13.1: Vector Functions and Space Curves
Vector Functions

2D

In 2D, a standard function maps each value in the domain to one output value in the range:
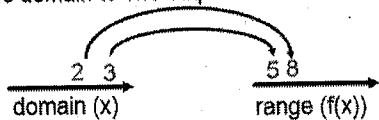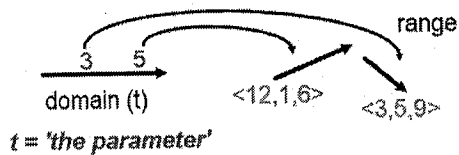

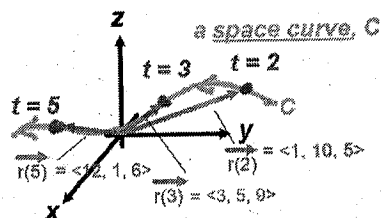| x	y = f(x)
2	8
3	5
6	12
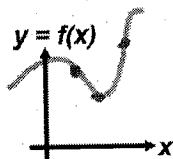
3D

In a 3D vector function, maps each value in the domain (a set of real numbers) to a *vector*.
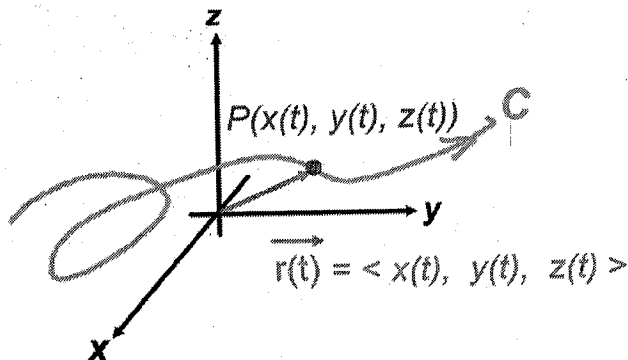

t = 'the parameter'

t	$\vec{r(t)} = \,<x(t), y(t), z(t)>$
2	<1, 10, 5>
3	<3, 5, 9>
5	<12, 1, 6>

a space curve, C

$x(t)$, $y(t)$, $z(t)$ are the parametric equations of C

A vector function has the form... $\vec{r(t)} = \,<x(t),\, y(t),\, z(t)>$

...and can have any number of dimensions, although we'll usually stay in 3D.

$$\vec{r(t)} = \,<x(t),\ y(t),\ z(t)>$$

We might be given a vector function and asked to sketch it, or given information about a curve and asked to find the functions x(t), y(t), z(t)..this is called 'parameterizing the curve'.

A vector function has the form... $\vec{r(t)} = \,<x(t),\, y(t),\, z(t)>$

...and can have any number of dimensions, although we'll usually stay in 3D.

$$\vec{r(t)} = \,<x(t),\ y(t),\ z(t)>$$
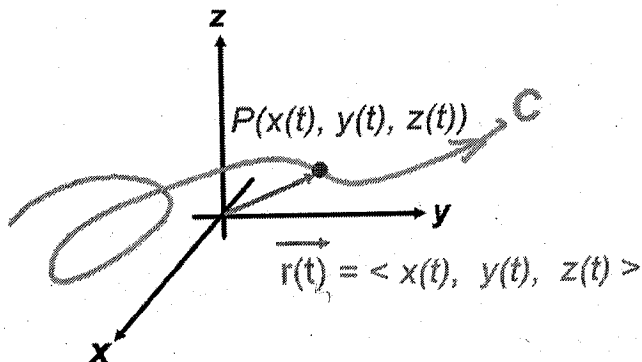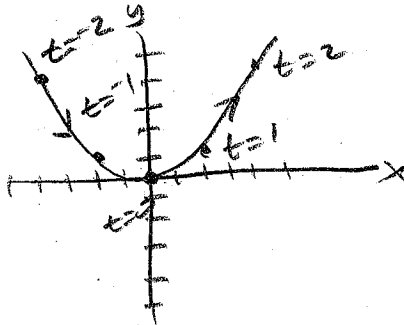
We might be asked to:
- state the domain of a vector function
- compute limits of vector functions
- sketch a given vector function's space curve
- parameterize a curve (find the parametric equations which define it)

Sketching Vector Functions

Make a table of values for selected t values:

Ex) Sketch $\vec{r(t)} = \langle 2t, t^2 \rangle$

t	x	y
-2	-4	4
-1	-2	1
0	0	0
1	2	1
2	4	4

Parametrization of a line

From last chapter...

For a line:
$$\vec{r} = \vec{r_0} + t\vec{v} \qquad -\infty < t < \infty$$
$$\langle x, y, z \rangle = \langle x_0, y_0, z_0 \rangle + t\langle a, b, c \rangle$$
$$\langle x, y, z \rangle = \langle x_0 + ta, \quad y_0 + tb, \quad z_0 + tc \rangle$$

so:
$$x(t) = x_0 + at$$
$$y(t) = y_0 + bt$$
$$z(t) = z_0 + ct$$
$$-\infty < t < \infty$$

For a line segment: $\vec{r} = (1-t)\vec{r_0} + t\vec{r_1} \qquad 0 \le t \le 1$

so:
$$x(t) = x_0 + at$$
$$y(t) = y_0 + bt$$
$$z(t) = z_0 + ct$$
$$0 \le t \le 1$$

Ex) Find a vector equation and parametric equations for the line segment that joins
$P(1, 0, 1)$ to $Q(2, 3, 1)$

$$\vec{r}(t) = (1-t)\langle 1, 0, 1 \rangle + t\langle 2, 3, 1 \rangle$$
$$= \langle 1-t, 0, 1-t \rangle + \langle 2t, 3t, t \rangle$$
$$= \langle 1+t, 3t, 1 \rangle$$

$$x(t) = 1+t$$
$$y(t) = 3t \qquad 0 \le t \le 1$$
$$z(t) = 1$$

Parametrization of an ellipse or circle in 2D

$$x(t) = a\cos t$$
$$y(t) = b\sin t$$

or

$$x = a\cos t$$
$$y = b\sin t$$

means

$$\cos t = \frac{x}{a}$$
$$\sin t = \frac{y}{b}$$

Since $\cos^2 t + \sin^2 t = 1$ (Pythagorean identity)

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$ (circle if a=b, ellipse otherwise)

So the parameterization is...
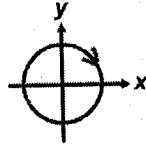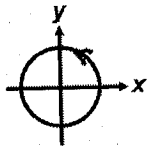
$$\vec{r(t)} = \langle a\cos t, b\sin t \rangle$$

$r(t) = \langle \sin t, \cos t \rangle$ for $0 \le t \le 2\pi$

$r(t) = \langle -\cos t, \sin t \rangle$ for $0 \le t \le 2\pi$

Ex) Sketch $\vec{r(t)} = \langle \cos t, \sin t \rangle$ for $0 \le t \le 2\pi$

t	x	y
0	1	0
$\pi/2$	0	1
π	-1	0
$3\pi/2$	0	-1

Parametrization of an ellipse or circle in 3D

What if we add a simple z(t)...

$$\vec{r(t)} = \langle \cos t, \sin t, t \rangle$$

t	x	y	z
0	1	0	0
$\pi/2$	0	1	$\pi/2$
π	-1	0	π
$3\pi/2$	0	-1	$3\pi/2$

What about this?

