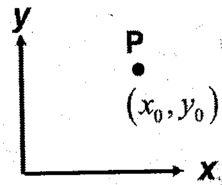
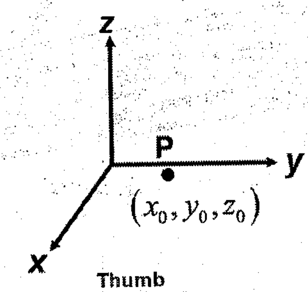
12.1 / 12.2: 3D-coordinate systems, intro to vectors

What makes this Calc '3' is that we are working with 3 or more variables, which correspond to 3 or more dimensions. (Mostly, we'll stick to 3D not higher dimensions)

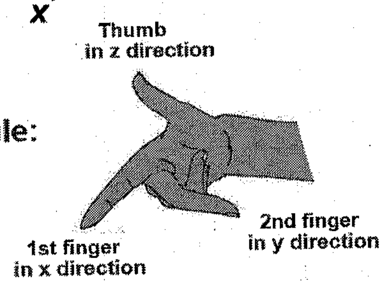
2D $\mathbb{R}^2$



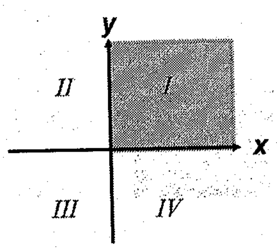
3D $\mathbb{R}^3$



The right-hand rule:

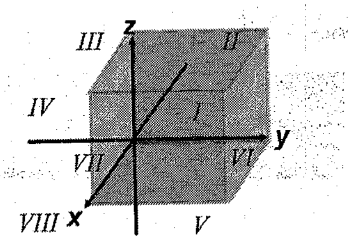


2D $\mathbb{R}^2$



4 quadrants

3D $\mathbb{R}^3$

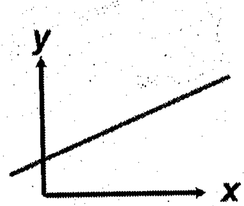


8 octants

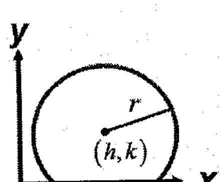
2D $\mathbb{R}^2$

$y = mx + b$ **is a <u>line</u>**

$x + 2y = 4$



$(x-h)^2 + (y-k)^2 = r^2$ **is a <u>circle</u>**

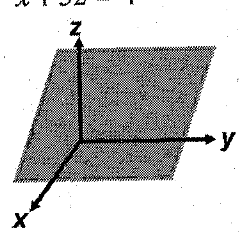


3D $\mathbb{R}^3$

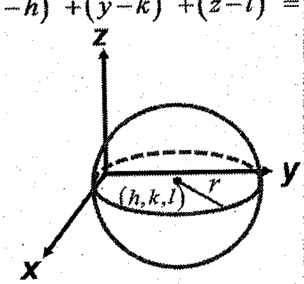
$y = mx + b$ **is a <u>plane</u>**

$x + 2y + 3z = 4$

$x + 3z = 4$



$(x-h)^2 + (y-k)^2 + (z-l)^2 = r^2$ **is a <u>sphere</u>**



2D $\mathbb{R}^2$

$(x-h)^2+(y-k)^2=r^2$ is a <u>circle</u>

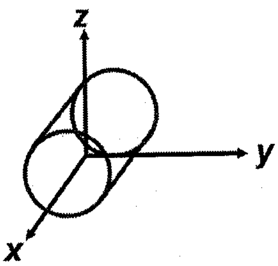
3D $\mathbb{R}^3$

$(x-h)^2+(y-k)^2+(z-l)^2=r^2$ is a <u>sphere</u>

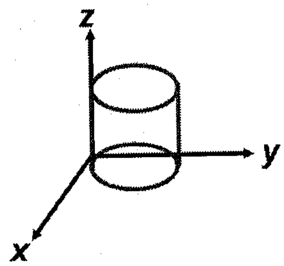
but $(x-h)^2+(y-k)^2=r^2$ is a <u>cylinder</u>

in $\mathbb{R}^3$ **...**

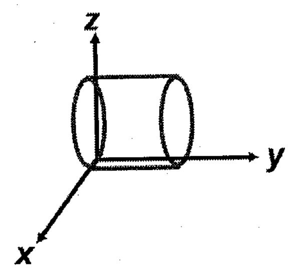
$(y-1)^2+(z-2)^2=4$
x can be anything...



$(x-1)^2+(y-2)^2=4$
z can be anything...



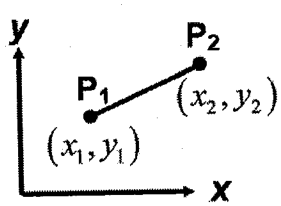
$(x-1)^2+(z-2)^2=4$
y can be anything...

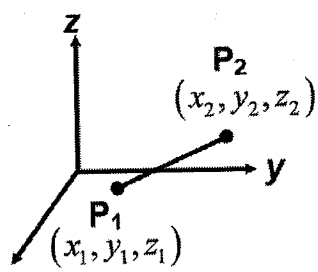


2D $\mathbb{R}^2$

3D $\mathbb{R}^3$

Distance between two points...





$$|P_1P_2|=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

$$|P_1P_2|=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$

Midpoint between two points...

$$\textit{midpoint of } P_1P_2=\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)$$

$$\textit{midpoint of } P_1P_2=\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2},\frac{z_1+z_2}{2}\right)$$

(review) Completing the square to put an equation in standard form:

Show that the equation represents a sphere, and find its center and radius:

$$4x^2+4y^2+4z^2+16y=8x$$

$$(4x^2-8x)+(4y^2+16y)+4z^2=0$$

$$4(x^2-2x+1)+4(y^2+4y+4)+4(z-0)^2=0+4+16$$

$$4(x-1)^2+4(y+2)^2+4(z-0)^2=20$$

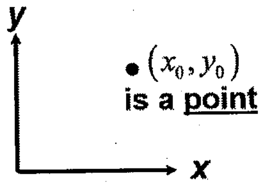
$$(x-1)^2+(y+2)^2+(z-0)^2=5$$

center: $(1,-2,0)$
radius: $\sqrt{5}$

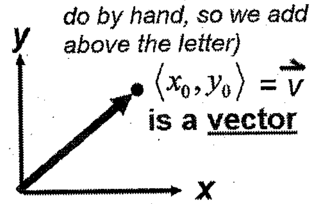
12.2 Vectors

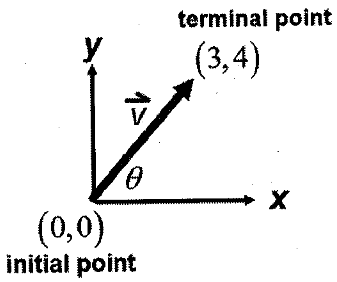
A **vector** is a directed line segment, and is characterized by its length and direction.

Let's go back to 2D to start...



textbook indicates a vector by making the letter **bold** *(too hard to do by hand, so we add an arrow above the letter)*





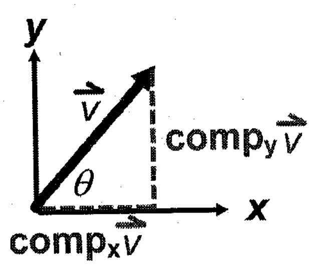
$$\vec{v} = \langle x_2 - x_1, y_2 - y_1 \rangle$$

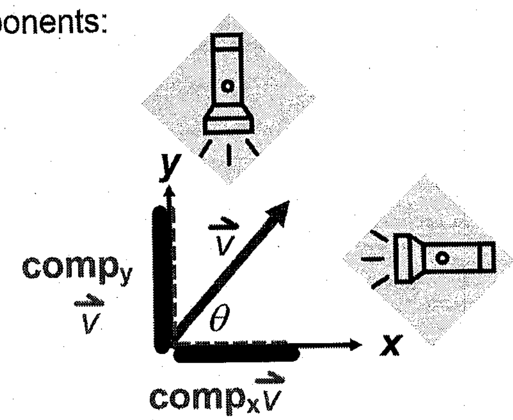
$$\vec{v} = \langle 3, 4 \rangle$$

length: $|\vec{v}| = \sqrt{3^2 + 4^2} = 5$

direction: $\theta = \tan^{-1}\left(\frac{4}{3}\right) = 53.15°$

The length of the vector along each axis direction is called a **component.** In 2D, each vector has two components:

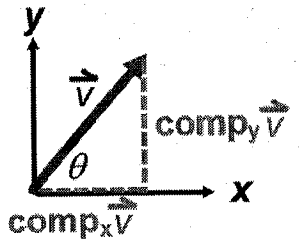




You can think of the components as being like the shadow of the vector on the x or y axis if you shined a flashlight on the vector.

This is called the **projection** of the vector onto the x-axis or y-axis.

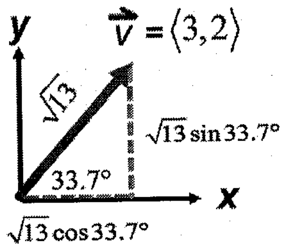
Converting between components and length, angle



$$\vec{v} = \langle comp_x, comp_y \rangle$$

$$|\vec{v}| = \sqrt{comp_x^{\ 2} + comp_y^{\ 2}} \qquad comp_x = |\vec{v}|\cos\theta$$

$$\theta = \tan^{-1}\left(\frac{comp_y}{comp_x}\right) \qquad comp_y = |\vec{v}|\sin\theta$$

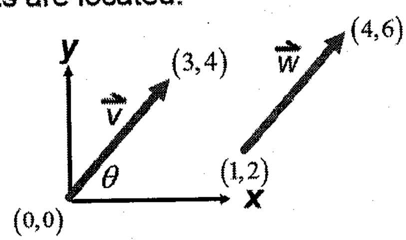


$$|\vec{v}| = \sqrt{3^2 + 2^2} = \sqrt{13} \qquad comp_x = \sqrt{13}\cos 33.7°$$

$$\theta = \tan^{-1}\left(\frac{2}{3}\right) = 33.7° \qquad comp_y = \sqrt{13}\sin 33.7°$$

Vector equality

Two vectors are considered 'equal' or 'the same vector' or 'equivalent' if their magnitudes and directions are the same, regardless of where the initial points are located:

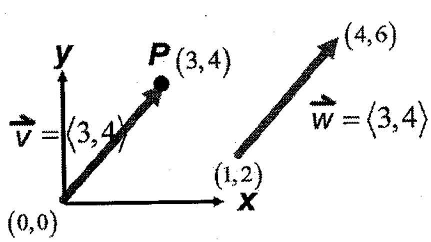


$$\vec{v} = \langle 3-0, 4-0 \rangle = \langle 3,4 \rangle \qquad \vec{w} = \langle 4-1, 6-2 \rangle = \langle 3,4 \rangle$$

$$\vec{v} = \vec{w}$$

Position vectors

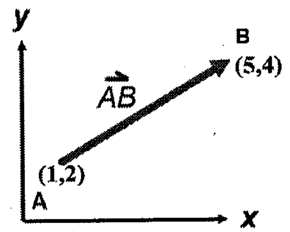
If the initial point of a vector is at the origin, it is called the **position vector** for the terminal point.



