2}\left(4-x^2-1+x^2\right)dx+\int_{1}^{2}\frac{3}{2}\left(4-x^2\right)dx$$

$$=\int_{-2}^{-1}\left(6-\frac{3}{2}x^2\right)dx+\int_{-1}^{1}\frac{9}{2}dx+\int_{1}^{2}\left(6-\frac{3}{2}x^2\right)dx$$

$$=\left[6x-\frac{1}{2}x^3\right]_{2}^{-1}+\left[\frac{9}{2}x\right]_{-1}^{1}+\left[6x-\frac{1}{2}x^3\right]_{1}^{2}$$

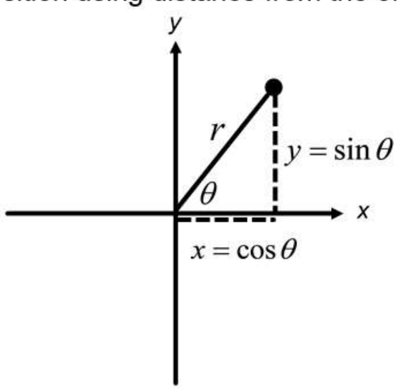
$$=-6+\frac{1}{2}+12-4+\frac{9}{2}+\frac{9}{2}+12-4-6+\frac{1}{2}$$

$$=14$$

Definition of Polar Coordinates

But there is a way we can take advantage of the circular nature of the region by defining a new coordinate system, called **polar coordinates**.

Instead of defining position in the domain as distances in the x and y direction, we define position using distance from the origin in a direction:



Converting...

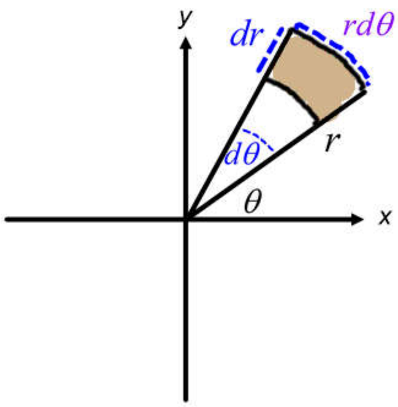
rectangular to polar	**polar to rectangular**
$r=\sqrt{x^2+y^2}$	$x=r\cos\theta$
$\theta=\tan^{-1}\left(\dfrac{y}{x}\right)$	$y=r\sin\theta$

What is *dA* in polar?

In order to integrate, we also need to know what area *dA* in the domain is equivalent to in polar coordinates.

$$\theta = \textit{angle in radians} = \frac{\textit{arclength}}{r}$$

$$\textit{therefore arclength} = r d\theta$$



When dr and $d\theta$ are infinitesimally small, this area is approximately rectangular, so the area is length * width:

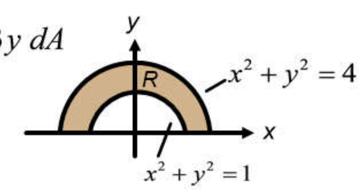
$$dA = (dr)(rd\theta)$$

$$\boxed{dA = rdrd\theta}$$

"argh - dee argh - dee theta"

Polar coordinates is superior when the domain region is circular

Let's try this problem again in polar coordinates... $\iint_R 3y\, dA$



$$\iint_R 3y\, dA = \int_0^{\pi}\int_1^2 3(r\sin\theta)\, r\, dr\, d\theta$$

$$= 3\left(\int_0^{\pi} \sin\theta\, d\theta\right)\left(\int_1^2 r^2\, dr\right)$$

$$= 3\left(\left[-\cos\theta\right]_0^{\pi}\right)\left(\left[\frac{1}{3}r^3\right]_1^2\right)$$

$$= 3(-\cos\pi + \cos 0)\left(\frac{1}{3}(2)^3 - \frac{1}{3}(1)^3\right)$$

$$= 3(2)\left(\frac{7}{3}\right)$$

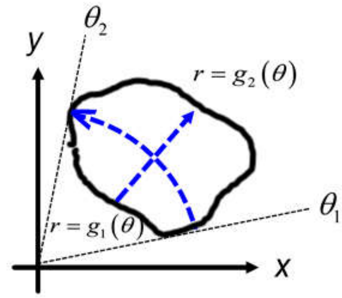
$$= 14$$

much easier!

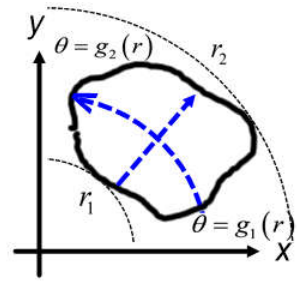
The inner integral limits of integration may be functions

Just as in the rectangular case, for more complicated regions, the inner integral may have limits of integration which are functions:

$$\iint_R f(x,y)\,dA = \int_{\theta_1}^{\theta_2}\int_{g_1(\theta)}^{g_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$



$$\iint_R f(x,y)\,dA = \int_{r_1}^{r_2}\int_{h_1(r)}^{h_2(r)} f(r\cos\theta, r\sin\theta)\, r\,d\theta\,dr$$



Examples

Ex) Sketch the region and evaluate $\displaystyle\int_0^{\pi/2}\int_0^{4\cos\theta} r\,dr\,d\theta$

Examples

Ex) Sketch the region and evaluate $\iint_R \sqrt{4-x^2-y^2}\, dA$ *where* $R=\{(x,y)\,|\,x^2+y^2\le 4,\ x\ge 0\}$

Ex #14) Sketch the region (and evaluate - for homework)

$\iint_D x\, dA$ **where D is the region in the 1st quadrant that lies between the circles $x^2+y^2=4$ and $x^2+y^2=2x$**

Examples

Ex #15) Use a double integral to find the area of one loop of the rose) $r = \cos 3\theta$
(setup - do the evaluation for homework)

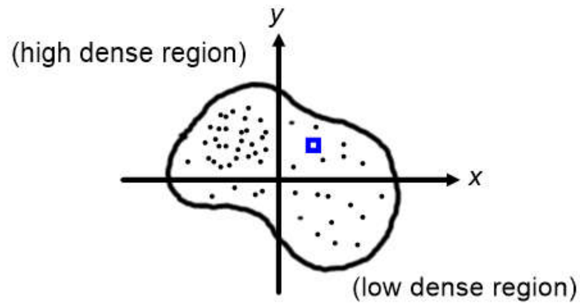
Ex #25) Volume above cone $z = \sqrt{x^2 + y^2}$ and below sphere $x^2 + y^2 + z^2 = 1$
(setup - do the evaluation for homework)

15.5: Applications of Double Integrals

Mass and Density

Define density as how much mass exists in an object over a small area:

$$\rho(x,y) = \frac{\Delta mass}{\Delta area}$$



...and this mass can vary over the object (is a function of *x* and *y*.

We can use a double-integral to sum the contributions over the entire object to obtain the total mass:

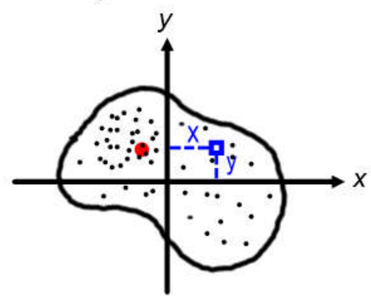
$$m = \iint_D \rho(x,y)\, dA$$

1st Moment (center of mass)

A **moment** is defined as the product of a distance and some other quantity. There are many moments which have useful applications.