$$\vec{r(t)} = \langle \cos t, \sin t, 3 \rangle$$

$a+ z=3$

a helix
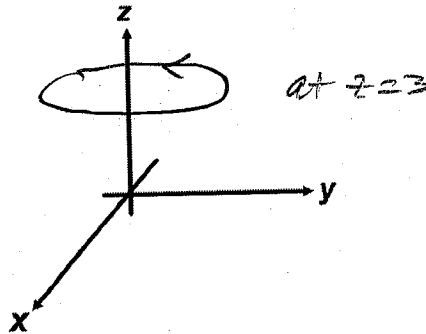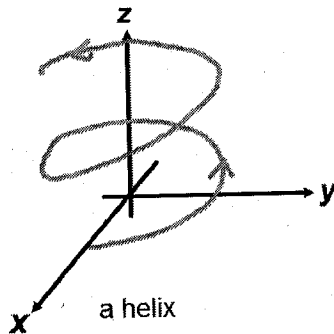
Parametrization of a parabola, polynomials in 2D

Not too hard to parameterize if you can solve for all variables in terms of just one of the variables:

$$y = x^2 \qquad \text{let } x = t$$
$$y = t^2$$

gives $\vec{r(t)} = \langle t, t^2 \rangle$

Ex) Find a parameterization if $y = 2x^3 + 3x^2 + 1$

$$z = x^4 - 3$$

$\vec{r(t)} = \langle t, 2t^3 + 3t^2 + 1, t^4 - 3 \rangle$

(just mace x=t)

Limits of Vector Functions

The limit of a vector function is defined by taking the limits of its component functions:

$$\lim_{t \to a} \overrightarrow{r(t)} = \left\langle \lim_{t \to a} x(t), \lim_{t \to a} y(t), \lim_{t \to a} z(t) \right\rangle$$

Ex) Find $\lim_{t \to 0} \left\langle e^{-3t}, \dfrac{t^2}{\sin^2 t}, \cos 2t \right\rangle = \boxed{<1, 1, 1>}$

$\lim_{t \to 0} e^{-3t}$

e^0

$\boxed{1}$

$\lim_{t \to 0} \dfrac{t^2}{\sin^2 t}$

$\dfrac{0}{0}$ (by L'Hopital)

$= \lim_{t \to 0} \dfrac{2t}{2 \sin t \cos t}$

$= \dfrac{0}{0}$

(by L'Hopital)

$= \lim_{t \to 0} \dfrac{2}{2(\sin t(-\sin t) + t(\cos t)(\cos t)}$

$= \dfrac{2}{2(-\sin^2 t + \cos^2 t)} \Big|_{t=0}$

$= \dfrac{2}{2(-(0)^2 + (1)^2)} = \boxed{1}$

$\lim_{t \to 0} \cos 2t$

$\cos(0)$

$\boxed{1}$

Domain of a vector function

Ex) Find the domain of $\overrightarrow{r(t)} = \left\langle \dfrac{t-2}{t+2}, \sin t, \ln(9 - t^2) \right\rangle$

$t \neq -2 \qquad -\infty < t < \infty$

$9 - t^2 > 0$ (no negative logs)

or zero

$-t^2 > -9$

$t^2 < 9$

$-3 < t < 3$

x:

y:

z:

$-\infty \qquad -3 \quad -2 \qquad 3 \qquad \infty$
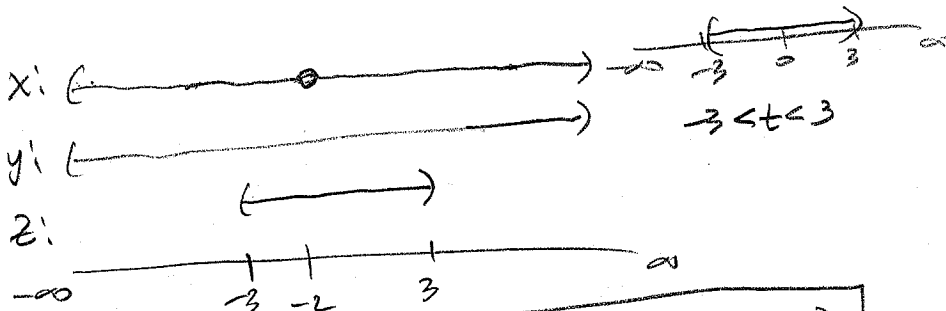
put all together: $\boxed{D: (-3, -2) \cup (-2, 3)}$

13.2: Derivatives and Integrals of Vector Functions
Derivative of a vector function

Recall from single-variable calculus:

$$y = f(x)$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

The derivative of a function (defined by the of the slope of the secant line) givens the slope of the tangent line to the curve at x.
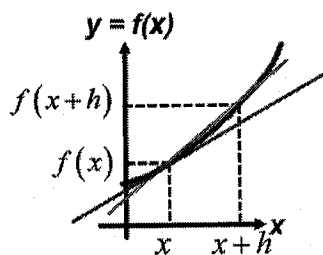
In multivariable calculus:
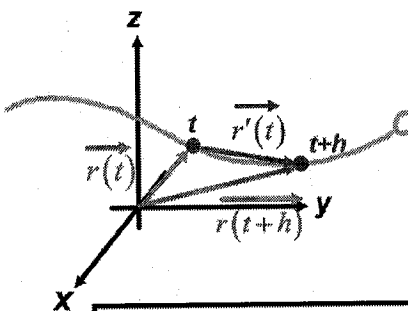
$$r'(t) = \lim_{h \to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}$$

A similar limit structure defines the derivative of the vector function. Before taking the limit, this is a vector between two points on the space curve which is roughly in the direction of the curve at the point where $t=t$.
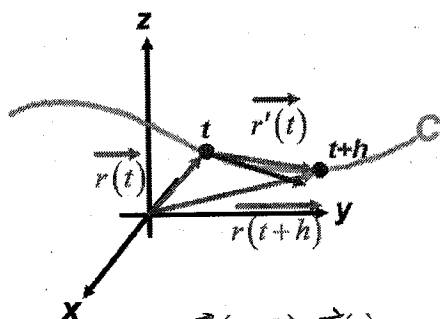
$$r'(t) = \lim_{h \to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}$$

Although it is harder to visualize, because we are taking the limit as h approaches 0, this derivative vector actually points exactly in the direction of the curve at $t=t$ so the derivative is called the **tangent vector** to the curve.