Vectors $\vec{v}$ and $\vec{w}$ are equivalent, but vector $\vec{v}$ is also a position vector for point *P*.

Finding a vector from 2 points

Ex) Find the vector from (1,2) to (5,4)

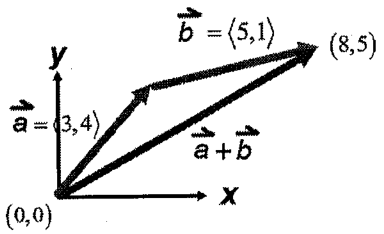


$$\overrightarrow{AB} = \langle 5-1, 4-2 \rangle$$
$$\overrightarrow{AB} = \langle 4,2 \rangle$$

Vector Addition

Adding two vectors is the equivalent of moving along the combined paths of both vectors to the new terminal point.

Geometric



Algebraic

$$\vec{a} = \langle 3,4 \rangle$$
$$\vec{b} = \langle 5,1 \rangle$$
$$\vec{a}+\vec{b} = \langle 3+5, 4+1 \rangle$$
$$\vec{a}+\vec{b} = \langle 8,5 \rangle$$

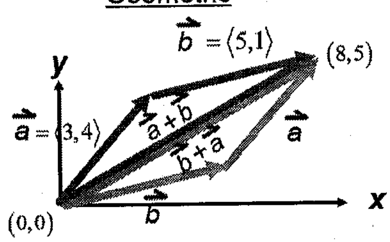
Placing one vector's tail to the other's tip results in a new terminal point for the addition vector

the 'triangle law'

Vector Addition is commutative

Reversing the order of the vectors being added gives the same result:

Geometric



Algebraic

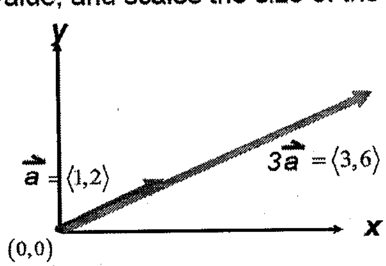
$$\vec{a} = \langle 3,4 \rangle$$
$$\vec{b} = \langle 5,1 \rangle$$
$$\vec{a}+\vec{b} = \langle 3+5, 4+1 \rangle$$
$$\vec{b}+\vec{a} = \langle 5+3, 1+4 \rangle$$

the 'parallelogram law'

the sum is also the diagonal of the parallelogram

Multiplying a vector by a scalar

Multiplying a vector by a scalar (number) multiplies all components by that value, and scales the size of the vector...

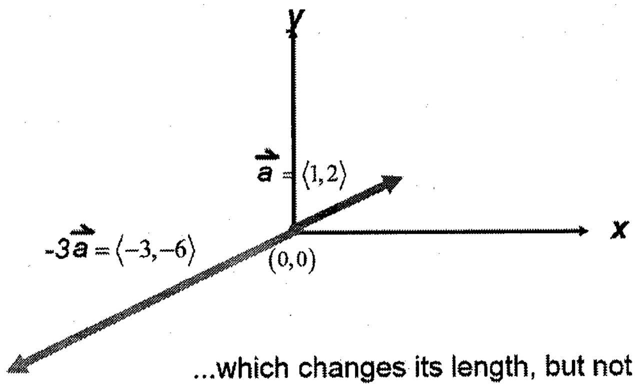


$$\vec{a} = \langle 1,2 \rangle$$
$$\overrightarrow{3a} = 3\langle 1,2 \rangle$$
$$\overrightarrow{3a} = \langle 3,6 \rangle$$

...which changes its length, but not its direction.

Negative vectors

However, if the scalar is negative, it changes the direction 180^0:



$\vec{a} = \langle 1,2 \rangle$

$-3\vec{a} = -3\langle 1,2 \rangle$

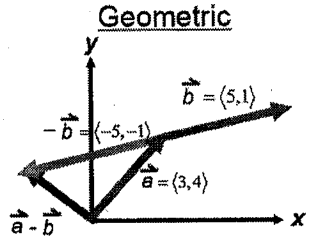
$-3\vec{a} = \langle -3,-6 \rangle$

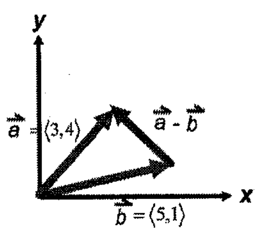
...which changes its length, but not its direction.

Vector Subtraction

Subtracting a vector is equivalent to adding a vector multiplied by -1:

Geometric





Algebraic

$\vec{a} = \langle 3,4 \rangle$

$\vec{b} = \langle 5,1 \rangle$

$\vec{a} - \vec{b} = \vec{a} + \left(-\vec{b}\right)$

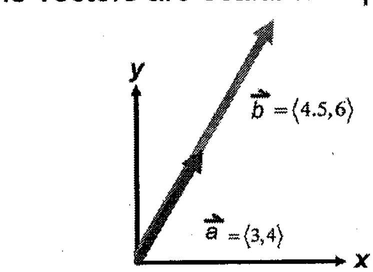
$\vec{a} - \vec{b} = \langle 3,4 \rangle + \langle -5,-1 \rangle$

$\vec{a} - \vec{b} = \langle -2,3 \rangle$

subtraction is also geometrically equivalent to combining vector 'tail-to-tail' (drawn from b to a)

Colinear vectors

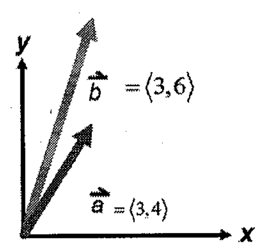
Vectors are called **colinear** if they are along the same line, which means that the vectors are scalar multiples of each other.



Compare scale factors for x and y components

$\frac{4.5}{3} = 1.5 \qquad \frac{6}{4} = 1.5$

$\vec{b}$ is a scalar multiple (1.5) of $\vec{a}$ so these vectors are colinear



Compare scale factors for x and y components

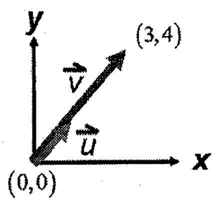
$\frac{3}{3} = 1 \qquad \frac{6}{4} = 1.5$

there isn't a single scalar multiple so $\vec{b}$ is not a scalar multiple (1.5) of $\vec{a}$ and these vectors are not colinear

Unit vectors

Unit vectors have a magnitude = 1

Ex) Find a unit vector $\vec{u}$ in the direction of $\vec{v}$



$$\vec{v} = \langle 3-0, 4-0 \rangle = \langle 3,4 \rangle$$

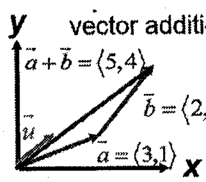
$$|\vec{v}| = \sqrt{3^2+4^2} = 5$$

divide by the length...

$$\vec{u} = \frac{\langle 3,4 \rangle}{5} = \frac{1}{5}\langle 3,4 \rangle = \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle$$

All these ideas are extendable into any number of dimensions

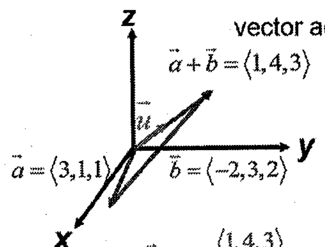
$\mathbb{R}^2$



$$\vec{u} = \frac{\langle 5,4 \rangle}{\sqrt{5^2+4^2}} = \frac{1}{\sqrt{41}}\langle 5,4 \rangle$$

unit vector in the direction of the sum vector

$\mathbb{R}^3$



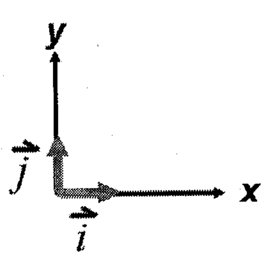
$$\vec{u} = \frac{\langle 1,4,3 \rangle}{\sqrt{1^2+4^2+3^2}} = \frac{1}{\sqrt{26}}\langle 1,4,3 \rangle$$

unit vector in the direction of the sum vector

Basis vectors

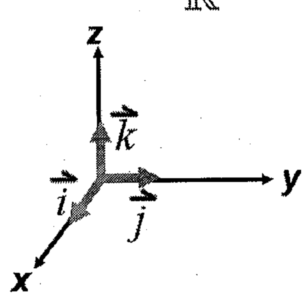
When unit vectors are in the direction of the axes, they are called **basis vectors**, and are sometimes given special symbols:

$\mathbb{R}^2$



$$\vec{i} = \langle 1,0 \rangle$$
$$\vec{j} = \langle 0,1 \rangle$$

$\mathbb{R}^3$



$$\vec{i} = \langle 1,0,0 \rangle$$
$$\vec{j} = \langle 0,1,0 \rangle$$
$$\vec{k} = \langle 0,0,1 \rangle$$

Other notation for vectors using basis vectors

Many textbooks, including ours, use an alternate notation for writing a vector:

$\mathbb{R}^2$	$\mathbb{R}^3$
$\vec{v} = \langle 3,2 \rangle$	$\vec{v} = \langle 3,2,4 \rangle$
$\vec{v} = 3\vec{i} + 2\vec{j}$	$\vec{v} = 3\vec{i} + 2\vec{j} + 4\vec{k}$

You need to know both of these methods, but we'll stick to the brackets in this class (a clearer notation).