The 1st moments of density (just called **moment**) is defined as the density times distance from one of the axes:

$$M_x = \iint_D y\rho(x,y)\, dA$$
$$M_y = \iint_D x\rho(x,y)\, dA$$



If we divide the moments by the mass, we obtain the position of the averages in the *x* and *y* directions, which is the location of the **center of mass**. This is the point where the object would balance...in the x (and y directions) have of the mass is on each side of this point.

$$\left(\bar{x},\bar{y}\right) = \frac{1}{m}\left(\iint_D x\rho(x,y)\, dA, \iint_D y\rho(x,y)\, dA\right)$$

center of mass

2nd Moment (moment of inertia)

The 2nd moments of density are defined as the density times distance squared from one of the axes and are called **moments of inertia:**

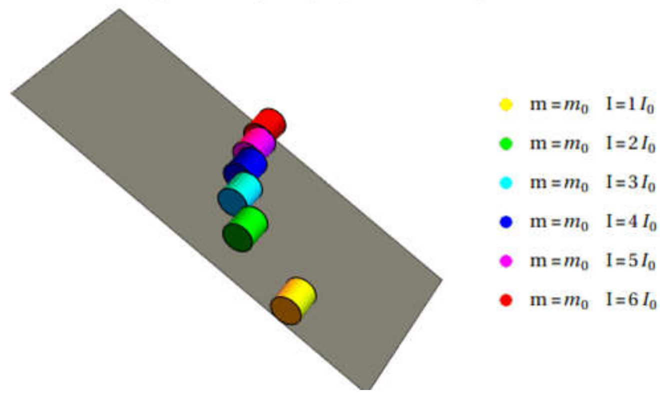
$$I_x = \iint_D y^2 \rho(x,y)\, dA$$

$$I_y = \iint_D x^2 \rho(x,y)\, dA$$

Mass is the aspect of an object which resists linear force - more mass means less movement (acceleration) for a given force: $F = ma$

Moment of inertia about an axis is like 'rotational mass', it is the aspect of an object which resists rotational motion which would be produced by a torque (alpha is angular acceleration): $\tau = I\alpha$

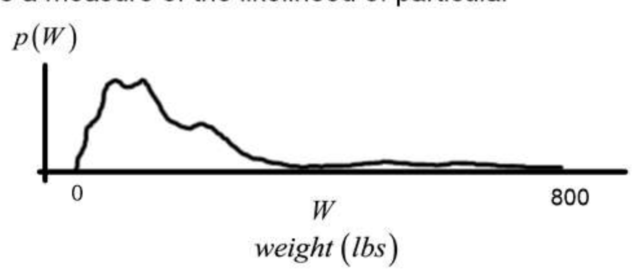
A **flywheel** has most of its mass around the outer edge which produces high moments of inertia. Such an object takes a lot of torque to start it spinning, but once spinning, resists friction well and keeps spinning for a long time.



Applications in Probability - Probability Density Functions

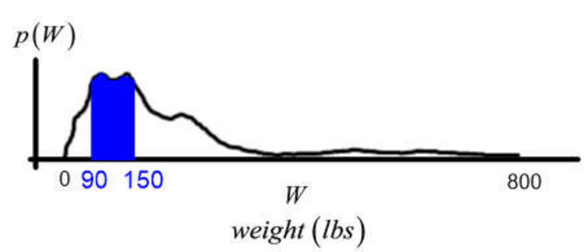
A **random variable** provides a numerical value of possible outcomes of a random process. For example, you could define a random variable for the weight of an individual person, *W*. This variable could take on values from 0 to some maximum weight, and if you selected a person at random, you would get a different value for *W* each time.

A **probability density function (PDF)** provides a measure of the likelihood of particular weights occurring in a population.



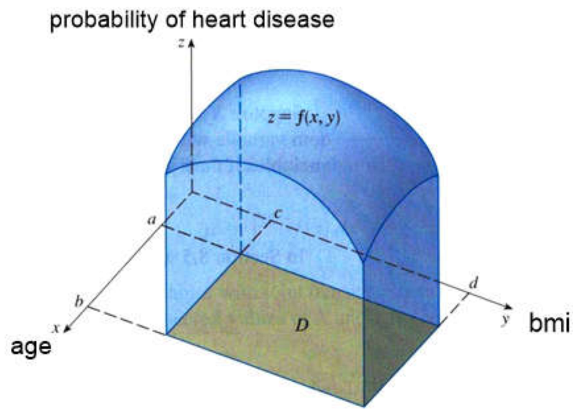
Summing all the probabilities (equivalent to 'mass' for matter density) would give the total probability which is always 1, but if you sum probabilities over a subregion of weights, you get the probability of a person randomly selected have a weight in this region:

$$P(90 \le W \le 150) = \int_{90}^{150} p(W)\, dW$$



Applications in Probability - Probability Density Functions

In general, the probability of an event might depend upon more than one variable, for example, likelihood of heart disease may depend upon age and also body mass index (bmi). We could define a general probability density function *f(x,y)* where *x* is age and *y* is bmi:



Summing all the probabilities (equivalent to 'mass' for matter density) would give the total probability which is always 1:

$$\iint_D f(x,y)\,dA = 1$$

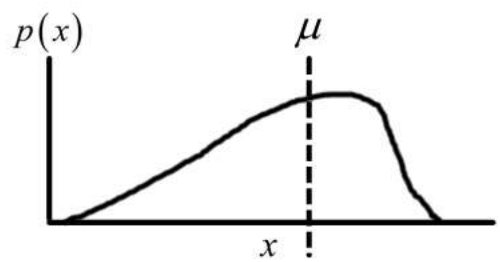
Summing probabilities over a subregion of the explanatory variables, you get the probability of a person with this range of ages and bmis having heart disease:

$$P(a \le x \le b, c \le y \le d) = \int_a^b \int_c^d f(x,y)\,dy\,dx$$

1st moment of a PDF: Expected Value (mean)

The moments for a probability density function (PDF) have specific meaning and uses. The 1st moments of a PDF gives the **Expected Values (or means):**

$$EV = \mu = \int_{-\infty}^{\infty} x\,p(x)\,dx$$



The expected value (or mean of the random variable probability distribution) represents the probability-weighted 'average' value.