To standardize the length of this vector, we divide the vector by it's magnitude to normalize it and define the **unit tangent vector** as:

$$\vec{T}(t) = \frac{\vec{r'}(t)}{|\vec{r'}(t)|}$$

Finding a unit tangent vector

We don't actually compute the derivative vector geometrically, we use the following theorem:

$$\text{If } \vec{r(t)} = \langle x(t), y(t), z(t) \rangle$$
$$\text{then } \vec{r'(t)} = \langle x'(t), y'(t), z'(t) \rangle$$

Ex) Find the unit tangent vector for $\vec{r(t)} = \langle 3t, t^2, 0 \rangle$ at the point where $t = 2$.

$$\vec{r'(t)} = \langle 3, 2t, 0 \rangle$$
$$@ \; t = 2$$
$$\vec{r'(2)} = \langle 3, 4, 0 \rangle$$
$$|\vec{r'(2)}| = \sqrt{3^2 + 4^2} = 5$$
$$\vec{T(2)} = \langle \frac{3}{5}, \frac{4}{5}, 0 \rangle$$

Derivative rules for vectors

3 THEOREM Suppose **u** and **v** are differentiable vector functions, c is a scalar, and f is a real-valued function. Then

1. $\dfrac{d}{dt}[\mathbf{u}(t) + \mathbf{v}(t)] = \mathbf{u}'(t) + \mathbf{v}'(t)$

2. $\dfrac{d}{dt}[c\mathbf{u}(t)] = c\mathbf{u}'(t)$

3. $\dfrac{d}{dt}[f(t)\mathbf{u}(t)] = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$

4. $\dfrac{d}{dt}[\mathbf{u}(t) \cdot \mathbf{v}(t)] = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t)$

5. $\dfrac{d}{dt}[\mathbf{u}(t) \times \mathbf{v}(t)] = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$

6. $\dfrac{d}{dt}[\mathbf{u}(f(t))] = f'(t)\mathbf{u}'(f(t))$ (Chain Rule)

Definite integral of a vector function

Because definite integrals are smoothed versions of Riemann sums which can be computed using limits, and limits are defined for vector functions, you can take definite integrals of vector functions:

$$\int_a^b r(t)\,dt = \lim_{n \to \infty} \sum_{i=1}^n r(t_i)\,\Delta t$$

$$= \lim_{n \to \infty} \left\langle \sum_{i=1}^n x(t_i)\,\Delta t, \; \sum_{i=1}^n y(t_i)\,\Delta t, \; \sum_{i=1}^n z(t_i)\,\Delta t \right\rangle$$

$$= \left\langle \lim_{n \to \infty} \sum_{i=1}^n x(t_i)\,\Delta t, \; \lim_{n \to \infty} \sum_{i=1}^n y(t_i)\,\Delta t, \; \lim_{n \to \infty} \sum_{i=1}^n z(t_i)\,\Delta t \right\rangle$$

$$= \left\langle \int_a^b x(t)\,dt, \; \int_a^b y(t)\,dt, \; \int_a^b z(t)\,dt \right\rangle$$

13.3: Arc Length and Curvature
Arc Length

Recall from last year, Ch8, we first defined arc length for a single-variable function:

$$L = \lim_{n \to \infty} \sum_{i=1}^{n} |P_{i-1}P_i|$$

$$= \lim_{n \to \infty} \sum_{i=1}^{n} \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2}$$

$$= \lim_{n \to \infty} \sum_{i=1}^{n} \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

By the mean value theorem: $f'(x^*) = \dfrac{\Delta y}{\Delta x}$, $\Delta y = f'(x^*)\Delta x$ so...

$$L = \lim_{n \to \infty} \sum_{i=1}^{n} \sqrt{(\Delta x)^2 + (f'(x^*)\Delta x)^2}$$

$$= \lim_{n \to \infty} \sum_{i=1}^{n} \sqrt{1 + (f'(x^*))^2} \sqrt{(\Delta x)^2}$$

$$= \lim_{n \to \infty} \sum_{i=1}^{n} \sqrt{1 + (f'(x^*))^2}\, \Delta x$$

$$= \int_a^b \sqrt{1 + (f'(x^*))^2}\, dx$$

$$\boxed{L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx}$$

Then, in Ch10 when parametric equations were introduced, the 2D arc length was expressed in terms of the parameter, t:

If C is described by parametric equations $x = f(t)$, $y = g(t)$

$$L = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_a^b \sqrt{1 + \left(\frac{dy/dt}{dx/dt}\right)^2}\, dx \frac{dt}{dt}$$

$$= \int_a^b \sqrt{1 + \left(\frac{dy/dt}{dx/dt}\right)^2}\, \frac{dx}{dt} dt$$

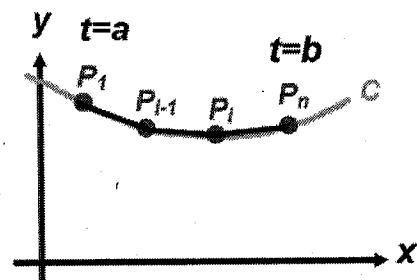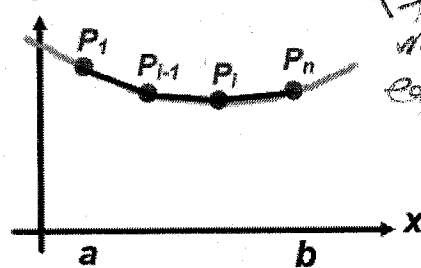$$= \int_a^b \sqrt{1 + \left(\frac{dy/dt}{dx/dt}\right)^2} \sqrt{\left(\frac{dx}{dt}\right)^2}\, dt$$

$$\boxed{L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt}$$
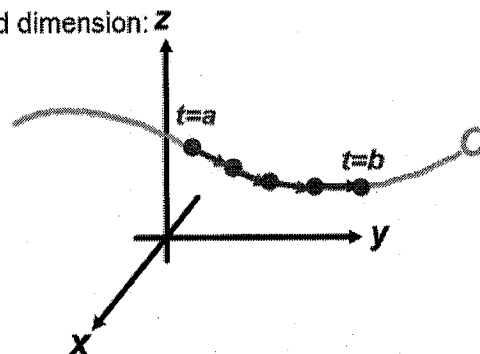
Arc Length

Now for multivariable calculus, we just extend this computation into a 3rd dimension: **z**

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$\boxed{L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt}$$

But we know that the tangent vector, $\vec{r'(t)}$ is given by:

$$\vec{r'(t)} = \langle x'(t), y'(t), z'(t) \rangle$$

and magnitude of the tangent vector is:

$$\left|\vec{r'(t)}\right| = \sqrt{\left(x'(t)\right)^2 + \left(y'(t)\right)^2 + \left(z'(t)\right)^2}$$

therefore:

$$\boxed{L = \int_a^b \left|r'(t)\right| dt}$$

...which makes sense intuitively. The arc length is the sum of the lengths of the tangent vectors.

Ex) Find the length of the arc of the circular helix with vector equation
$$\vec{r(t)} = \langle \cos t, \sin t, t \rangle \text{ from the point } (1,0,0) \text{ to the point } (1, 0, 2\pi)$$

$$\vec{r'(t)} = \langle -\sin t, \cos t, 1 \rangle$$

$$\left|\vec{r'(t)}\right| = \sqrt{\left(-\sin t\right)^2 + \left(\cos t\right)^2 + \left(1\right)^2} = \sqrt{2}$$

Interval is from $t = 0$ *to* $t = 2\pi$ (look at z coordinates)

So arc length is given by:

$$L = \int_0^{2\pi} \sqrt{2}\, dt$$

$$= \left[\sqrt{2}\, t\right]_0^{2\pi}$$

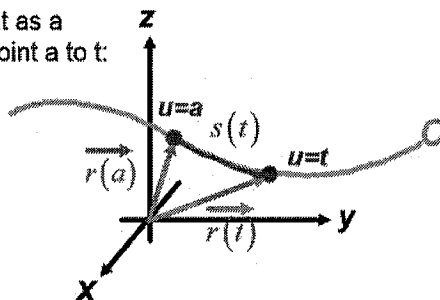$$= \left[\sqrt{2}(2\pi)\right] - \left[\sqrt{2}(0)\right]$$

$$\boxed{L = 2\sqrt{2}\pi}$$
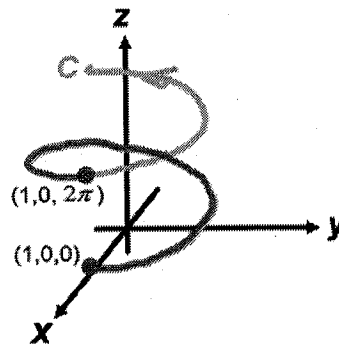
Arc Length Function