$\langle 3,2,4 \rangle$ Angle brackets denote a **vector**

$(3,2,4)$ Curved parentheses denote a **point**

Properties of vectors

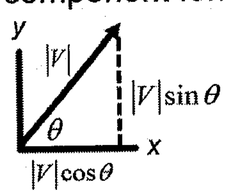
PROPERTIES OF VECTORS If **a**, **b**, and **c** are vectors in V_n and c and d are scalars, then

1. $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$
2. $\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$
3. $\mathbf{a} + \mathbf{0} = \mathbf{a}$
4. $\mathbf{a} + (-\mathbf{a}) = \mathbf{0}$
5. $c(\mathbf{a} + \mathbf{b}) = c\mathbf{a} + c\mathbf{b}$
6. $(c + d)\mathbf{a} = c\mathbf{a} + d\mathbf{a}$
7. $(cd)\mathbf{a} = c(d\mathbf{a})$
8. $1\mathbf{a} = \mathbf{a}$

Applications of vectors

Things to know...

- Rewrite all vectors in component form (make sure angle is from standard position):



$$\vec{V} = \langle |V|\cos\theta,\ |V|\sin\theta \rangle$$

- If objects are not moving, then the sum of all force vectors = 0.

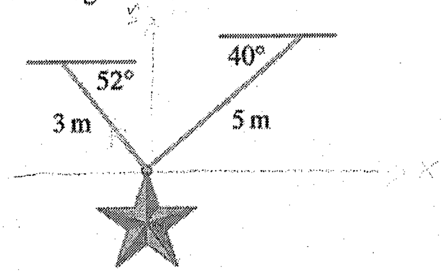
$$\sum H(\textit{horiz components}) = 0$$
$$\sum V(\textit{vert components}) = 0$$

- Weight is a force (not mass): F=ma, W=mg
- Units:

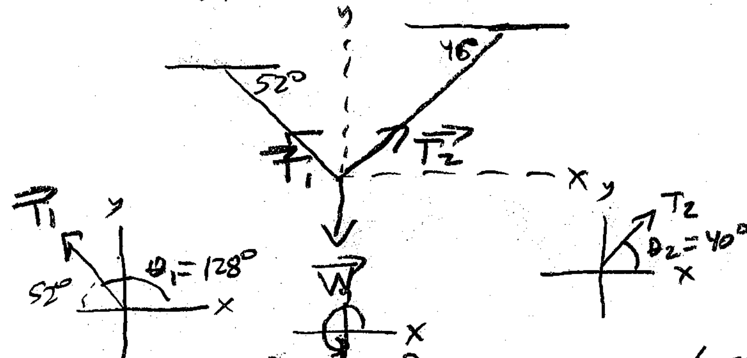
	m	W	g
Imperial	slugs	lbs	32.2 ft/sec^2
metric	kg	N	9.81 m/sec^2

Examples

32. Ropes 3 m and 5 m in length are fastened to a holiday decoration that is suspended over a town square. The decoration has a mass of 5 kg. The ropes, fastened at different heights, make angles of 52° and 40° with the horizontal. Find the tension in each wire and the magnitude of each tension.



here, the vectors are tensions (forces) – length of wire is irrelevant



$|\vec{W}| = mg = 5(9.8) = 49\ N$

object is not moving, so $\Sigma \vec{F} = 0$

$$\vec{T_1} + \vec{T_2} + \vec{W} = \vec{0}$$

$$\langle |T_1|\cos 128°, |T_1|\sin 128° \rangle + \langle |T_2|\cos 40°, |T_2|\sin 40° \rangle + \langle 0, -49 \rangle = \langle 0, 0 \rangle$$

$$\Sigma x = 0: \quad |T_1|\cos 128° + |T_2|\cos 40° + 0 = 0$$
$$\Sigma y = 0: \quad |T_1|\sin 128° + |T_2|\sin 40° - 49 = 0$$

$$\begin{cases} (\cos 128°)|T_1| + (\cos 40°)|T_2| = 0 \\ (\sin 128°)|T_1| + (\sin 40°)|T_2| = 49 \end{cases}$$

solve system by RREF!

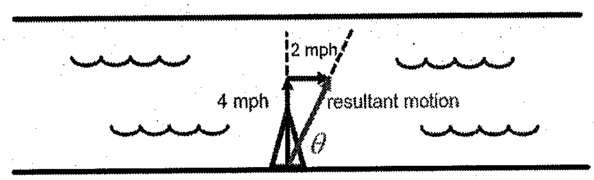
$$\left[\begin{array}{cc|c} (\cos 128°) & (\cos 40°) & 0 \\ (\sin 128°) & (\sin 40°) & 49 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{cc|c} 1 & 0 & 37.55906 \\ 0 & 1 & 30.1858 \end{array}\right] \begin{array}{l} = |T_1| \\ = |T_2| \end{array}$$

$$\vec{T_1} = \langle 37.55906\cos 128°, 37.55906\sin 128° \rangle = \langle -23.124, 29.597 \rangle\ N$$
$$\vec{T_2} = \langle 30.1858\cos 40°, 30.1858\sin 40° \rangle = \langle 23.124, 19.403 \rangle\ N$$

sums to: $\langle 0, 49 \rangle$

to exactly balance the weight force

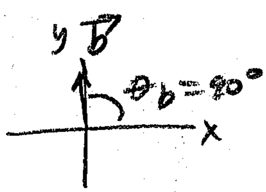
A boat heads straight across a river at a speed of 4 mph, but the water in the river is flowing a 2 mph (as in the figure). What is the resultant and direction of the boat?



Here, the object is moving, so the vectors represent velocities and to find the overall motion, we just sum the individual vectors...

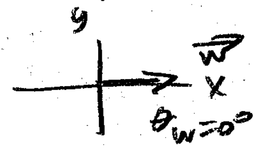
resultant $\vec{r} = \underset{(boat)}{\vec{b}} + \underset{(water)}{\vec{w}}$

boat:



$\vec{b} = \langle 4\cos 90°, 4\sin 90° \rangle = \langle 0, 4 \rangle$

water:



$\vec{w} = \langle 2\cos 0°, 2\sin 0° \rangle = \langle 2, 0 \rangle$

resultant, $\vec{r} = \vec{b} + \vec{w}$

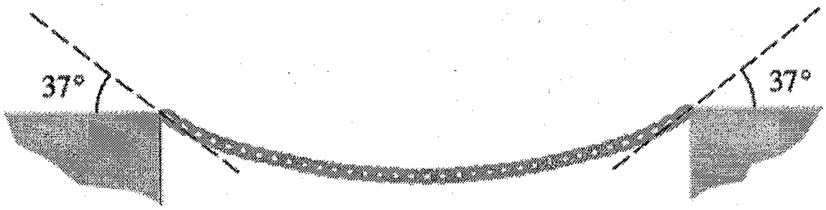
$\vec{r} = \langle 0,4 \rangle + \langle 2,0 \rangle = \boxed{\langle 2,4 \rangle}$

direction: $\tan\theta = \frac{y}{x} = \frac{4}{2} = 2$

$\theta = \tan^{-1}(2) = \boxed{63.4348°}$

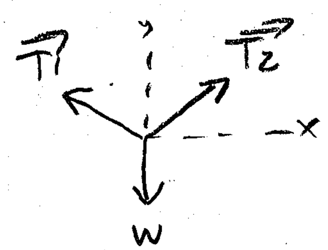
speed $= |\vec{v}| = |\langle 2,4 \rangle| = \sqrt{2^2+4^2} = \sqrt{20} = \boxed{4.472 \text{ mph}}$

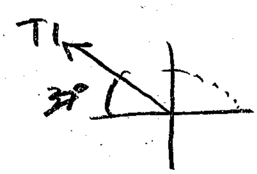
34. The tension **T** at each end of the chain has magnitude 25 N. What is the weight of the chain?

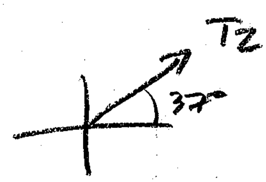


(Setup only, finish in homework...)

you can model entire chain as a force acting on a point, and due to symmetry can assume this force acts in center of the chain:







$$\vec{T}_1 + \vec{T_2} + \vec{w} = \vec{0}$$

$$\langle 25\cos 143°, 25\sin 143° \rangle + \langle 25\cos 37°, 25\sin 37° \rangle + \vec{w} = \vec{0}$$

Solve the system for $\vec{w}$, then find $|\vec{w}|$

12.3: Dot Product

We've talked about how to add and subtract vectors and multiply a vector by a scalar. How do we multiply two vectors together? There are two different ways:

Dot product	**Cross product**
$\vec{a} \bullet \vec{b}$	$\vec{a} \times \vec{b}$
the result is a <u>scalar</u> (number)	**the result is a <u>vector</u>** (covered in the next section

A dot product example...

$\vec{a} = \langle 1, -2, 0 \rangle$

$\vec{b} = \langle 3, 2, 4 \rangle$

$\vec{a} \bullet \vec{b} = (1)(3) + (-2)(2) + (0)(4)$

$= 3 - 4 + 0$

$= \boxed{-1}$

Dot product is defined to be...

$$\vec{a} \bullet \vec{b} = \langle a_1, a_2, a_3 \rangle \bullet \langle b_1, b_2, b_3 \rangle = a_1 b_1 + a_2 b_2 + a_3 b_3$$

Dot product is also known as the 'scalar product' or 'inner product'

Properties of the Dot Product

2 PROPERTIES OF THE DOT PRODUCT If **a**, **b**, and **c** are vectors in V_3 and c is a scalar, then

1. $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$
2. $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$
3. $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$
4. $(c\mathbf{a}) \cdot \mathbf{b} = c(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (c\mathbf{b})$
5. $\mathbf{0} \cdot \mathbf{a} = 0$

The dot product has properties similar to those for multiplying real numbers.