If the function is multivariable, then:

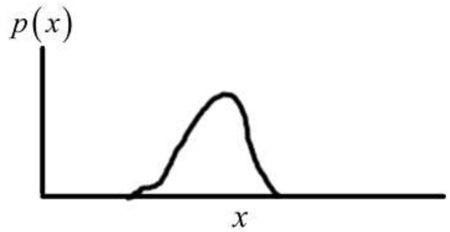
$$EV_x = \mu_x = \iint_D x\,p(x,y)\,dA$$

$$EV_y = \mu_y = \iint_D y\,p(x,y)\,dA$$

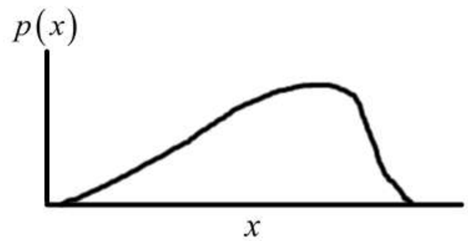
2nd moment of a PDF: Variance

The 2nd moment of a PDF gives its **Variance:**

$$Var(x) = \sigma^2 = \int_{-\infty}^{\infty} x^2 \; p(x)\, dx$$



low variance



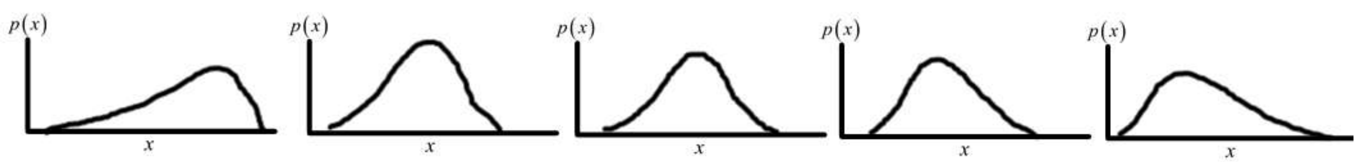
high variance

The variance is the standard-deviation squared...both are measures of our widely dispersed the probability is about the mean.

3rd moment of a PDF: Skewness

The 3rd moment of a PDF gives its **Skewness:**

$$skewness = \int_{-\infty}^{\infty} x^3 \; p(x)\, dx$$



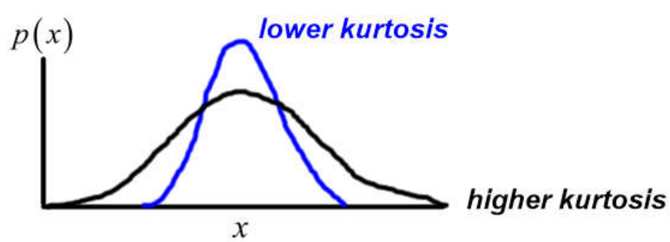
high negative skewness ***low negative skewness*** ***zero skewness*** ***low positive skewness*** ***high positive skewness***

Skewness is a measure of deviation from symmetry and can be positive or negative.

4th moment of a PDF: Kurtosis

The 4th moment of a PDF gives its **Kurtosis:**

$$kurtosis = \int_{-\infty}^{\infty} x^4 \; p(x)\, dx$$



Kurtosis gives a measure of 'tailedness' - how much probability is in the tails instead of the central grouping. Higher kurtosis means there is a higher likelihood of the outlying values occurring.

Examples (setup only...redo and compute values for hw)

#5) Find the mass and center of mass for $\rho(x,y) = x + y$
for the lamina occupying the triangular region with vertices of (0,0), (2,1), and (0,3)

#11) A lamina occupies the part of the disk $x^2 + y^2 \leq 1$ in the first quadrant.
Find its center of mass if the density at any point is proportional to its distance from the x-axis.

Examples (setup only...redo and compute values for hw)

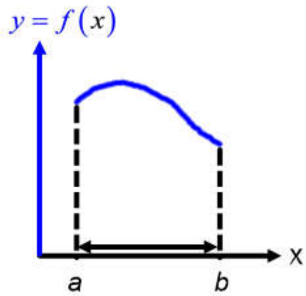
#12) A lamina occupies the part of the disk $x^2 + y^2 \leq 1$ in the first quadrant.
Find its center of mass if the density at any point is proportional to the square of its distance from the origin.

#16) A lamina occupies the region inside the circle $x^2 + y^2 = 2y$ but outside the circle $x^2 + y^2 = 1$
Find its center of mass if the density at any point is inversely proportional to its distance from the origin.

15.6: Triple Integrals

What is a Triple Integral?

Single integral



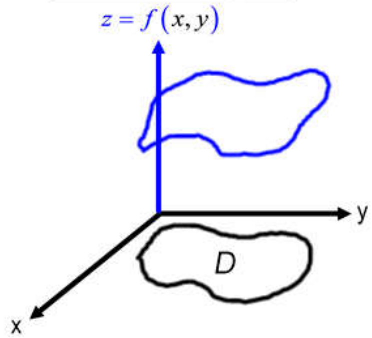
Domain is a 1D region in x

Output is $y = f(x)$

Single integral is a summation of output values over the 1D domain line values from a to b:

$$\int_a^b f(x)\,dx$$

Double integral



Domain is a 2D region in x,y

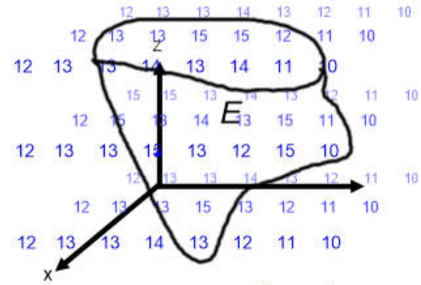
Output is $z = f(x,y)$

Double integral is a summation of output values over the 2D domain area:

$$\iint_D f(x,y)\,dA$$

$$dA = dx\,dy,\ dy\,dx$$
$$= r\,dr\,d\theta$$

Triple integral



air pressure $P = f(x,y,z)$

Domain is a 3D region in x,y,z

Output is $P = f(x,y,z)$

Triple integral is a summation of output values over the 3D domain volume:

$$\iiint_E f(x,y,z)\,dV$$

$$dV = dx\,dy\,dz,\ dx\,dz\,dy$$
$$= dy\,dx\,dz,\ dy\,dz\,dx$$
$$= dz\,dx\,dy,\ dz\,dy\,dx$$
$$= \text{others (next sections)}$$

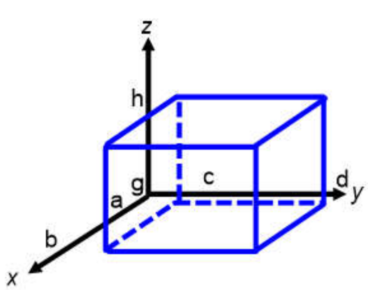
Applications of Triple Integrals

- Physics: Density, center of mass, moments of physical 3D objects
- Probability (expected value, variance, skewness, kurtosis of probability distributions with 3 explanatory variables)
- Flux integrals (subject of our last chapter in Vector Calculus)

In this section, we are going to focus on how to evaluate (can be tricky)...