Now that we have a way to calculate length of the arc, we could write that as a function of the parameter t by considering the arc length from an initial point a to t:

$$s(t) = L = \int_a^t |\overrightarrow{r'(u)}| \, du = \int_a^t \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2} \, du$$

By the fundamental theorem of calculus part I: $g(x) = \int_0^x f(u) \, du$, then $g'(x) = f(x)$

differentiating both sides of the length expression above WRT t gives:

$$\frac{ds}{dt} = |\overrightarrow{r'(t)}|$$

Another way to say/remember this is that the magnitude of the tangent vector represents the rate of change of arc length as it moves along the curve (and we can find a function for arc length by taking the antiderivative of this derivative).

mainly used for this

Parametrizing a curve with respect to arc length

We can use the derivative expression to change the parameter to arc length which is sometimes useful because it doesn't depend upon having any particular coordinate system.

Here is how that would look with the helix example:

Ex) Reparametrize the helix $\overrightarrow{r(t)} = \langle \cos t, \sin t, t \rangle$

with respect to arc length measured from (1,0,0) in the direction of increasing t.
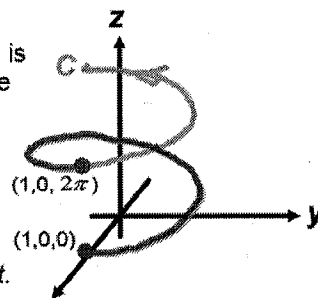
Starting with the magnitude of the tangent vector (from previous example):

$$\frac{ds}{dt} = |\overrightarrow{r'(t)}| = \sqrt{2}$$

$$s = s(t) = \int_0^t |r'(u)| \, du = \int_0^t \sqrt{2} \, du = \sqrt{2}t$$

So $s = \sqrt{2}t, \quad t = \frac{s}{\sqrt{2}}$

We then substitute this into the original vector expression:

$$r(t) = \langle \cos(t), \sin(t), t \rangle$$

$$\boxed{r(s) = \left\langle \cos\left(\frac{s}{\sqrt{2}}\right), \sin\left(\frac{s}{\sqrt{2}}\right), \frac{s}{\sqrt{2}} \right\rangle}$$

$\vec{r}(t) = \langle \cos t, \sin t, t \rangle$

$\vec{r'}(t) = \langle -\sin t, \cos t, 1 \rangle$

$|\vec{r'}(t)| = \sqrt{\sin^2 t + \cos^2 t + 1}$

$= \sqrt{2}$

$s = s(t) = \int_0^t \sqrt{2} \, du = \sqrt{2}t \quad t = \frac{s}{\sqrt{2}}$

$r(t) = \langle \cos t, \sin t, t \rangle$

$r(s) = \left\langle \cos\left(\frac{s}{\sqrt{2}}\right), \sin\left(\frac{s}{\sqrt{2}}\right), \frac{s}{\sqrt{2}} \right\rangle$

Arc length computation with reparametrized curve

If we now compute the arc length again, it should give us the same value:

$$r(s) = \left\langle \cos\left(\frac{s}{\sqrt{2}}\right), \sin\left(\frac{s}{\sqrt{2}}\right), \frac{s}{\sqrt{2}} \right\rangle$$

What are the new start/end values for s? (using the z coordinates again)

$$\frac{s}{\sqrt{2}} = 0 \Rightarrow s = 0 \qquad \frac{s}{\sqrt{2}} = 2\pi \Rightarrow s = 2\sqrt{2}\pi$$

$$r'(s) = \left\langle -\frac{1}{\sqrt{2}}\sin\left(\frac{s}{\sqrt{2}}\right), \frac{1}{\sqrt{2}}\cos\left(\frac{s}{\sqrt{2}}\right), \frac{1}{\sqrt{2}} \right\rangle$$

$$|r'(s)| = \sqrt{\left(-\frac{1}{\sqrt{2}}\sin\left(\frac{s}{\sqrt{2}}\right)\right)^2 + \left(\frac{1}{\sqrt{2}}\cos\left(\frac{s}{\sqrt{2}}\right)\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2} = \sqrt{\frac{1}{2}\sin^2\left(\frac{s}{\sqrt{2}}\right) + \frac{1}{2}\cos^2\left(\frac{s}{\sqrt{2}}\right) + \frac{1}{2}} = 1$$

$$L = \int_0^{2\sqrt{2}\pi} 1 \, ds = \boxed{2\sqrt{2}\pi}$$

The point is, there are many possible parametrizations for a curve, but the arc length is a physical property and will always be the same if computed correctly.
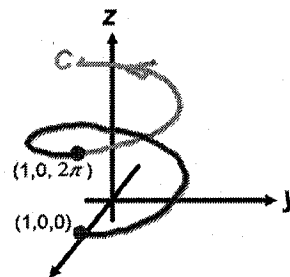
So what's the difference?

We can parametrize a curve using the vector function parameter, t, or we can parametrize a curve using this alternative variable, s...what is the difference?

If we parametrize using t and find the arc length...

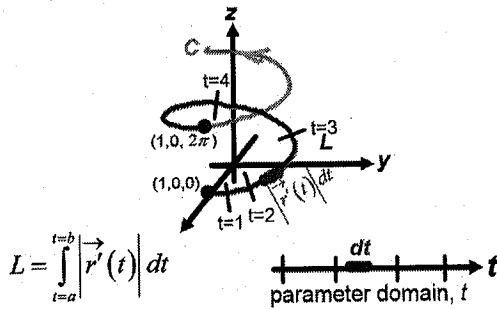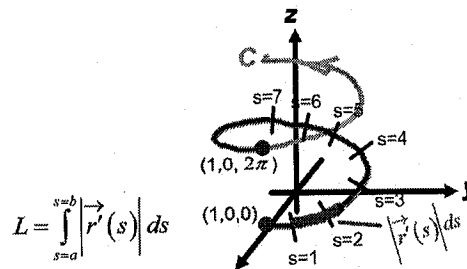$$L = \int_{t=a}^{t=b} \left|\overrightarrow{r'}(t)\right| dt$$

parameter domain, t

L is the arc length...

...and to find it, we take a small part of the domain, dt,

...multiply it by $\left|\overrightarrow{r'}(t)\right|$ which produces a small length along the curve, in the direction of the curve

...and then use an integral to sum these lengths = L

If we parametrize using s and find the arc length...

$$L = \int_{s=a}^{s=b} \left|\overrightarrow{r'}(s)\right| ds$$

L is still the arc length...

...and to find it, we take a small part of a new domain, ds, which is along the curve itself (like a new, curving, axis as the domain)

...multiply it by $\left|\overrightarrow{r'}(s)\right|$ which produces a small length along the curve, which scales each unit in this new s domain to an actual length along the curve

...and then use an integral to sum these lengths = L

Why would we do this?

Right now, there is no advantage to reparametrizing, so we won't do this. But later, we may need to sum other things up using an integral besides just the length of the curve itself.