Dot product can be used to find the angle between 2 vectors

Given the definition of dot product, using the Law of Cosines and properties of the dot product it can be proved that (see textbook for proof):

$$\vec{a} \bullet \vec{b} = |\vec{a}|\,|\vec{b}| \cos(\theta)$$

where θ is the angle between the vectors

and

$$\cos(\theta) = \frac{\vec{a} \bullet \vec{b}}{|\vec{a}|\,|\vec{b}|}$$

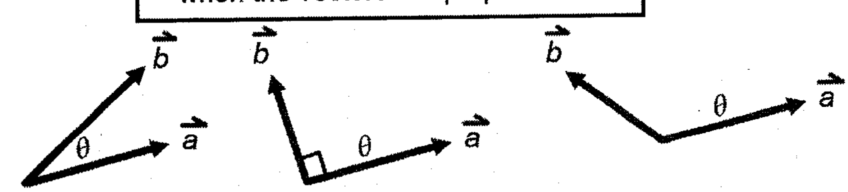
Dot product can be used to test whether vectors are perpendicular

If two vectors are perpendicular, the angle between them is 90°.

$$\vec{a} \bullet \vec{b} = |\vec{a}|\,|\vec{b}| \cos(90^\circ)$$

$$\vec{a} \bullet \vec{b} = 0$$

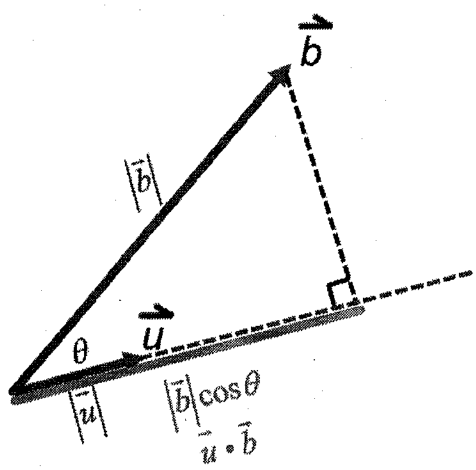
when the vectors are perpendicular.



$\vec{a} \bullet \vec{b}$ is positive $\quad \vec{a} \bullet \vec{b} = 0 \quad \vec{a} \bullet \vec{b}$ is negative

(To show whether vectors are parallel, test is one is a scalar multiple of the other)

Geometric interpretation of dot product

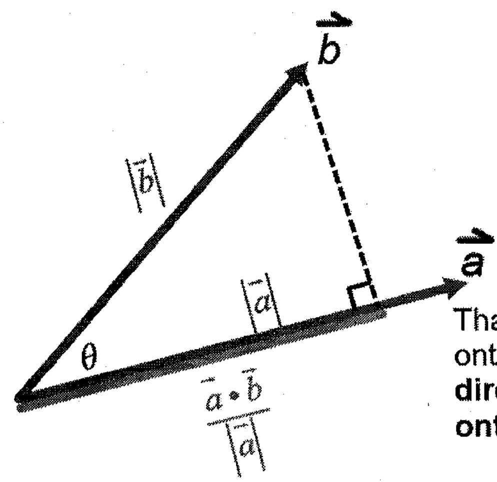


If $\vec{u}$ is a unit vector:

$$\vec{u} \bullet \vec{b} = |\vec{u}||\vec{b}|\cos\theta = (1)|\vec{b}|\cos\theta$$

$$\vec{u} \bullet \vec{b} = |\vec{b}|\cos\theta$$

...then the dot product of $\vec{b}$ and $\vec{u}$ represents the component of the $\vec{b}$ vector that points in the $\vec{u}$ direction.



For a non-unit vector, $\vec{a}$:

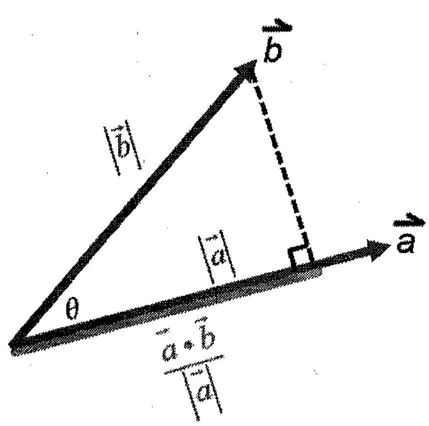
$$\vec{a} \bullet \vec{b} = |\vec{a}||\vec{b}|\cos\theta$$

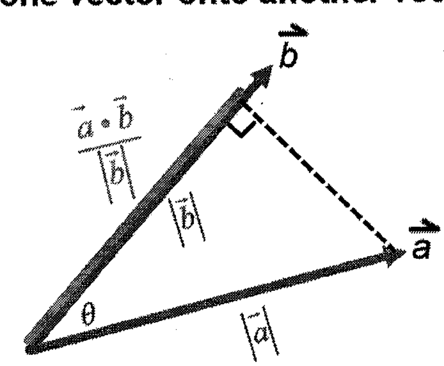
...and the dot product still represents the component of the $\vec{b}$ vector that points in the $\vec{a}$ direction, but multiplied by the magnitude of $\vec{a}$.

That means the length of the projection of $\vec{b}$ onto $\vec{a}$ (called the '**component of $\vec{b}$ in the direction of $\vec{a}$**' or the '**scalar projection of $\vec{b}$ onto $\vec{a}$**') is given by:

$$comp_a \vec{b} = \frac{\vec{a} \bullet \vec{b}}{|\vec{a}|}$$

Component (scalar projection) of one vector onto another vector

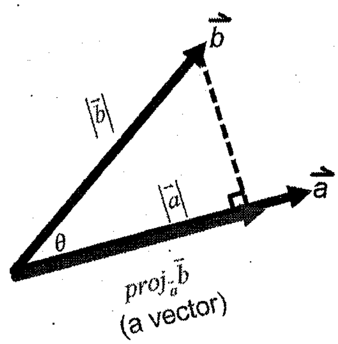




By dividing the dot product by the magnitude of a vector you are finding the scalar projection of the *other* vector onto that vector.

Vector projection of one vector onto another vector

The scalar projection of $\vec{b}$ onto $\vec{a}$ is a scalar (number) representing the length of $\vec{b}$ in the direction of $\vec{a}$. If you multiply this scalar projection by a unit vector in the direction of $\vec{a}$...



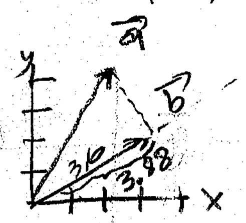
$$proj_{\vec{a}}\vec{b} = \left(\frac{\vec{a}\bullet\vec{b}}{|\vec{a}|}\right)\frac{\vec{a}}{|\vec{a}|}$$

...you get a vector called the **vector projection of $\vec{b}$ onto $\vec{a}$** which represents the component of $\vec{b}$ in the direction of $\vec{a}$ (with the direction of $\vec{a}$ preserved).

A 2D example...

$\vec{a} = \langle 2, 4\rangle$

$\vec{b} = \langle 3, 2\rangle$



Find scalar projection of $\vec{a}$ onto $\vec{b}$

$|\vec{b}| = \sqrt{3^2+2^2} = \sqrt{13} = 3.6$

$\vec{u}_b = \langle \frac{3}{\sqrt{13}}, \frac{2}{\sqrt{13}} \rangle$

$\vec{a}\cdot\vec{u}_b = \langle 2,4\rangle \cdot \langle \frac{3}{\sqrt{13}}, \frac{2}{\sqrt{13}}\rangle$

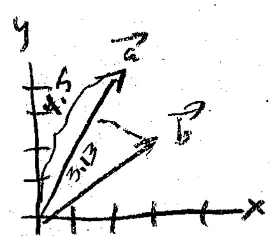
$= (2)(\frac{3}{\sqrt{13}}) + (4)(\frac{2}{\sqrt{13}}) = \frac{14}{\sqrt{13}} = 3.88$

Find vector projection of $\vec{a}$ onto $\vec{b}$

$\frac{14}{\sqrt{13}} \langle \frac{3}{\sqrt{13}}, \frac{2}{\sqrt{13}} \rangle$

$\langle \frac{42}{13}, \frac{28}{13} \rangle$

$\langle 3.2, 2.15 \rangle$



Find scalar projection of $\vec{b}$ onto $\vec{a}$

$|\vec{a}| = \sqrt{2^2+4^2} = \sqrt{20} = 4.5$

$\vec{u}_a = \langle \frac{2}{\sqrt{20}}, \frac{4}{\sqrt{20}} \rangle$

$\vec{b}\cdot\vec{u}_a = \langle 3,2\rangle \cdot \langle \frac{2}{\sqrt{20}}, \frac{4}{\sqrt{20}}\rangle$

$= (3)(\frac{2}{\sqrt{20}}) + (2)(\frac{4}{\sqrt{20}})$

$= \frac{14}{\sqrt{20}} = 3.13$

Find vector projection of $\vec{b}$ onto $\vec{a}$

$\frac{14}{\sqrt{20}} \langle \frac{2}{\sqrt{20}}, \frac{4}{\sqrt{20}} \rangle$

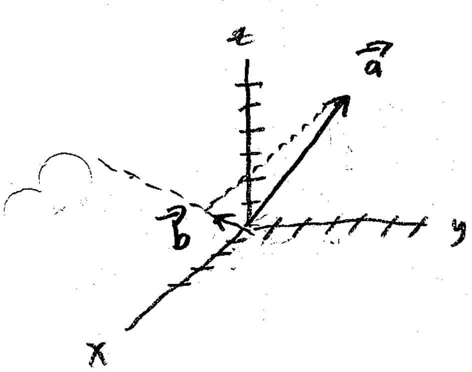
$\langle \frac{28}{20}, \frac{56}{20} \rangle$

$\langle 1.4, 2.8 \rangle$

A 3D example...

$\vec{a} = \langle 1, 4, 6\rangle$

$\vec{b} = \langle 3, 1, 2\rangle$



Find scalar projection of $\vec{a}$ onto $\vec{b}$

$|\vec{b}| = \sqrt{3^2+1^2+2^2} = \sqrt{14} = 3.7$

$\vec{u}_b = \langle \frac{3}{\sqrt{14}}, \frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}} \rangle$

$\vec{a}\cdot\vec{u}_b = \langle 1,4,6\rangle \cdot \langle \frac{3}{\sqrt{14}}, \frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}\rangle$

$= (1)(\frac{3}{\sqrt{14}}) + (4)(\frac{1}{\sqrt{14}}) + (6)(\frac{2}{\sqrt{14}})$

$= \frac{19}{\sqrt{14}} = 5.08$

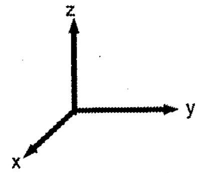
Find vector projection of $\vec{a}$ onto $\vec{b}$

$\frac{19}{\sqrt{14}} \langle \frac{3}{\sqrt{14}}, \frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}} \rangle$

$\langle \frac{27}{14}, \frac{19}{14}, \frac{38}{14} \rangle$

Physics application of dot product: Work

In physics, **work** is defined as the product of the component of a force applied to an object in a direction and the distance the object is moved in that direction.