Fubini's Theorem for Triple Integrals (domain is a box)

Just an with double integrals, if the domain region has no dependencies between the variables (the limits of integration are all constants), then you can evaluate the integral iteratively in any order:



$$\iiint_E f(x,y,z)\,dV = \int_g^h\int_c^d\int_a^b f(x,y,z)\,dx\,dy\,dz$$

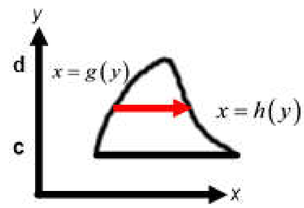
$$= \int_a^b\int_c^d\int_g^h f(x,y,z)\,dz\,dy\,dx$$

$$\vdots$$

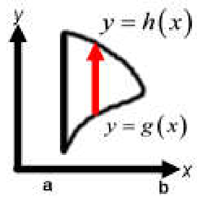
(6 possible orderings)

If there are dependencies, you may have to integrate in a particular order

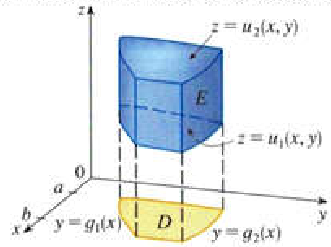
The textbook discusses different 'types'/'cases' that dictate the order you need to integrate, but the main idea here is that if an edge of the domain region is not constant, you must write it as a function of a variable 'outside' this variable:



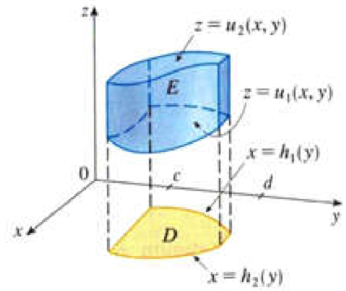
$$\iint_R f(x,y)\,dA = \int_c^d \int_{g(y)}^{h(y)} f(x,y)\,dx\,dy$$



$$\iint_R f(x,y)\,dA = \int_a^b \int_{g(x)}^{h(x)} f(x,y)\,dy\,dx$$



$$\iiint_E f(x,y,z)\,dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\,dy\,dx$$



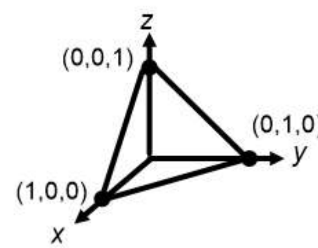
$$\iiint_E f(x,y,z)\,dV = \int_c^d \int_{h_1(y)}^{h_2(y)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\,dx\,dy$$

Order of integration (examples)

The best way to understand this is just to work a lot of examples...

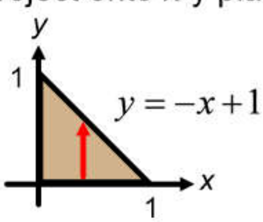
Evaluate $\iiint_E z\,dV$ *where E is the solid tetrahedron bounded by $x=0$, $y=0$, $z=0$ and $x+y+z=1$*

(evaluate all 6 ways)



(domain)

1) project onto x-y plane (this will be the outside two integrals, do y first):

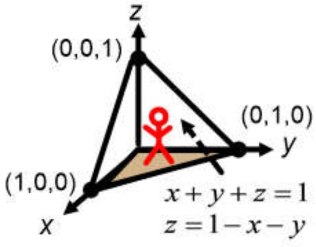


The outer integrals:

$$\int_0^1 \int_0^{-x+1} \square \, dy\,dx$$

The inner integral:

Imagine a person standing on the projection and jumping...they will hit their head on a ceiling which determines height (z in this case).



$$\int_0^1 \int_0^{-x+1} \int_0^{1-x-y} (z)\,dz\,dy\,dx$$

Now evaluate:

$$\int_0^1 \int_0^{-x+1} \int_0^{1-x-y} (z)\,dz\,dy\,dx$$

$$\int_0^1 \int_0^{-x+1} \left[\frac{1}{2}z^2\right]_0^{1-x-y} dy\, dx$$

$$\int_0^1 \int_0^{-x+1} \left(\frac{1}{2}(1-x-y)^2\right)-\left(\frac{1}{2}(0)^2\right) dy\, dx$$

$$\frac{1}{2}\int_0^1 \int_0^{-x+1} (1-x-y)^2\, dy\, dx \qquad \begin{aligned} u &= 1-x-y \\ du &= -dy \end{aligned}$$

$$\frac{1}{2}\int_0^1 \left[-\frac{1}{3}(1-x-y)^3\right]_0^{-x+1} dx$$

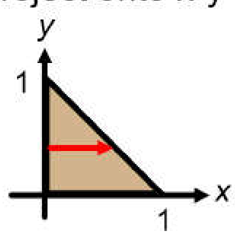
$$-\frac{1}{6}\int_0^1 \left(\left(1-x-(-x+1)\right)^3-\left(1-x-(0)\right)^3\right) dx$$

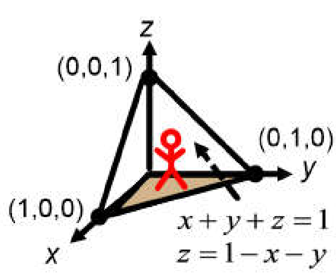
$$\frac{1}{6}\int_0^1 (1-x)^3\, dx$$

$$\frac{1}{6}\left[-\frac{1}{4}(1-x)^4\right]_0^1 = \frac{1}{6}\left(-\frac{1}{4}(1-(1))^4+\frac{1}{4}(1-0)^4\right)$$

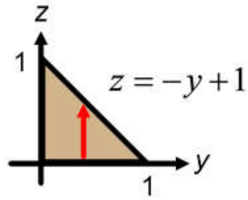
$$\frac{1}{24}$$

2) project onto x-y plane (this will be the outside two integrals, do x first):





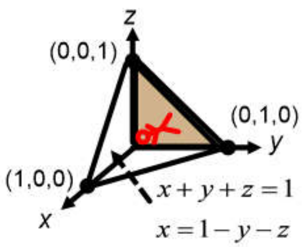
3) project onto y-z plane (this will be the outside two integrals, do z first):



The outer integrals:

$$\int_0^1 \int_0^{-y+1} \Box \, dz \, dy$$

The inner integral:



Imagine a person standing on the projection and jumping...they will hit their head on a ceiling which determines height (y in this case).

$$\int_0^1 \int_0^{-y+1} \int_0^{1-y-z} (z) \, dx \, dz \, dy$$

(will also evaluate to 1/24)

Setup the last 3 orderings...

4) project onto y-z plane (this will be the outside two integrals, do y first):

5) project onto x-z plane (this will be the outside two integrals, do z first):

6) project onto x-z plane (this will be the outside two integrals, do x first):

More examples...

#15) Evaluate $\iiint\limits_T x^2 \, dV$ where T is the solid tetrahedron with vertices (0,0,0), (1,0,0), (0,1,0), and (0,0,1).

More examples...

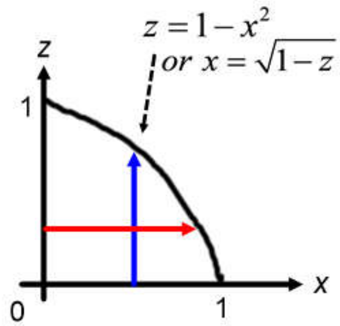
#21) Find volume of the solid enclosed by the cylinder $x^2 + y^2 = 9$ and the planes $y + z = 5$ and $z = 1$

More examples...

#34) The figure shows the region of integration for the integral $\int_0^1 \int_0^{1-x^2} \int_0^{1-x} f(x,y,z)\,dy\,dz\,dx$

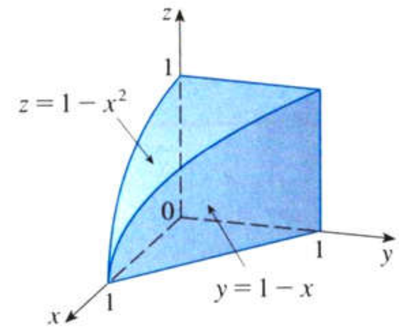
Rewrite this integral as an equivalent iterated integral in the five other orders.