For example, maybe we want to sum up the total electric field along a curved path, and we would have a function to express the electric field in a direction.

$$L = \int_{t=a}^{t=b} \left(some\ expression\ for\ electric\ field\right)\left|\overrightarrow{r'}(t)\right| dt$$

$$L = \int_{s=a}^{s=b} \left(some\ expression\ for\ electric\ field\right)\left|\overrightarrow{r'}(s)\right| ds$$

The expression for electric field might be simpler to write using the parameter t, or it might be simpler expressed as a function along the direction of the curve, s, which would make computing the integral simpler.

Curvature

How curvy is a curve?

gradual change in tangent direction (low curvature)

rapid change in tangent direction (high curvature)
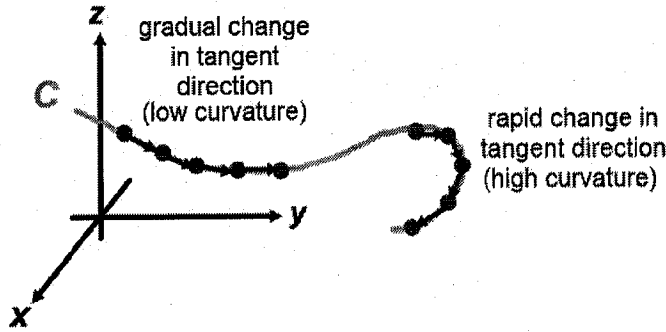
Curvature at a given point is a measure of how quickly the curve changed direction at that point, and is defined as the magnitude of the rate of change of the unit tangent vector with respect to arc length. (Arc length is used so curvature will be independent of the parametrization).

Definition:

$$\text{curvature} \quad \kappa = \left| \frac{d\vec{T}}{ds} \right|$$

'kappa'

Curvature is easier to compute if expressed in terms of t instead of s, so by the Chain Rule:

$$\frac{d\vec{T}}{dt} = \frac{d\vec{T}}{ds} \frac{ds}{dt}$$

$$\kappa(t) = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|}$$

In practice, this equivalent formula (proved in the textbook) is usually used instead:

$$\kappa(t) = \frac{|r'(t) \times r''(t)|}{|r'(t)|^3}$$

Ex) Find the curvature of $\vec{r}(t) = \langle t, t, 1+t^2 \rangle$

$$\vec{r}'(t) = \langle 1, 1, 2t \rangle$$

$$\vec{r}''(t) = \langle 0, 0, 2 \rangle$$

$$|\vec{r}'(t)|^3 = \left(\sqrt{1^2 + 1^2 + (2t)^2} \right)^3$$

$$= (4t^2 + 2)^3$$

$$\vec{r}'(t) \times \vec{r}''(t)$$

$$= \begin{vmatrix} + & - & + \\ 1 & 1 & 2t \\ 0 & 0 & 2 \end{vmatrix}$$

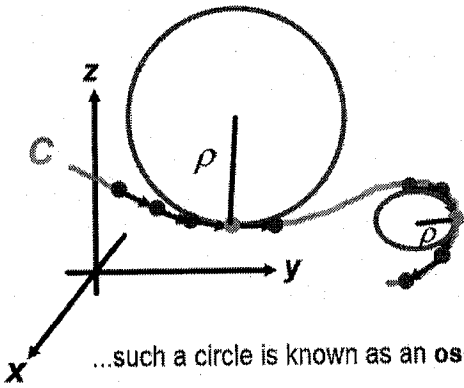$$= \langle 2-0, -(2-0), 0-0 \rangle$$

$$= \langle 2, -2, 0 \rangle$$

$$|\vec{r}'(t) \times \vec{r}''(t)| = \sqrt{2^2 + 2^2 + 0^2} = \sqrt{8}$$

Then curvature, $K = \dfrac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^3}$

$$\boxed{K = \frac{\sqrt{8}}{(4t^2+2)^3}}$$

Osculating circles - geometry interpretation of curvature

You can imagine creating a circle which is tangent to the curve at a point, and whose radius reflects a curve roughly 'as curvy' as the curve:
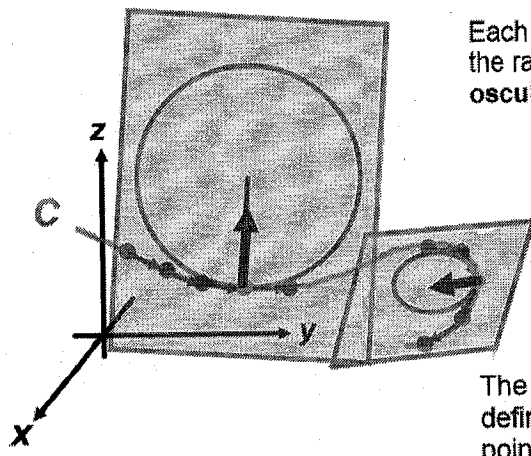
osculating
circle radius

$$\rho = \frac{1}{\kappa}$$
'rho'

...such a circle is known as an **osculating circle** (from the Latin *osculum*, meaning 'kiss').

You can imagine creating a circle which is tangent to the curve at a point, and whose radius reflects a curve roughly 'as curvy' as the curve:

Each osculating circle exists on a plane formed by the radius and the tangent vector, called the **osculating plane**

The **Normal** vector to a curve at a point is defined as in the direction of the radius from the point to the center of the osculating circle.

The TNB Frame and the 5 things we know at a point on a curve

The **Normal vector, $\vec{N}$**, points to the radius of curvature (is perpendicular to the tangent vector and in the osculating plane). It can be calculated from the unit tangent vector:

$$N(t) = \frac{\vec{T'}(t)}{|\vec{T'}(t)|}$$

The derivative of the position vector, tangent $\vec{r'}$, is in the direction of travel (direction of curve with t increasing).

The **Position vector, $\vec{r}$**, is from the origin to the point.

The **Unit Tangent vector, T**, is r', but normalized to unit length.

The **Binormal vector, B**, is defined as $\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$ and is perpendicular to the tangent and normal vectors.

The planes at a point on a curve

The Binormal and Normal vectors form the **Normal plane**, which is perpendicular to direction of movement.
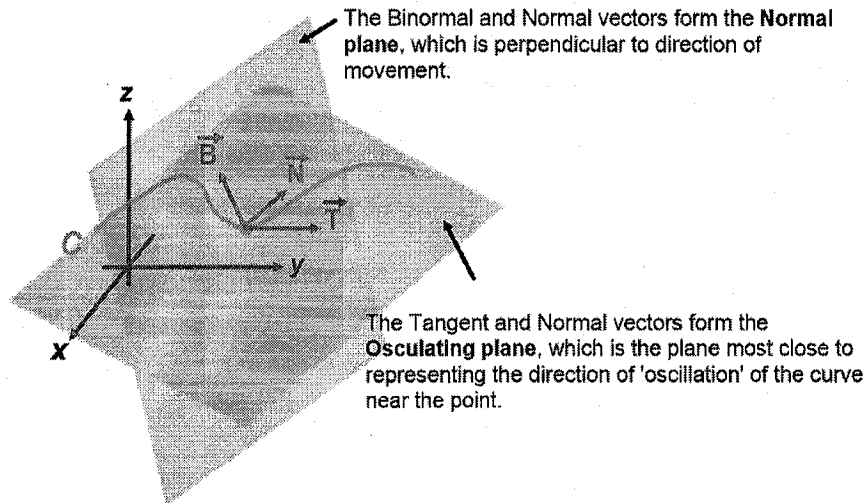
The Tangent and Normal vectors form the **Osculating plane**, which is the plane most close to representing the direction of 'oscillation' of the curve near the point.