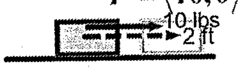
$$Work = comp_{\vec{F}}\vec{D} = \left(\left|\vec{F}\right|\cos\theta\right)\left|\vec{D}\right| = \vec{F}\bullet\vec{D}$$

Examples...

Find the work done if a 10 lb force is exerted on a block by a rope which pulls the block horizontally 2 ft...

(force vector)

$\vec{F} = \langle 10, 0\rangle$

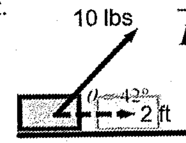


$\vec{D} = \langle 2, 0\rangle$

(displacement vector)

$$Work = \vec{F}\bullet\vec{D} = \langle 10,0\rangle\bullet\langle 2,0\rangle$$
$$= (10)(2)+(0)(0)$$
$$= 20\ ft-lbs$$

Find the work done if a 10 lb force is exerted on a block by a rope which pulls in a direction 42° above the horizontal and moves the block 2 ft.

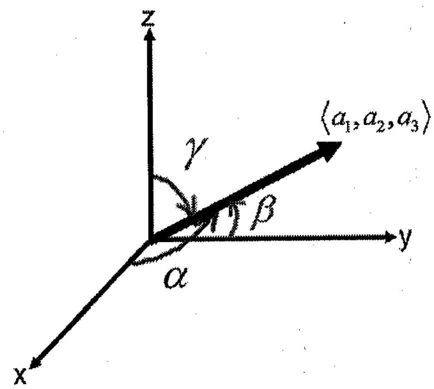


$$\vec{F} = \langle 10\cos 42°, 10\sin 42°\rangle$$
$$= \langle 7.431, 6.691\rangle$$

$\vec{D} = \langle 2, 0\rangle$

$$Work = \vec{F}\bullet\vec{D} = \langle 7.431, 6.691\rangle\bullet\langle 2,0\rangle$$
$$= (7.431)(2)+(6.691)(0)$$
$$= 14.862\ ft-lbs$$

Direction Angles and Direction Cosines



The angles between a vector and the axes are called **direction angles:**

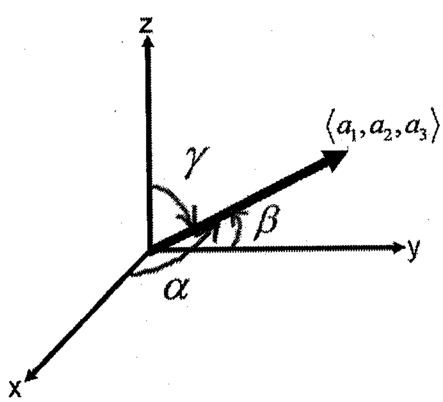
x-axis: alpha α
y-axis: beta β
z-axis: gamma γ

...and the cosines of these angles are called the **direction cosines.**

Because:

$$\cos(\theta) = \frac{\vec{a} \bullet \vec{b}}{|\vec{a}||\vec{b}|} \qquad \cos(\alpha) = \frac{\vec{a} \bullet \vec{i}}{|\vec{a}||\vec{i}|} = \frac{\langle a_1, a_2, a_3 \rangle \cdot \langle 1,0,0 \rangle}{|\vec{a}|(1)} = \frac{a_1}{|\vec{a}|}$$

which is equivalent to the x-component of a unit vector in the direction of $\vec{a}$.



$$\cos(\alpha) = \frac{\vec{a} \bullet \vec{i}}{|\vec{a}||\vec{i}|} = \frac{\langle a_1, a_2, a_3 \rangle \cdot \langle 1,0,0 \rangle}{|\vec{a}|(1)} = \frac{a_1}{|\vec{a}|}$$

..meaning the direction cosines are the components of the unit vector in the direction of $\vec{a}$:

$$\frac{a}{|\vec{a}|} = \frac{\langle a_1, a_2, a_3 \rangle}{|\vec{a}|} = \langle \cos\alpha, \cos\beta, \cos\gamma \rangle$$

12.4: Cross Product

We've talked about how to add and subtract vectors and multiply a vector by a scalar. How do we multiply two vectors together? There are two different ways:

Dot product

$\vec{a} \bullet \vec{b}$

the result is a scalar (number)
(covered in the last section)

Dot product is defined for 2D, 3D+

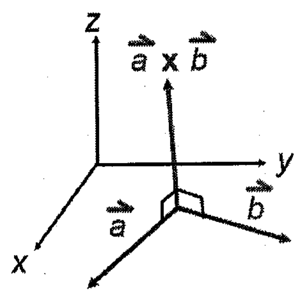
Cross product

$\vec{a} \times \vec{b}$

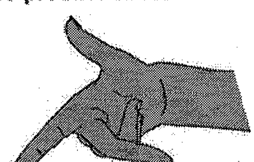
the result is a vector

Cross product is defined only for 3D (in this class)

Multiplying two 3D vectors to form the cross product creates a new vector which is perpendicular to both original vectors...



...and whose direction is found using the right-hand-rule:



Definition of how to find the components of the cross-product vector:

$$\text{If } \vec{a} = \langle a_1, a_2, a_3 \rangle \text{ and } \vec{b} = \langle b_1, b_2, b_3 \rangle$$
$$\text{then } \vec{a} \times \vec{b} = \langle a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_1b_2 - a_2b_1 \rangle$$

But in practice we use an equivalent determinant to compute cross-product:

$$\begin{matrix} (+) & (-) & (+) \end{matrix}$$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\vec{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\vec{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\vec{k}$$

$$= \left\langle \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}, -\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}, \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \right\rangle$$

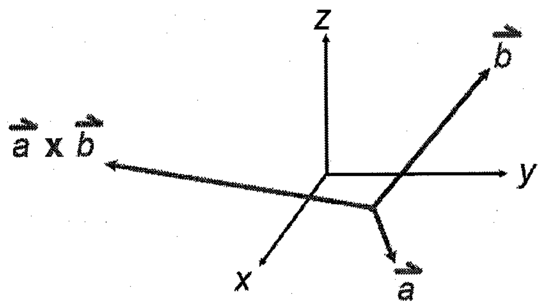
Ex: Given $\vec{a} = \langle 1, 1, -1 \rangle$ $\vec{b} = \langle 2, 4, 6 \rangle$

Find $\vec{a} \times \vec{b}$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & -1 \\ 2 & 4 & 6 \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 4 & 6 \end{vmatrix} \vec{i} - \begin{vmatrix} 1 & -1 \\ 2 & 6 \end{vmatrix} \vec{j} + \begin{vmatrix} 1 & 1 \\ 2 & 4 \end{vmatrix} \vec{k}$$

$$= \langle (1)(6)-(-1)(4),\ -[(1)(6)-(-1)(2)],\ (1)(4)-(1)(2) \rangle$$

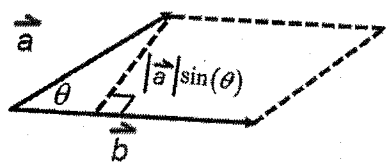
$$= \langle 10,\ -8,\ 2 \rangle$$



Physical interpretation of cross-product:

It can be shown (proof is in the textbook) that:

$$|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}|\sin(\theta)$$

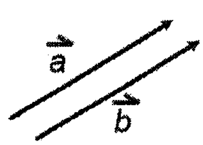


Which means that the magnitude of the cross-product corresponds to the **area of the parallelogram** formed by the two original vectors.

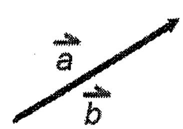
area of the parallelogram = $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}|\sin(\theta)$

Showing vectors are parallel or perpendicular:

Since any two parallel vectors would form a parallelogram with zero area...



(align initial points)



zero parallelogram area

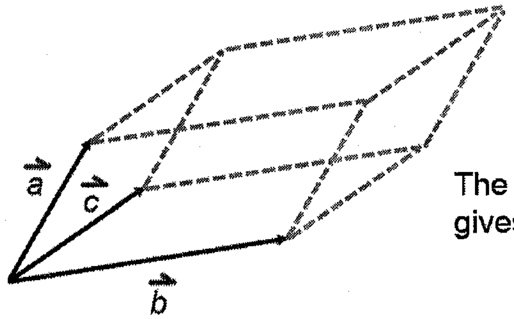
If the cross-product of two vectors is zero, the vectors are parallel.

And recall...

If the dot-product of two vectors is zero, the vectors are perpendicular.

A related physical relationship: the Scalar Triple Product

If you have three, 3D vectors, you can form the Scalar Triple Product:



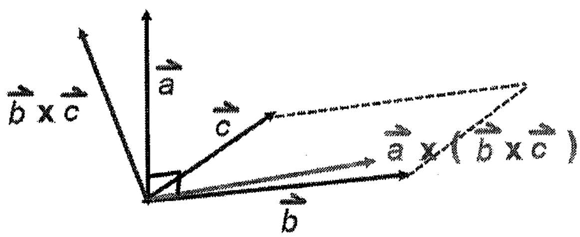
$$\left|\vec{a} \bullet (\vec{b} \times \vec{c})\right|$$

The magnitude of the Scalar Triple Product gives the volume of the parallelepiped.

If the Scalar Triple Product for 3 vectors is zero, that means the 3 vectors must all lie in the same plane.

The Vector Triple Product

There is also a Vector Triple Product: $\vec{a} \times (\vec{b} \times \vec{c})$



The vector triple product produces a vector which is in the plane containing $\vec{b}$ and $\vec{c}$, but is also perpendicular to $\vec{a}$.

...but it is used less frequently. Usually, if people say 'triple product' they mean the scalar triple product.