Original ordering is x-z projection:

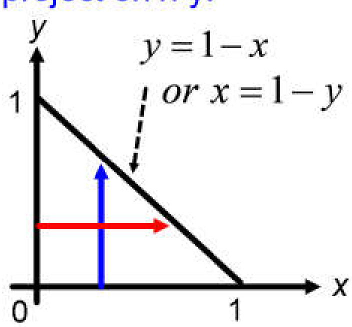


For this projection, other direction:

$$\int_0^1 \int_0^{\sqrt{1-z}} \int_0^{1-x} f(x,y,z)\,dy\,dx\,dz$$



Can project on x-y:



(from equation of top surface)

$$\int_0^1 \int_0^{1-x} \int_0^{1-x^2} f(x,y,z)\,dz\,dy\,dx$$

Other direction:

$$\int_0^1 \int_0^{1-y} \int_0^{1-x^2} f(x,y,z)\,dz\,dx\,dy$$

#34) The figure shows the region of integration for the integral $\int_0^1 \int_0^{1-x^2} \int_0^{1-x} f(x,y,z)\, dy\, dz\, dx$

Rewrite this integral as an equivalent iterated integral in the five other orders.

$y = 0$ *to* $y = 1 - x$

$z = 0$ *to* $z = 1 - x^2$

$y = 1 - x$ and $z = 1 - x^2$

are the surfaces on the side and top of the region.

If you find the intersection of these:

$x = 1 - y$

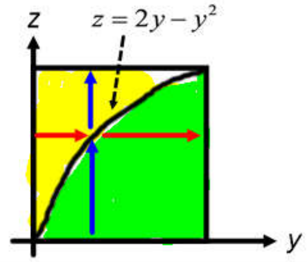
$z = 1 - (1-y)^2$

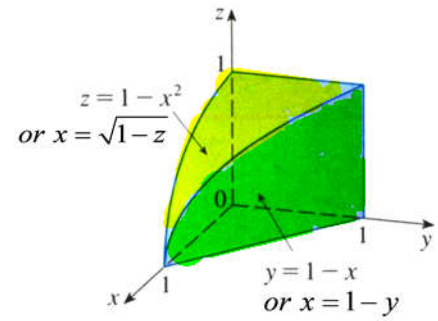
$z = 1 - (1 - 2y + y^2)$

$z = 2y - y^2$

This splits the region into two subregions so we need to add two integrals, each with a different 'ceiling' establishing the x value.

Project on y-z:





$$\int_0^1 \int_{2y-y^2}^1 \int_0^{\sqrt{1-z}} f(x,y,z)\, dx\, dz\, dy + \int_0^1 \int_0^{2y-y^2} \int_0^{1-y} f(x,y,z)\, dx\, dz\, dy$$

For the final direction, we need to reverse the variables at the boundary:

$z = 2y - y^2$

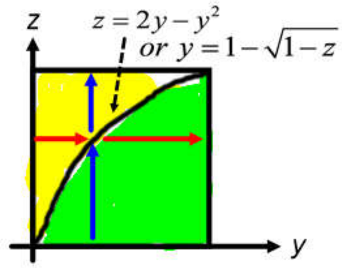
$y^2 - 2y = -z$

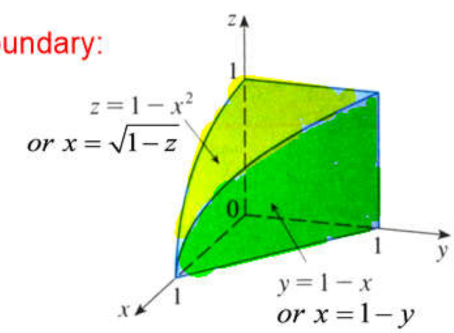
$y^2 - 2y + 1 = -z + 1$

$(y-1)^2 = 1 - z$

$y - 1 = \pm\sqrt{1-z}$

$y = 1 - \sqrt{1-z}$





...and here are the two integrals:

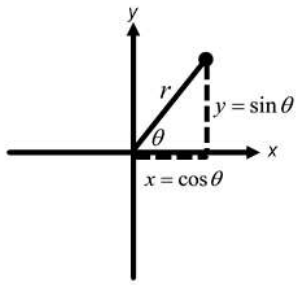
$$\int_0^1 \int_0^{1-\sqrt{1-z}} \int_0^{\sqrt{1-z}} f(x,y,z)\, dx\, dy\, dz + \int_0^1 \int_{1-\sqrt{1-z}}^1 \int_0^{1-y} f(x,y,z)\, dx\, dy\, dz$$

15.7: Cylindrical Coordinates

Defining Cylindrical Coordinates

Double integral

$$\iint_D f(x,y)\, dA$$



Converting...

rectangular to polar

$$r = \sqrt{x^2 + y^2}$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

polar to rectangular

$$x = r\cos\theta$$

$$y = r\sin\theta$$

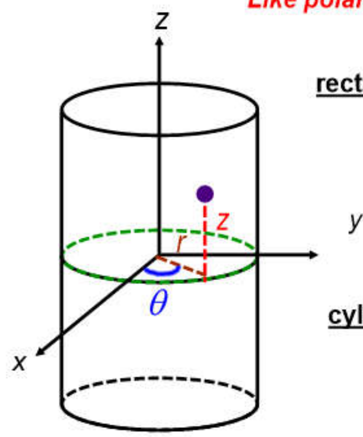
$dA = dx\,dy,\ dy\,dx$ **rectangular**

$= r\,dr\,d\theta$ **polar**

Triple integral

$$\iiint_E f(x,y,z)\, dV$$

Like polar in x-y plane, plus z



Converting...

rectangular to cylindrical

$$r = \sqrt{x^2 + y^2}$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$$z = z$$

cylindrical to rectangular

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$z = z$$

$dV = dx\,dy\,dz,\ dx\,dz\,dy$

$= dy\,dx\,dz,\ dy\,dz\,dx$ **rectangular**

$= dz\,dx\,dy,\ dz\,dy\,dx$

$= r\,dz\,dr\,d\theta$ **cylindrical**

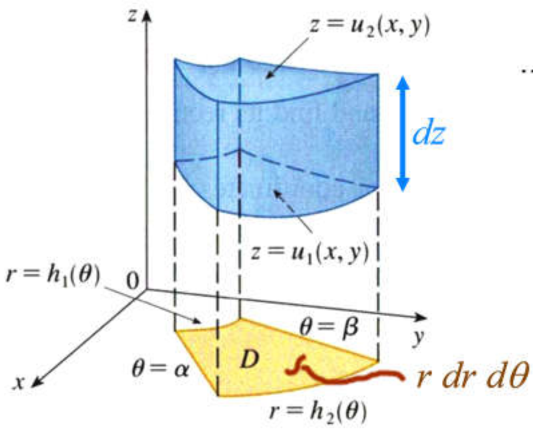
Cylindrical coordinates are best for...