Applets for viewing TNB frame

https://www.geogebra.org/m/kaGKS9BB

https://www.geogebra.org/m/rs4wmyex

Summary of formulas

$$\vec{T}(t) = \frac{\vec{r}'(t)}{\left|\vec{r}'(t)\right|} \qquad \vec{N}(t) = \frac{\vec{T}'(t)}{\left|\vec{T}'(t)\right|} \qquad \vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$$

$$\kappa(t) = \left|\frac{d\vec{T}}{ds}\right| = \frac{\left|\vec{T}'(t)\right|}{\left|\vec{r}'(t)\right|} = \frac{\left|\vec{r}'(t) \times \vec{r}''(t)\right|}{\left|\vec{r}'(t)\right|^3} \qquad \rho = \frac{1}{\kappa}$$

13.4: Motion in Space: Velocity and Acceleration

Velocity and Acceleration with vectors

Motion in one direction only

In earlier classes we said that velocity is the derivative of displacement and acceleration the derivative of velocity:

distance (displacement): $s(t) = t^2 - 5t + 8 \ m$

average velocity: $v_{avg} = \dfrac{s_2 - s_1}{t_2 - t_1} = \dfrac{s(t+h) - s(t)}{h}$
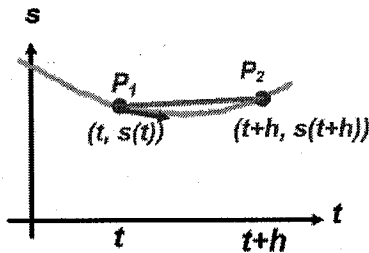
instantaneous velocity: $v(t) = \lim\limits_{h \to 0} \dfrac{s(t+h) - s(t)}{h} = s'(t)$

$v(t) = 2t - 5 \ m/s$

acceleration: $a(t) = v'(t)$

$a(t) = 2 \ m/s^2$
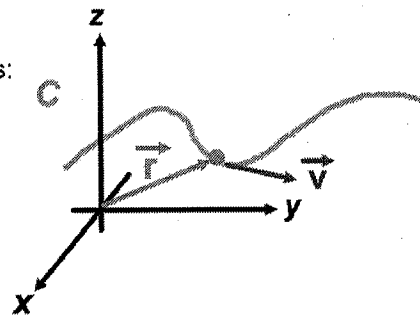
displacement
s

Motion in 3D

To analyze motion in 3 dimensions, we use vectors for the variables:

position: $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$

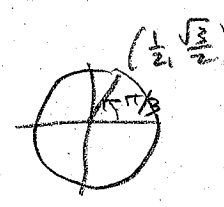**velocity:** $\vec{v}(t) = \vec{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle$

acceleration: $\vec{a}(t) = \vec{v}'(t) = \langle x''(t), y''(t), z''(t) \rangle$

Without vectors, we would have to compute effects in x, y, and z directions separately. Vectors allow us to do this in a single calculation.

Ex) Find the velocity and acceleration vectors for $\vec{r}(t) = \langle 3\cos t, 2\sin t, t \rangle$
and evaluate at $t = \dfrac{\pi}{3}$

$\left(\dfrac{1}{2}, \dfrac{\sqrt{3}}{2} \right)$

	at any t	*at* $t = \dfrac{\pi}{3}$
position: (derivatives)	$\vec{r}(t) = \langle 3\cos t, 2\sin t, t \rangle$	$\vec{r}\left(\dfrac{\pi}{3}\right) = \langle 3\cos\dfrac{\pi}{3}, 2\sin\dfrac{\pi}{3}, \dfrac{\pi}{3} \rangle$
		$= \langle \dfrac{3}{2}, \sqrt{3}, \dfrac{\pi}{3} \rangle$
velocity:	$\vec{v}(t) = \langle -3\sin t, 2\cos t, 1 \rangle$	$\vec{v}\left(\dfrac{\pi}{3}\right) = \langle -3\sin\dfrac{\pi}{3}, 2\cos\dfrac{\pi}{3}, 1 \rangle$
		$= \langle \dfrac{-3\sqrt{3}}{2}, 1, 1 \rangle$
acceleration:	$\vec{a}(t) = \langle -3\cos t, -2\sin t, 0 \rangle$	$\vec{a}\left(\dfrac{\pi}{3}\right) = \langle 3\cos\dfrac{\pi}{3}, -2\sin\dfrac{\pi}{3}, 0 \rangle$
		$= \langle -\dfrac{3}{2}, -\sqrt{3}, 0 \rangle$

Meaning/interpretation of velocity and acceleration

Ex) Find the velocity and acceleration vectors for $\vec{r}(t) = \langle 3\cos t, 2\sin t, t \rangle$

and evaluate at $t = \dfrac{\pi}{3}$

https://www.geogebra.org/m/rs4wmyex

position: $\vec{r}(t) = \langle 3\cos t, 2\sin t, t \rangle$

$$\vec{r}\left(\frac{\pi}{3}\right) = \left\langle \frac{3}{2}, \sqrt{3}, \frac{\pi}{3} \right\rangle$$

the position of the point in space

velocity: $\vec{v}(t) = \langle -3\sin t, 2\cos t, 1 \rangle$
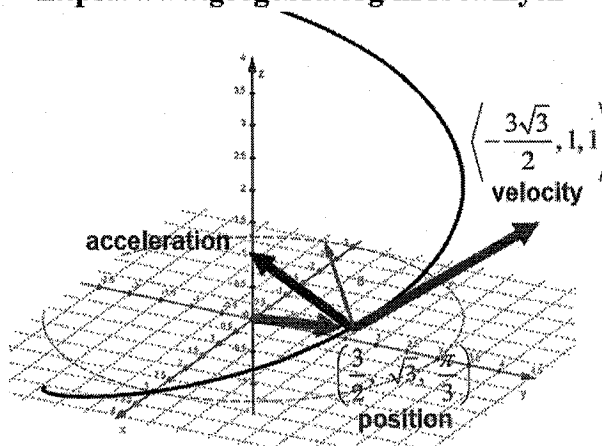
$$\vec{v}\left(\frac{\pi}{3}\right) = \left\langle \frac{-3\sqrt{3}}{2}, 1, 1 \right\rangle$$

the speed and direction of the point

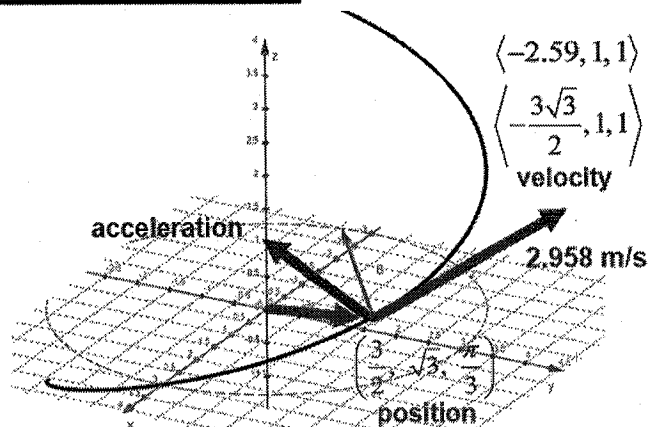acceleration: $\vec{a}(t) = \langle -3\cos t, -2\sin t, 0 \rangle$

$$\vec{a}\left(\frac{\pi}{3}\right) = \left\langle -\frac{3}{2}, -\sqrt{3}, 0 \right\rangle$$

related to the force which is acting on the point to cause the change in velocity