Properties of the Cross Product

8 THEOREM If **a**, **b**, and **c** are vectors and c is a scalar, then

1. $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$
2. $(c\mathbf{a}) \times \mathbf{b} = c(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (c\mathbf{b})$
3. $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$
4. $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$
5. $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$
6. $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$

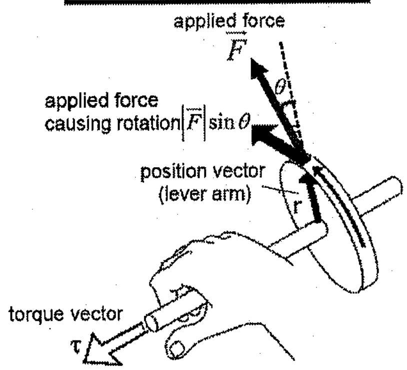
You can distribute scalar numbers with vectors in cross-product multiplication, but reversing the order of two vectors in a cross-product produces a 'negative' vector (in opposite direction).

Physics application of cross product: Torque

In physics, when a force is applied with a 'lever arm' such that it causes rotation around a central point, the force times lever arm distance is called **torque**, and is formally defined as the cross-product of the position and force vectors:

$$\text{Torque, } \vec{\tau} = \vec{r} \times \vec{F}$$

Where $\vec{r}$ is the position vector starting at the pivot point and ending where the force is applied.

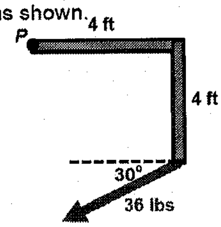


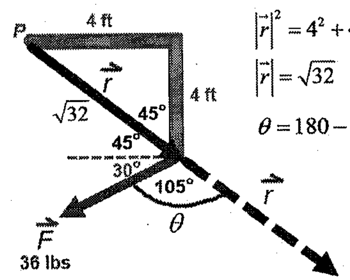
$$|\vec{\tau}| = |\vec{r} \times \vec{F}| = |\vec{r}||\vec{F}|\sin\theta$$

Notice that the magnitude of torque is only influenced by $|\vec{F}|\sin\theta$ which is the component of F perpendicular to r which causes rotation.

Physics application of cross product: Torque

Ex #40. Find the magnitude of the torque about *P* if a 36-lb force is applied as shown.





$$|\vec{r}|^2 = 4^2 + 4^2$$

$$|\vec{r}| = \sqrt{32}$$

$$\theta = 180 - 74 = 105$$

we can solve using the right side of the equation...

$$|\vec{\tau}| = |\vec{r} \times \vec{F}| = |\vec{r}||\vec{F}|\sin\theta$$

$$= (\sqrt{32})(36)\sin 105°$$

$$\approx 196.7\ ft-lbs$$

...or the left side...

$$|\vec{\tau}| = |\vec{r} \times \vec{F}|$$

$$\vec{r} = \langle\sqrt{32}\cos 315°, \sqrt{32}\sin 315°, 0\rangle = \langle 4, -4, 0\rangle$$

$$\vec{F} = \langle 36\cos 210°, 36\sin 210°, 0\rangle = \langle -31.1776, -18, 0\rangle$$

$$\vec{r} \times \vec{F} = \begin{vmatrix} + & - & + \\ 4 & -4 & 0 \\ -31.1776 & -18 & 0 \end{vmatrix} = \langle 0-0, -(0-0), -18(4) - 4(31.1776)\rangle$$

$$\vec{r} \times \vec{F} = \langle 0, 0, -196.7\rangle \quad |\vec{\tau}| = |\vec{r} \times \vec{F}| = \sqrt{0^2 + 0^2 + (-196.7)^2} = 196.7\ ft-lbs$$

Newton's laws of motion give us this equation for how force creates linear motion:

$$\vec{F} = m\vec{a}$$

...and you can think of 'mass' as the aspect of an object which is resisting the change the force is applying...the larger the mass, the smaller the amount of linear acceleration for a given force.

The magnitude of torque corresponds to a rotational force and there is a similar equation for rotational motion:

$$\vec{\tau} = I\vec{\alpha}$$

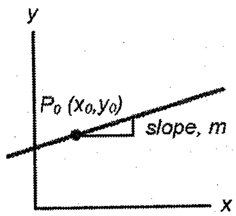
...where the torque is the rotational force and alpha is the angular acceleration (how fast the rotational speed is changing). *I* is the 'moment of inertia' which corresponds to mass for rotation - it is the aspect of an object which resists rotation in the same way mass resists linear motion.

There can be multiple force or torque vectors acting on a body which is free to move. By using vectors, the effects in each linear direction or in each direction of rotation are automatically combined appropriately.

12.5: Equations of Lines and Planes

Lines in 3D

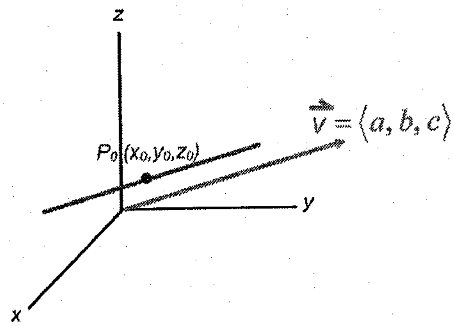
in 2D...



$$(y - y_0) = m(x - x_0)$$

In 2D, to define a line we need a point on the line and the slope.

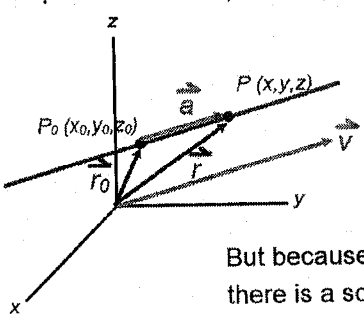
in 3D...



In 3D, to define a line we need a point on the line and the direction of the line in 3D. Direction can be represented by a vector parallel to the line, $\vec{v}$

Lines in 3D - Vector equation of a line

There are two forms for defining a line in 3D: the **vector equation** and **parametric equations.** For the vector form, we start by defining two points on the line, P_0 (the known point on the line) and P (any other point on the line at an arbitrary position).



We then define position vectors to each of these points, $\vec{r_0}$ and $\vec{r}$. We can then define another vector, $\vec{a}$ between the points.

Using vector addition:

$$\vec{r} = \vec{r_0} + \vec{a}$$

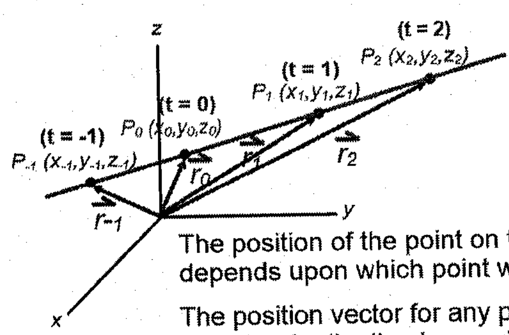
But because $\vec{a}$ is parallel to the direction vector $\vec{v}$ there is a scalar t such that: $\vec{a} = t\vec{v}$
which means we can write:

$$\boxed{\vec{r} = \vec{r_0} + t\vec{v}}$$

This is called the **vector equation** of a line.

Lines in 3D - The parameter, *t*

The scalar constant *t* is called the **parameter**, and varying the parameter will cause the arbitrary point *P* to move along the line.



(In AP Calc BC, do you remember talking about the bug crawling along the line?)

The position of the point on the line when t=0 is arbitrary and depends upon which point we decided to use to define the line.

The position vector for any point on the line is given by the vector equation for the line by providing a value for parameter t...

$$\vec{r} = \vec{r_0} + t\vec{v} \qquad -\infty < t < \infty$$

...using all the values from $-\infty$ to ∞ to sweep through the entire line.

Lines in 3D - Parametric Equations of a line

If we write the vector equation of line in component form...

$$\vec{r} = \vec{r_0} + t\vec{v} \qquad -\infty < t < \infty$$

$$\langle x, y, z\rangle = \langle x_0, y_0, z_0\rangle + t\langle a, b, c\rangle$$

$$\langle x, y, z\rangle = \langle x_0 + ta, \quad y_0 + tb, \quad z_0 + tc\rangle$$

...because two vectors are equal when their components are equal, this gives three scalar equations:

$$\boxed{\begin{aligned} x &= x_0 + at \\ y &= y_0 + bt \qquad -\infty < t < \infty \\ z &= z_0 + ct \end{aligned}}$$

These are called the **parametric equations** of a line.

Lines in 3D - Direction numbers and symmetric equations

Direction numbers...

$$\vec{r} = \vec{r_0} + t\vec{v} \qquad -\infty < t < \infty$$

$$\langle x, y, z\rangle = \langle x_0, y_0, z_0\rangle + t\langle \underline{a, b, c}\rangle$$

The component of the direction vector, a, b, and c, are called the **direction numbers** of the line.

Symmetric Equations...

If you start with the parametric equations of a line, and solve each for the parameter *t*:

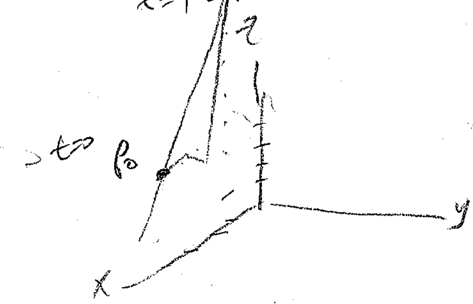
$$\begin{aligned} x &= x_0 + at \\ y &= y_0 + bt \\ z &= z_0 + ct \end{aligned} \longrightarrow t = \frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

These resulting equations are called the **symmetric equations** of the line.

If a direction number is zero, for example, b = 0, then that equation is written in the form: $b = 0,\ so\ y = y_0 + bt = y_0,\ \boxed{y = y_0}$

Lines in 3D - example questions

Find equation of a line given a point and a direction vector.



Find a vector equation and the parametric equations of a line passing through (3,-1,4) and parallel to the vector <-2,1,6>.

$\vec{r_0} = \langle 3,-1,4 \rangle$

$\vec{v} = \langle -2,1,6 \rangle$

$\vec{r} = \vec{r_0} + t\vec{v}$

$$\vec{r} = \langle 3,-1,4 \rangle + t\langle -2,1,6 \rangle$$

$y = mx + b$

$y = b + mx$

$$x = 3 - 2t \qquad y = -1 + t \qquad z = 4 + 6t$$

where $-\infty < t < \infty$

Find equation of a line through 2 given points

Find a vector equation and the parametric equations of a line passing through (6, 1, -3) and (2, 4, 5).