- Cylindrical domain regions (define z so it is pointing in whichever direction is the cylinder's axis)
- Cones
- Circular paraboloids

Anything where the domain region has circular cross-sections, even if the radius of the circle is changing in the z-direction.

Triple integrals in cylindrical coordinates

The volume of any 'extruded' shape is the base area times the height...



...so for cylindrical coordinates, *dV* becomes:

$$dV = (r\ dr\ d\theta)\ dz$$
$$dV = r\ dz\ dr\ d\theta$$

$$\iiint_E f(x,y,z)\ dx\ dy\ dz = \int_{\alpha}^{\beta}\int_{h_1(\theta)}^{h_2(\theta)}\int_{u_1(r\cos\theta)}^{u_2(r\cos\theta)} f(r\cos\theta, r\sin\theta, z)\ r\ dz\ dr\ d\theta$$

Examples...

Ex) Graph $\left(2, \frac{2\pi}{3}, 1\right)$ and convert to rectangular coordinates.

Ex) Convert $(3, -3, -7)$ to cylindrical coordinates.

Examples...

Ex) Setup the integral for $\iiint_E xyz\, dV$ where *E* is a right circular cylinder with the z-axis as the axis of symmetry, with radius = 2, from z = 0 to z = 3.

#22) Find the volume of the solid that lies within both the cylinder $x^2 + y^2 = 1$ and the sphere $x^2 + y^2 + z^2 = 4$

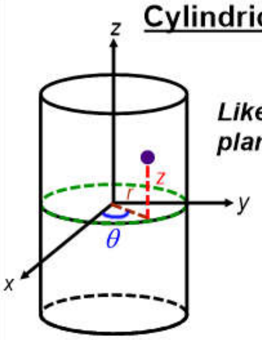
15.8: Spherical Coordinates

Defining Spherical Coordinates

Rectangular

$$dV = dxdydz,\ dxdzdy,\ dydxdz,\ dydzdx,\ dzdxdy,\ dzdydx$$

Cylindrical



Like polar in x-y plane, plus z

Converting...

rectangular to cylindrical

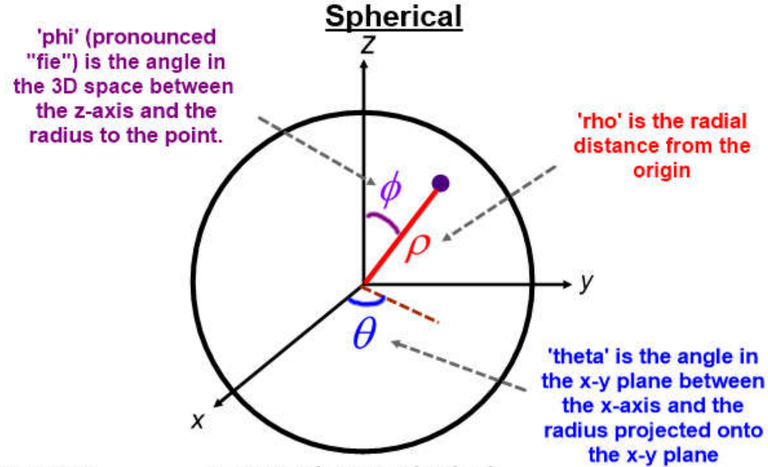
$$r = \sqrt{x^2 + y^2} \quad \theta = \tan^{-1}\left(\frac{y}{x}\right) \quad z = z$$

cylindrical to rectangular

$$x = r\cos\theta \quad y = r\sin\theta \quad z = z$$

$$dV = r\ dz\ dr\ d\theta$$

Spherical



Converting... **rectangular to spherical**

$$\rho^2 = x^2 + y^2 + z^2 \quad \phi = \cos^{-1}\left(\frac{z}{\rho}\right) \quad \theta = \cos^{-1}\left(\frac{x}{\rho\sin\phi}\right) = \sin^{-1}\left(\frac{y}{\rho\sin\phi}\right)$$

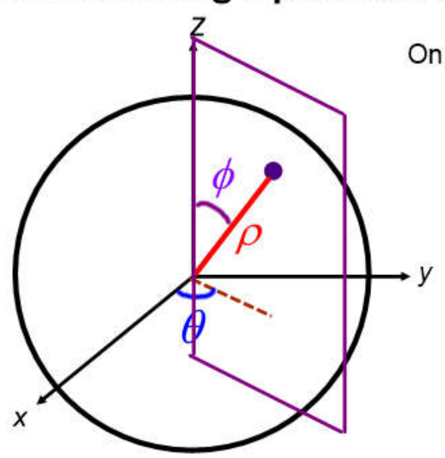
spherical to rectangular

$$x = \rho\sin\phi\cos\theta \qquad y = \rho\sin\phi\sin\theta \qquad z = \rho\cos\phi$$

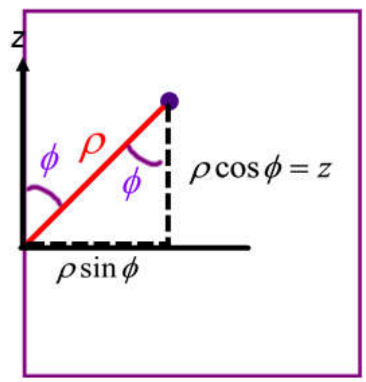
in x-y plane: $r = \rho\sin\phi$

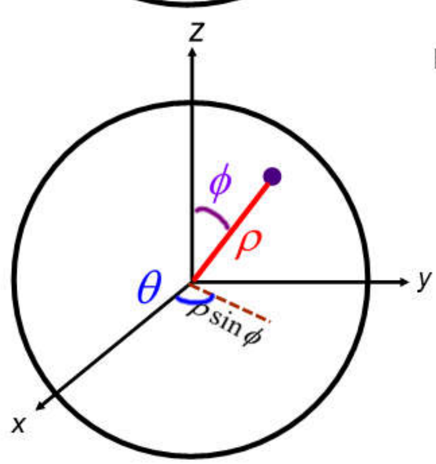
$$dV = \rho^2\sin\phi\ d\rho\ d\theta\ d\phi$$

Understanding Spherical Coordinate Geometry

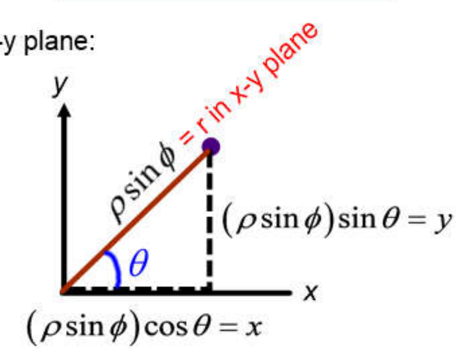


On the plane formed by z-axis and the radius to the point:

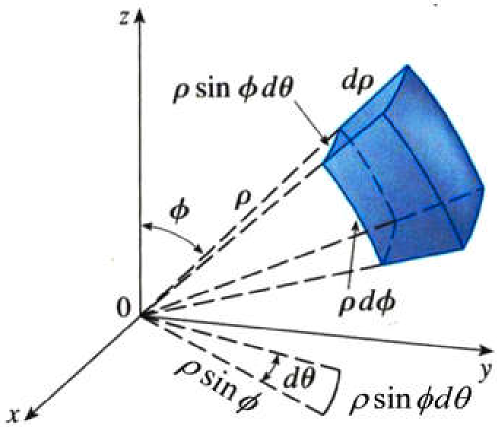




Now on the x-y plane:



Triple integrals in spherical coordinates



...so for spherical coordinates, *dV* becomes:

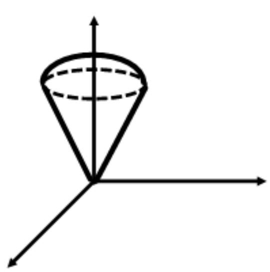
$$dV = (length)(width)(height)$$
$$= (\rho d\phi)(\rho \sin\phi d\theta)(d\rho)$$
$$= \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

$$\iiint_E f(x,y,z)\, dx\, dy\, dz$$
$$= \int_c^d \int_\alpha^\beta \int_a^b f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\, \rho^2 \sin\phi \, d\rho\, d\theta\, d\phi$$

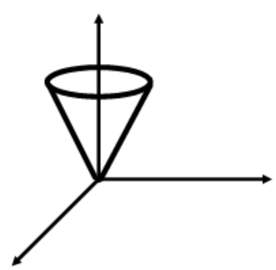
Spherical coordinates are best for...

- Spheres
- Cones where we want volume to the top of the 'snow cone'

use spherical for this...



use cylindrical for this...



Examples...

#2a) Plot the point whose spherical coordinates are $\left(5, \frac{\pi}{2}, \frac{\pi}{3}\right)$ then find the rectangular coordinates of the point.

Examples...

#4b) Change from rectangular to spherical coordinates: $\left(-1, 1, \sqrt{6}\right)$

#9a) Write the equation in spherical coordinates: $z^2 = x^2 + y^2$

Examples...

#13) Sketch the solid described by: $\rho \le 1, \quad \dfrac{3\pi}{4} \le \phi \le \pi$

#15) A solid lies above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = z$ write a description of the solid in terms of inequalities involving spherical coordinates (find volume of this solid).

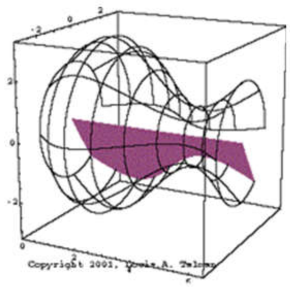
15.9 (extra content): General Transformations, Jacobians, Disk/Shell method for volumes

Optional, extra topics

The following topics are not officially part of the course, but I wanted to mention them for completeness.

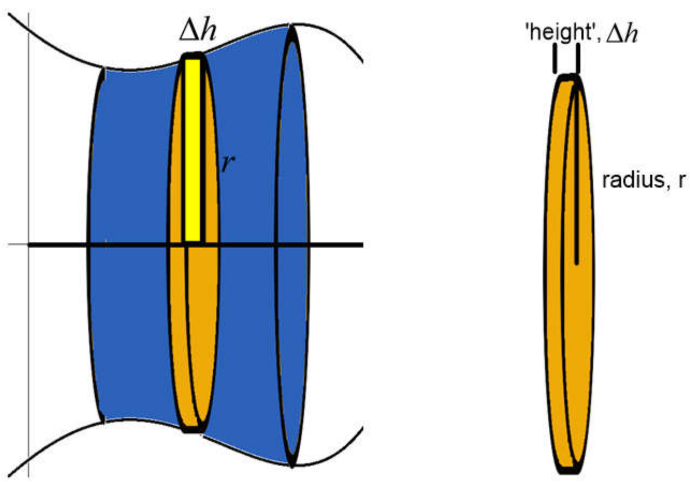
The first is that we had a method to find volumes back in Calc 1/Calc 2 and even in Brief Calculus: **the disk and shell methods for finding volumes of revolution**.

If we rotate this area around the x-axis, we form a 3 dimensional volume called a 'solid of revolution':



Volumes of Solids of Revolution, Disc Method

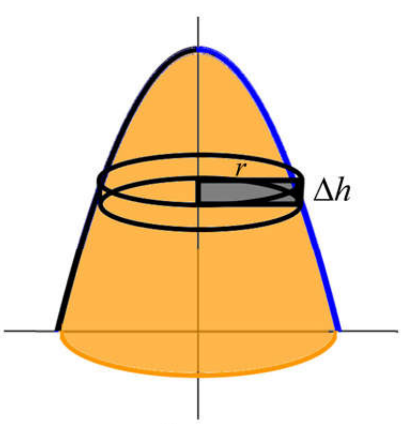
We can use an integral to find the 3-D volume of a solid of revolution, by computing the summation of an infinite number of small shapes, but instead of the shapes being 2-D rectangles, the 2-D rectangles would also revolve around the axis and form 3-D cylinders called 'discs':



$$V = \sum (volume\ of\ cylinder)$$
$$V = \sum \pi r^2 \bullet height$$
$$V = \int_a^b \pi r^2 \Delta h$$

Volumes of Solids of Revolution, Disc and Shell Methods

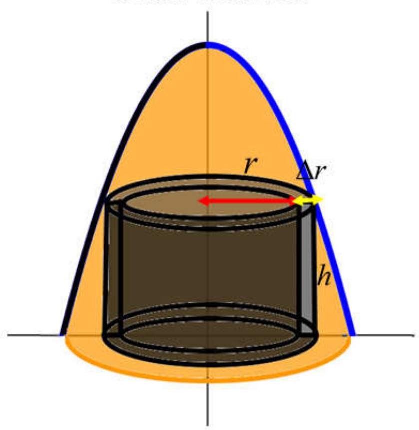
'Disc Method'



$$V = \int \pi r^2 \Delta h$$

$rect \perp axis\ of\ rotation$

'Shell Method'



$$V = \int 2\pi r h \Delta r$$

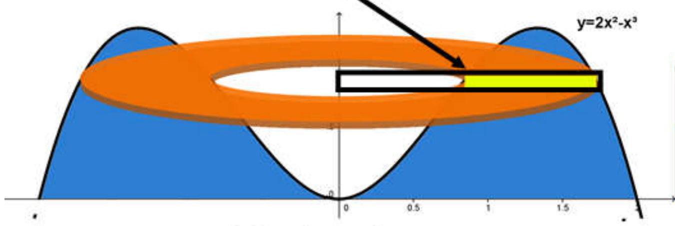
$rect\ ||\ axis\ of\ rotation$

Volumes of Solids of Revolution, Disk and Shell Methods

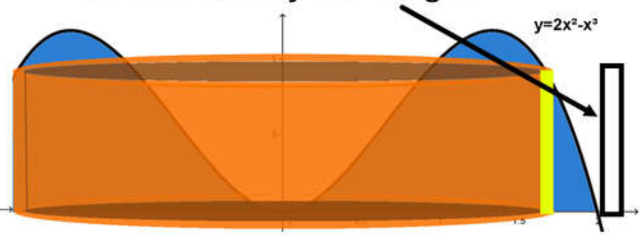
Sometimes, the solid didn't extend all the way to the axis or rotation so we had to use the 'washer' variation (or similar for shell method)...

rectangle doesn't go all the way to axis there is a 'hole' so we need two integrals

rectangle goes all the way to axis no 'hole' so only one integral



Disc/washer



Shell

$$V = \int \pi r_2{}^2 \Delta h - \int \pi r_1{}^2 \Delta h$$

$$V = \int_y \pi x_2{}^2 dy - \int_y \pi x_1{}^2 dy$$

$$V = \int_x \pi y_2{}^2 dx - \int_x \pi y_1{}^2 dx$$

$$V = \int 2\pi h \Delta r$$

$$V = \int_x 2\pi x \left[f(x) \right] dx$$

$$V = \int_y 2\pi y \left[f(y) \right] dy$$

This method only works when there is a symmetry, such that all variations are functions of only one variable. We can then express the volume as a function of this variable, and integrate along that one direction.