$\left\langle -\frac{3\sqrt{3}}{2}, 1, 1 \right\rangle$ velocity

acceleration

position $\left\langle \frac{3}{2}, \sqrt{3}, \frac{\pi}{3} \right\rangle$

The magnitude of velocity is speed:

$$speed = |\vec{v}| = \sqrt{\left(\frac{-3\sqrt{3}}{2}\right)^2 + (1)^2 + (1)^2} = 2.958 \ m/s$$

$\langle -2.59, 1, 1 \rangle$

$\left\langle \frac{3\sqrt{3}}{2}, 1, 1 \right\rangle$ velocity

acceleration

2.958 m/s

position $\left\langle \frac{3}{2}, \sqrt{3}, \frac{\pi}{3} \right\rangle$

acceleration: $a(t) = \langle -3\cos t, -2\sin t, 0 \rangle$

$$a\left(\frac{\pi}{3}\right) = \left\langle -\frac{3}{2}, -\sqrt{3}, 0 \right\rangle$$

related to the force which is acting on the point to cause the change in velocity and is in the direction of the Normal vector.

Acceleration is related to the force on an object by <u>Newton's Second Law of Motion</u>:

$$\vec{F} = m\vec{a}$$

$\left\langle -\frac{3\sqrt{3}}{2}, 1, 1 \right\rangle$ velocity

acceleration

2.958 m/s

position $\left\langle \frac{3}{2}, \sqrt{3}, \frac{\pi}{3} \right\rangle$
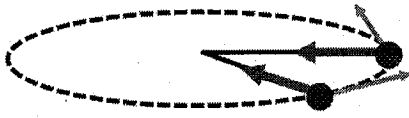
$$\vec{a} = \frac{\vec{F}}{m}$$

The amount that an object moves, when acted upon by a force, depends upon its *mass*, m. The more mass, the less change in velocity (smaller acceleration).
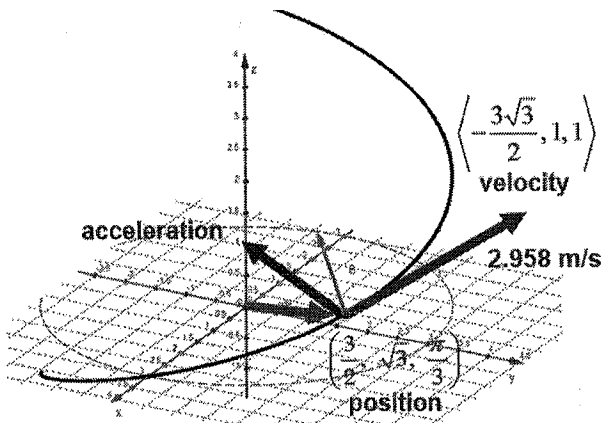
Meaning/interpretation of acceleration

This allows us to explain things like circular motion. Imagine a ball connected to a string, and twirling the ball around in a circle at constant speed:

Although the speed (magnitude) of the velocity vector is constant, the direction of velocity is constantly changing for the ball to move around the circle.

The acceleration vector is always towards the center of the circle and is caused by the force the string exerts on the ball.

$$\overrightarrow{F} = m\overrightarrow{a}$$

Projectile motion in 3D

Projectile motion is a special case of motion in which an object starts at some initial position, is 'launched' with an initial velocity, but then is only acted upon by the force of gravity, which is conventionally directed in the negative z direction.

Let's develop the equations for projectile motion by starting with an acceleration vector with a negative z component. This component is called the **gravitational constant, g**:

$$\overrightarrow{a}(t) = \langle 0, 0, -g \rangle$$

Acceleration is the derivative of velocity, so we can find velocity by taking the antiderivative:

$$\overrightarrow{v}(t) = \left\langle \int 0\, dt, \int 0\, dt, \int (-g)\, dt \right\rangle = \langle 0, 0, -gt \rangle + \overrightarrow{C}$$

The integration constant is the velocity when $t=0$, or the **initial velocity, $\overrightarrow{v_0}$**, which is also a vector:

$$\overrightarrow{v}(t) = \langle 0, 0, -gt \rangle + \overrightarrow{v_0} = \langle 0, 0, -gt \rangle + \langle v_{0x}, v_{0y}, v_{0z} \rangle$$

$$\overrightarrow{v}(t) = \langle v_{0x}, v_{0y}, v_{0z} - gt \rangle$$

Velocity is the derivative of position, so we can find position by taking the antiderivative:

$$\overrightarrow{r}(t) = \left\langle \int v_{0x} dt, \int v_{0y} dt, \int (v_{0z} - gt)\, dt \right\rangle + \overrightarrow{C}$$

Again, the integration constant is the position when $t=0$, or the **initial position, $\overrightarrow{r_0}$**, which is also a vector:

$$\overrightarrow{r}(t) = \left\langle v_{0x}t, v_{0y}t, v_{0z}t - \frac{1}{2}gt^2 \right\rangle + \overrightarrow{r_0} = \left\langle v_{0x}t, v_{0y}t, v_{0z}t - \frac{1}{2}gt^2 \right\rangle + \langle r_{0x}, r_{0y}, r_{0z} \rangle$$

The resulting vectors are:

$$\overrightarrow{a}(t) = \langle 0, \quad 0, \quad -g \rangle$$

$$\overrightarrow{v}(t) = \langle v_{0x}, \quad v_{0y}, \quad v_{0z} - gt \rangle$$

$$\overrightarrow{r}(t) = \left\langle r_{0x} + v_{0x}t, \quad r_{0y} + v_{0y}t, \quad r_{0z} + v_{0z}t - \frac{1}{2}gt^2 \right\rangle$$

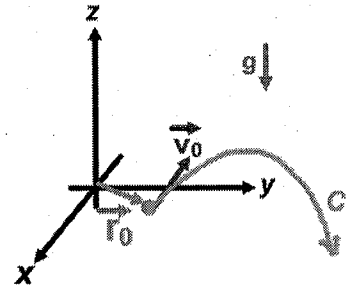Looking at the parametric equations for the position (the 'trajectory'):

$$x(t) = r_{0x} + v_{0x}t$$
$$y(t) = r_{0y} + v_{0y}t$$

...the object starts at an initial position, and in the x and y directions, continues moving steadily with whatever initial velocity is imparted.

$$z(t) = r_{0z} + v_{0z}t - \frac{1}{2}gt^2$$

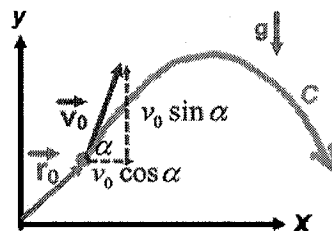But the z-position is affected both by the initial velocity and gravity.

Projectile motion in 2D

To keep things more manageable, we often consider cases of 2D motion (and then define gravity in the negative y direction). We can also define an **angle of elevation**, α, which allows us to specify the initial velocity with x and y components.