$\vec{r_0} = \langle 6,1,-3 \rangle$

$\vec{v} = \langle (2-6), (4-1), (5+3) \rangle = \langle -4,3,8 \rangle$

$$\vec{r} = \langle 6,1,-3 \rangle + t\langle -4,3,8 \rangle$$

$$x = 6 - 4t \qquad y = 1 + 3t \qquad z = -3 + 8t$$

where $-\infty < t < \infty$

Note: because the choice of point for P_0 and proportion scale factors for a,b,c are arbitrary, there are many possible valid equations.

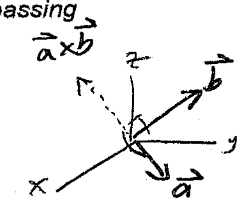
Find equation given one point and related in some way to other lines

*Find a vector equation and the parametric equations of a line passing through (2, 1, 0) and perpendicular to both **i+j** and **j+k**.*

$\vec{r_0} = \langle 2,1,0 \rangle$

$\vec{a} = \langle 1,1,0 \rangle$

$\vec{b} = \langle 0,1,1 \rangle$



$$\vec{v} = \vec{a} \times \vec{b} = \begin{vmatrix} + & - & + \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = \langle (1-0), -(1-0), (1-0) \rangle = \langle 1,-1,1 \rangle$$

$$\vec{r} = \langle 2,1,0 \rangle + t\langle 1,-1,1 \rangle$$

$$x = 2 + t \qquad y = 1 - t \qquad z = t$$

where $-\infty < t < \infty$

Other ways to use these techniques...

Is the line through (-4,-6,1) and (-2,0,-3) parallel to the line through (10,18,4) and (5,3,14)?

direction vectors:

$\vec{v_1} = \langle (-2+4), (0+6), (-3-1) \rangle = \langle 2, 6, -4 \rangle$

$\vec{v_2} = \langle (5-10), (3-18), (14-4) \rangle = \langle -5, -15, 10 \rangle$

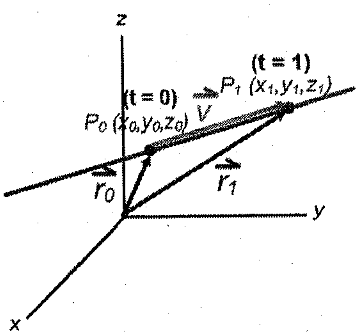
Scalar multiples of each other?

check $\frac{\vec{v_2}}{\vec{v_1}}$: $\frac{-5}{2}$, $\frac{-15}{6} = \frac{-5}{2}$, $\frac{10}{-4} = \frac{-5}{2}$

yes, the lines are parallel

Lines in 3D - Parametric Equations of a line segment

What if only need a line segment instead of the whole line? Usually, this would be defining equations for a line segment between 2 given points, P_0 and P_1:



If given points occur when t=0 and t=1, then the direction vector is just between P_0 and P_1:

$$\vec{v} = \langle x_1 - x_0,\ y_1 - y_0,\ z_1 - z_0 \rangle$$

...and the vector and parametric equations become:

$$\langle x, y, z \rangle = \langle x_0, y_0, z_0 \rangle + t\langle x_1 - x_0, y_1 - y_0, z_1 - z_0 \rangle$$

$$x = x_0 + t(x_1 - x_0) = x_0 + tx_1 - tx_0 = (1-t)x_0 + tx_1$$

$$y = x_0 + t(y_1 - y_0) = y_0 + ty_1 - ty_0 = (1-t)y_0 + ty_1$$

$$z = z_0 + t(z_1 - z_0) = z_0 + tz_1 - tz_0 = (1-t)z_0 + tz_1$$

$$\vec{r} = (1-t)\vec{r_0} + t\vec{r_1} \qquad 0 \le t \le 1$$

Lines in 3D - example questions

Find equation given one point and related in some way to other lines

Find parametric equations for the line segment from (10,3,1) to (5,6, -3).

$\vec{r_0} = \langle 10, 3, 1 \rangle$

$\vec{r_1} = \langle 5, 6, -3 \rangle$

$\vec{r} = (1-t)\vec{r_0} + t\vec{r_1}$

$= (1-t)\langle 10, 3, 1 \rangle + t\langle 5, 6, -3 \rangle$

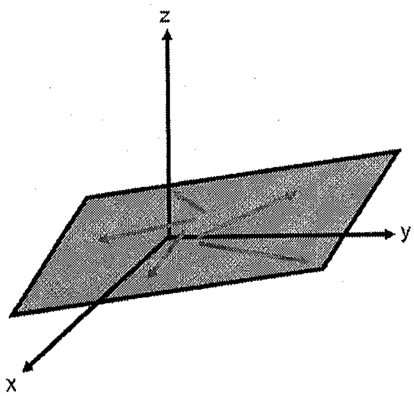
$= \langle 10 - 10t + 5t,\ 3 - 3t + 6t,\ 1 - t - 3t \rangle$

$= \langle 10 - 5t,\ 3 + 3t,\ 1 - 4t \rangle$

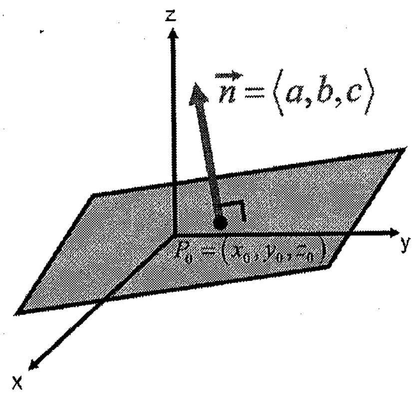
$x = 10 - 5t$
$y = 3 + 3t$
$z = 1 - 4t$
where $0 \le t \le 1$

Equations of Planes

A line is defined by 1) a point on the line and 2) the direction vector of the line.
A plane is also defined by 1) a point on the line and 2) the direction vector for the plane.

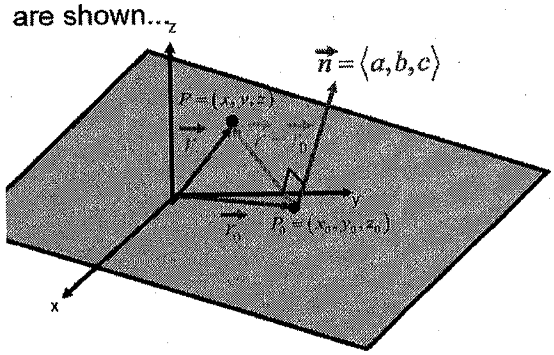


We can't use a vector in the plane to define it's direction (there are many of them)



The only way to get a unique vector is to use a vector normal to the plane...this vector, n, defines the plane's direction.

If we define another generic point, P(x,y,x), on the plane, direction vectors $\vec{r}$ and $\vec{r_0}$ are shown...



We can then define vector $\vec{r}-\vec{r_0}$ which lies in the plane.

This vector is therefore perpendicular to the normal vector, which means its dot-product is zero:

$$\vec{n}\bullet(\vec{r}-\vec{r_0})=0$$

or

$$\vec{n}\bullet\vec{r}=\vec{n}\bullet\vec{r_0}$$

Either of these is called the **vector equation of the plane.**

However, in practice, we usually write the equation of a plane in a different form...

Starting from the vector form equation, if we expand each vector into its components: $\vec{n}\bullet(\vec{r}-\vec{r_0})=0$

$$\langle a,b,c\rangle\bullet\langle(x-x_o,y-y_0,z-z_o)\rangle=0$$

$$a(x-x_0)+b(y-y_0)+c(z-z_0)=0$$

$$ax-ax_0+by-by_0+cz-cz_0=0$$

$$ax+by+cz=ax_0+by_0+cz_0$$

and since $\vec{n}=\langle a,b,c\rangle$ *and* $\vec{r_0}=\langle x_0,y_0,z_0\rangle$

$$ax+by+cz=\vec{n}\bullet\vec{r_0}$$

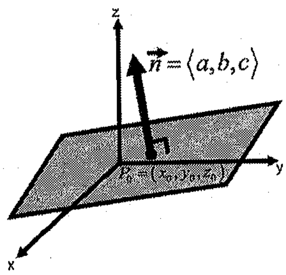
which is called the **scalar equation of the plane through *P_0*** and is the version we more typically use.

Notice the left side constants are the direction numbers for the normal vector, and the right side is a number (set by the point the plane is going through along with the direction).

Equations of Planes vs. Lines

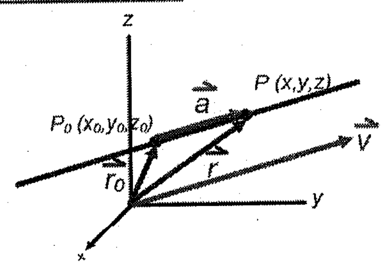
Something important to notice...

For planes, the equation of the plane is defined by a point on the plane and a vector which is **perpendicular** to the plane:



$$ax + by + cz = \vec{n} \cdot \vec{r}_0$$

For lines, the equation of the line is defined by a point on the line and a vector which is **in the direction of the line**:



$$\vec{r} = \vec{r}_0 + t\vec{v}$$

Equations of planes - example questions

Find an equation of the plane for the plane through the point (6,3,2) and perpendicular to the vector <-2,1,5>.

need: point (6,3,2) $\vec{n} = \langle -2,1,5 \rangle$

so $\vec{r_0} = \langle 6,3,2 \rangle$

$ax+by+cz = \vec{n} \cdot \vec{r_0}$

$-2x+y+5z = \langle -2,1,5 \rangle \cdot \langle 6,3,2 \rangle$

$= (-2)(6)+(1)(3)+(5)(2)$

$= -12+3+10 = 1$

$\boxed{-2x+y+5z=1}$

Find an equation of the plane for the plane through the point (-2,8,10) and perpendicular to the line x=1+t, y=2t, z=4-3t.