In Brief Calc, we stuck to shapes with circular cross-section, but in Calc1/Calc2 other shapes are considered (for example pyramids with rectangular cross-section).

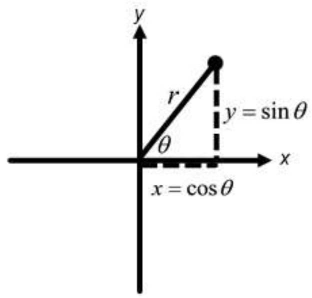
What we've covered in this chapter gives a more flexible approach that can be used for all solids.

General Coordinate Transformations

The other optional topic, which is covered in our textbook, section 15.9, is Change of Variables for General Coordinate Transformations.

We found that for double and triple integrals, it is sometimes easier to cover the region in the domain using polar, cylindrical, or spherical coordinates:

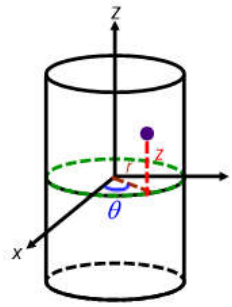
Polar (2D)



$$x = r\cos\theta$$
$$y = r\sin\theta$$

$$\iint_D f(x, y)\, dx\, dy$$
$$= \iint_D f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

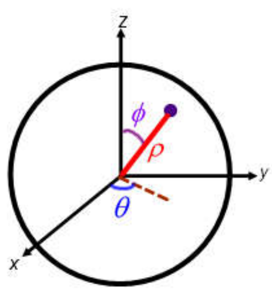
Cylindrical (3D)



$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$

$$\iiint_E f(x, y, z)\, dx\, dy\, dz$$
$$= \iiint_E f(r\cos\theta, r\sin\theta, z)\, r\, dr\, d\theta\, dz$$

Spherical (3D)



$$x = \rho\sin\phi\cos\theta$$
$$y = \rho\sin\phi\sin\theta$$
$$z = \rho\cos\phi$$

$$\iiint_E f(x, y, z)\, dx\, dy\, dz$$
$$= \iiint_E f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\, \rho^2 \sin\phi\, d\rho\, d\theta\, d\phi$$

General Coordinate Transformations

These particular coordinate transformations are extremely useful and by far the most widely used, however, in some specialized circumstances, you may wish to transform the coordinates in some other, more general, way.

For example, you may wish to transform something in coordinates *x,y,z* to coordinates *u,v,w* and therefore define some other way to express one coordinate system in terms of the other system's variables:

Polar (2D) $x = r\cos\theta$ $y = r\sin\theta$

Cylindrical (3D) $x = r\cos\theta$ $y = r\sin\theta$ $z = z$

Spherical (3D) $x = \rho\sin\phi\cos\theta$ $y = \rho\sin\phi\sin\theta$ $z = \rho\cos\phi$

General (3D) $x = x(u,v,w)$ $y = y(u,v,w)$ $z = z(u,v,w)$

The tricky thing is that the volume (or area) element is not just *du dv dw*, it generally must be multiplied by something to account for how the shape of the translation forms the volume:

Polar (2D) $\iint_D f(r\cos\theta, r\sin\theta)\boxed{r}\,dr\,d\theta$

Cylindrical (3D) $\iiint_E f(r\cos\theta, r\sin\theta, z)\boxed{r}\,dr\,d\theta\,dz$

Spherical (3D) $\iiint_E f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\boxed{\rho^2\sin\phi}\,d\rho\,d\theta\,d\phi$

General (3D) $\iiint_E f(x(u,v,w), y(u,v,w), z(u,v,w))\boxed{???}\,du\,dv\,dw$

This extra multiplier is called the **Jacobian of the transformation.**

General Coordinate Transformations - Jacobian of a Transformation

The following result is derived in section 15.9 of our textbook, but there is a method you can use to determine the formula for the Jacobian of a tranformation, given the transformation equations, and it has a unique symbol:

$$\text{Jacobian} = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

This pattern can be extended to any number of dimensions. In 2D, it reduces to:

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

We could, for example, use the Jacobian to verify our usual multiplier for spherical coordinates:

$$x = \rho \sin\phi \cos\theta \qquad y = \rho \sin\phi \sin\theta \qquad z = \rho\cos\phi$$

$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix} = \begin{vmatrix} (\sin\phi\cos\theta) & (-\rho\sin\phi\sin\theta) & (\rho\cos\phi\cos\theta) \\ (\sin\phi\sin\theta) & (\rho\sin\phi\cos\theta) & (\rho\cos\phi\sin\theta) \\ (\cos\phi) & (0) & (-\rho\sin\phi) \end{vmatrix}$$

(use this row to find the determinant)

$$= (\cos\phi)\left(-\rho^2\sin\phi\cos\phi\sin^2\theta - \rho^2\sin\phi\cos\phi\cos^2\theta\right) + (-\rho\sin\phi)\left(\rho\sin^2\phi\cos^2\theta + \rho\sin^2\phi\sin^2\theta\right)$$

$$= -\rho^2\sin\phi\cos^2\phi\sin^2\theta - \rho^2\sin\phi\cos^2\phi\cos^2\theta - \rho^2\sin\phi\left(\sin^2\phi\cos^2\theta + \sin^2\phi\sin^2\theta\right)$$

$$= -\rho^2\sin\phi\cos^2\phi\left(\cos^2\theta + \sin^2\theta\right) - \rho^2\sin\phi\sin^2\phi\left(\cos^2\theta + \sin^2\theta\right)$$

$$= -\rho^2\sin\phi\cos^2\phi - \rho^2\sin\phi\sin^2\phi$$

$$= -\rho^2\sin\phi\left(\cos^2\phi + \sin^2\phi\right)$$

$$= \left|-\rho^2\sin\phi\right| = \boxed{\rho^2\sin\phi}$$

You will likely not need anything other than polar, cylindrical, or spherical, but for certain specialized fields you could potentially need Jacobians.

Mathematicians - if you become a mathematician, you'll likely need to derive and prove things even in your intermediate and advanced calculus courses. You could end up working in topology or manifold theory which explore the properties of higher dimensions.

Theoretical Physicists - if you end up studying the (currently popular) String Theory, you'll need Jacobians because this is based on the assumption that the universe actually inhabits an 11-dimensional space.