Now, acceleration is:

$$\vec{a}(t) = \langle 0, \ -g \rangle$$

Acceleration is the derivative of velocity, so we can find velocity by taking the antiderivative:

$$\vec{v}(t) = \left\langle \int 0\, dt, \int (-g)\, dt \right\rangle = \langle 0, -gt \rangle + \langle v_{0x}, v_{0y} \rangle$$

$$\vec{v}(t) = \langle v_0 \cos\alpha, \ v_0 \sin\alpha - gt \rangle$$

Velocity is the derivative of position, so we can find position by taking the antiderivative:

$$\vec{r}(t) = \left\langle \int (v_0 \cos\alpha)\, dt, \int (v_0 \sin\alpha - gt)\, dt \right\rangle = \left\langle (v_0 \cos\alpha)t, (v_0 \sin\alpha)t - \frac{1}{2}gt^2 \right\rangle + \langle r_{0x}, r_{0y} \rangle$$

$$\vec{r}(t) = \left\langle (v_0 \cos\alpha)t, (v_0 \sin\alpha)t - \frac{1}{2}gt^2 \right\rangle + \vec{r_0}$$

Units for Projectile motion

The numerical values depend upon whether we are working in metric or Imperial units:

	Imperial Units	Metric Units
distance:	r in ft	r in m
velocity:	v in ft/s	v in m/s
acceleration:	a in ft/s^2	a in m/s^2
mass:	m in slugs	m in kg
force:	F in lbs	F in N
gravitational constant:	g = 32 ft/s^2	g = 9.81 m/s^2

Ex) Find the velocity and position vectors of a particle that has the given acceleration and the given initial velocity and position: $\quad \vec{a}(t) = \langle 2, 6t, 12t^2 \rangle \quad \vec{v}(0) = \langle 1,0,0 \rangle \quad \vec{r}(0) = \langle 0,1,-1 \rangle$

$$\vec{v}(t) = \int \vec{a}(t)\, dt = \langle 2t, 3t^2, 4t^3 \rangle + \vec{v_0}$$

$$\boxed{\vec{v}(t) = \langle 2t+1, 3t^2, 4t^3 \rangle}$$

$$\vec{r}(t) = \int \vec{v}(t)\, dt = \langle t^2+t, t^3, t^4 \rangle + \vec{r_0}$$

$$\boxed{\vec{r}(t) = \langle t^2+t, t^3+1, t^4-1 \rangle}$$

Ex) A projectile is fired with an initial speed of 500 m/s and an angle of elevation of 30° from a starting position 200 m above the ground. Find...
(a) functions for position (height), velocity, and acceleration as functions of time,
(b) the range of the projectile,
(c) the maximum height reached, and
(d) the speed at impact.
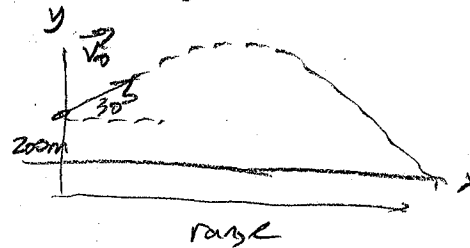
(a) $\vec{s}(0) = \langle 0, 200 \rangle$, $\vec{v}(0) = \langle 500\cos 30°, 500\sin 30° \rangle$
$$= \langle 433.0127, 250 \rangle$$

$$\boxed{\vec{a}(t) = \langle 0, -g \rangle = \langle 0, -9.81 \rangle \; m/s^2}$$

$\vec{v}(t) = \langle \int 0 \, dt, \int -9.81 \, dt \rangle = \langle 0 + c_1, -9.81t + c_2 \rangle$

at $t=0$: $\langle 433.0127, 250 \rangle = \langle c_1, -9.81(0) + c_2 \rangle$ $c_1 = 433.0127$
$c_2 = 250$

So $\boxed{\vec{v}(t) = \langle 433.0127, -9.81t + 250 \rangle \; m/s}$

$s(t) = \langle \int 433.9127 \, dt, \int (-9.81t + 250) \, dt \rangle$

$$= \langle 433.0127t + c_3, -4.905t^2 + 250t + c_4 \rangle$$

at $t=0$: $\langle 0, 200 \rangle = \langle 433.0127(0) + c_3, -4.905(0)^2 + 250(0) + c_4 \rangle$

$c_3 = 0, \quad c_4 = 200$

So $\boxed{s(t) = \langle 433.0127t, -4.905t^2 + 250t + 200 \rangle \; m}$

(b) range is x of position when height (y) = 0
height = 0 when $-4.905t^2 + 250t + 200 = 0$ (by calculator, when $t = 51.7 \, s$)
$x(51.7) = 433.0127(51.7) = \boxed{22386.756 \, m}$

(c) at max height $\frac{dy}{dt} = 0$ so y coordinate of velocity, $-9.81t + 250 = 0$
$9.81t = 250$, $t = \frac{250}{9.81} = 25.484 \, sec$

height at this time is position y: $-4.905(25.484)^2 + 250(25.484) + 200$
so max height = $\boxed{3385.525 \, m}$

(d) speed = |velocity| from (b), impact at $t = 51.7$
So $\vec{v}(51.7) = \langle 433.0127, -9.81(51.7) + 250 \rangle$
$$= \langle 433.0127, -257.177 \rangle$$

speed = $\sqrt{433.972^2 + 257.177^2} = \boxed{504.413 \, m/s}$

Homework hints:

#19) find expression for speed, and remember that things are minimized when 1st derivative = 0.

#22) to make this more manageable, consider a helical path: $r(t) = \langle \cos t, \sin t, t \rangle$

calculate T and N vectors, and show they are orthogonal by checking dot product = 0.

<u>for a difficult challenge</u>: if you want to show this is true for all cases, use this structure...
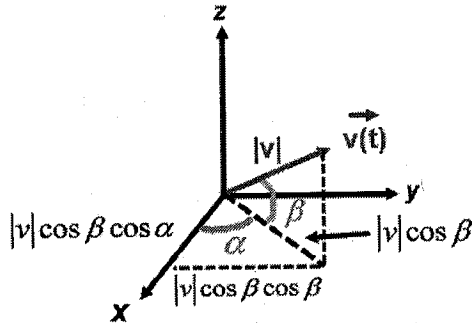
First show the projections of v are true as shown, then find the T and N vectors (you don't need to normalize). Remember, even though $|v|$ is a constant, α and β are functions of t.

#29) Use a coordinate system with catapult at origin:
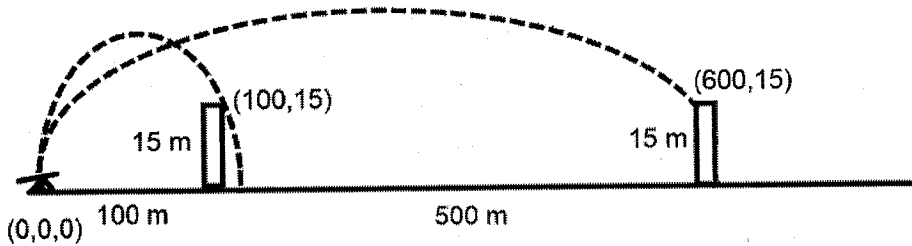
Solve two cases: find range of angles of elevation to just barely go far enough to get inside the first wall, and range of angles of elevation to just barely not go past second wall.

You should get a quadratic in $\tan \alpha$ if you use the trig identity: $\sec^2 \alpha = \tan^2 \alpha + 1$