$\langle 1+t, 0+2t, 4-3t \rangle$

$\langle 1,0,4 \rangle + t\langle 1,2,-3 \rangle$

this means $\vec{n} = \langle 1,2,-3 \rangle$

so $x+2y-3z = \langle 1,2,-3 \rangle \cdot \langle -2,8,10 \rangle$

$= (1)(-2)+(2)(8)+(-3)(10)$

$= -2+16-30$

$= -16$

$\boxed{x+2y-3z=-16}$

"⊥ to line" $x=1+t$, $y=2t$, $z=4-3t$

2 points: $t=0$: $x=1$, $y=0$, $z=4$ $(1,0,4)$

$t=1$: $x=2$, $y=2$, $z=1$ $(2,2,1)$

$\vec{v} = \langle (2-1),(2-0),(1-4) \rangle$

$= \langle 1,2,-3 \rangle$

$x = 1+1t$

$y = 0+2t$

$z = 4-3t$

Find an equation of the plane for the plane through the points (2,-4,6), (5,1,3) and (0,1,2).

need $\vec{n}$ (from 2 vectors in the plane): $\vec{PQ} = \langle (5-2), (1+4), (3-6) \rangle = \langle 3, 5, -3 \rangle$

$\vec{PR} = \langle (0-2), (1+4), (2-6) \rangle = \langle -2, 5, -4 \rangle$

then $\vec{n} = \vec{PQ} \times \vec{PR} = \begin{vmatrix} + & - & + \\ 3 & 5 & -3 \\ -2 & 5 & -4 \end{vmatrix} = \langle (-20+15), -(-12-6), (15+10) \rangle$

$= \langle -5, 18, 25 \rangle$

choose $\vec{r_0} = \langle 0, 1, 2 \rangle$

then $ax + by + cz = \vec{n} \cdot \vec{r_0}$

$-5x + 18y + 25z = \langle -5, 18, 25 \rangle \cdot \langle 0, 1, 2 \rangle$

$= (-5)(0) + (18)(1) + (25)(2)$

$= 0 + 18 + 50$

$= 68$

$$\boxed{-5x + 18y + 25z = 68}$$

Distances in 3D

Distance between two points:

$$D = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

Distance between a point and a plane:

For a plane defined by: $ax + by + cz = -d$

so $ax + by + cz + d = 0$

The distance between point $P_1(x_1, y_1, z_1)$ and the plane is given by:

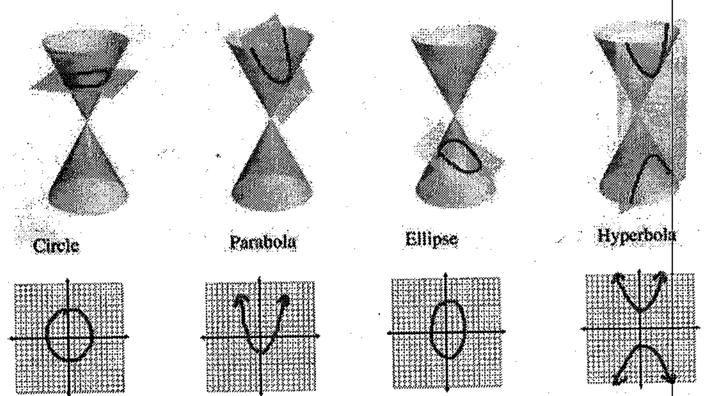
$$D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

(This is derived in section 12.5 in the textbook, if you're interested.)

12.6: Cylinders and Quadric Surfaces

First, a quick review of conic sections and how to sketch them...

A conic section is a 2D curve which is the intersection of a plane with a cone...

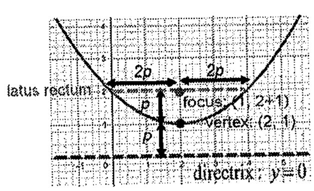


...and all have equations of the general form:

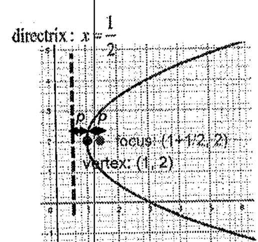
$$Ax^2 + \underline{Bxy} + Cy^2 + Dx + Ey + F = 0$$

If the *xy* term is present, the conic section is not aligned with the x-y axes (is rotated)
We will not consider this case (it is solved with a rotational coordinate transformation)

Parabolas



$(x-2)^2 = 4(y-1)$

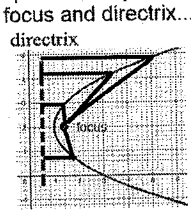


$(y-2)^2 = 2(x-1)$

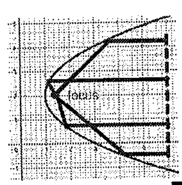
Standard form:	$(x-h)^2 = 4p(y-k)$ $(y-k)^2 = 4p(x-h)$	*p = distance from vertex to focus* *and from vertex to directrix*

x^2? (like $y=x^2$)

Geometrically, all points on a parabola are equidistant from focus and directrix...



...which makes this shape perfect for focusing energy (antenna dishes, flashlights, etc.)



all elements of a wavefront arrive at the same time at the focus

Parabola examples

$(x-h)^2 = 4p(y-k)$ $(y-k)^2 = 4p(x-h)$	*p = distance from vertex to focus* *and from vertex to directrix*

$x^2 - 6x - 8y - 7 = 0$

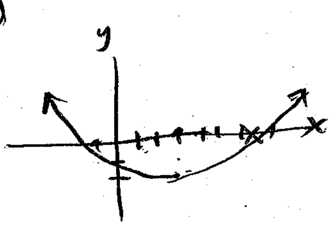
$(x^2 - 6x + \underline{9}) = 8y + 7 + 9$

$(x-3)^2 = 8y + 16 = 8(y+2)$

vertex $(3, -2)$

$4p = 8$

$p = 2$



$4x - y^2 - 2y - 9 = 0$

$-y^2 - 2y = -4x + 9$

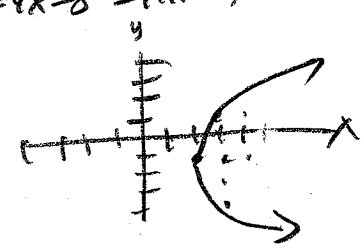
$(y^2 + 2y + \underline{1}) = 4x - 9 + \underline{1}$

$(y+1)^2 = 4x - 8 = 4(x-2)$

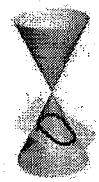
vertex: $(2, -1)$

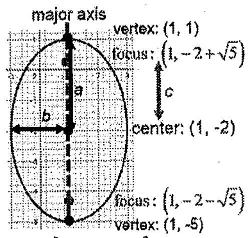
$4p = 4$

$p = 1$

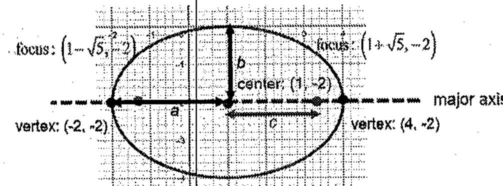


Ellipses





$$\frac{(x-1)^2}{4}+\frac{(y+2)^2}{9}=1$$



$$\frac{(x-1)^2}{9}+\frac{(y+2)^2}{4}=1$$

bigger denominator = longer direction

Standard form:

$c^2=a^2-b^2$

$c=$ *distance from center to foci*

$$\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1$$ *vertical major axis*

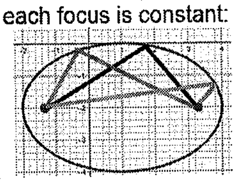
$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$$ *horizontal major axis*

eccentricity $=e=\frac{c}{a}$

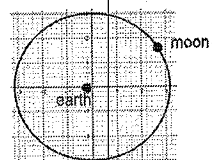
$e=1$ *is a circle,*

higher $e\Rightarrow$ *more oval*

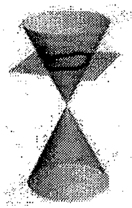
Geometrically, the sum of the distances from a point on an ellipse to each focus is constant:



In astronomical orbits, the object follows an elliptical path around the object being orbited which is at a focus (although the ellipse is usually close to circular):



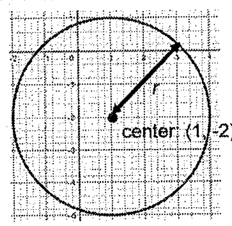
Circles



Circles are special cases of ellipses where $a = b$:

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{a^2}=1$$

$$(x-h)^2+(y-k)^2= ~~a^2~~ \; r^2$$



$$(x-1)^2+(y+2)^2=9$$

Standard form: $(x-h)^2+(y-k)^2=r^2$

Ellipse/Circle examples

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$$ *horizontal major axis*

$$\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1$$ *vertical major axis*

$c^2=a^2-b^2$

$c=$ *distance from center to foci*

eccentricity $=e=\frac{c}{a}$

$$9x^2+4y^2-36x+8y+4=0$$

$(9x^2-36x\quad)+(4y^2+8y\quad)=-4$

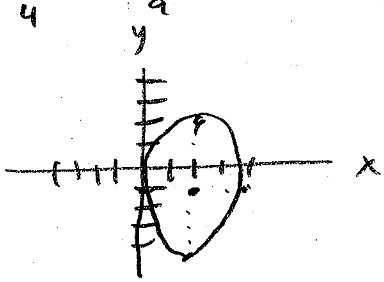
$9(x^2-4x+4)+4(y^2+2y+1)=-4+36+4$

$9(x-2)^2+4(y+1)^2=36$

$\frac{9(x-2)^2}{36}+\frac{4(y+1)^2}{36}=1$ center: $(2,-1)$

width$_x=2$

$\frac{(x-2)^2}{4}+\frac{(y+1)^2}{9}=1$ width$_y=3$



$$2x^2+2y^2+12x-16y+40=0$$

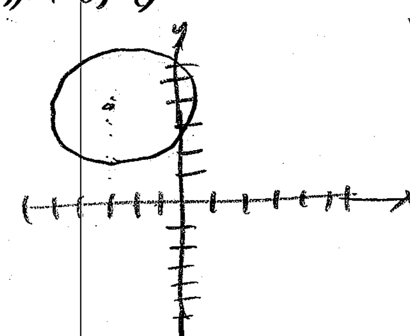
$(2x^2+12x\quad)+(2y^2-16y\quad)=-40$

$2(x^2+6x+9)+2(y^2-8y+16)=-40+18+32$

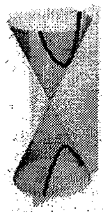
$2(x+3)^2+2(y-4)^2=10$

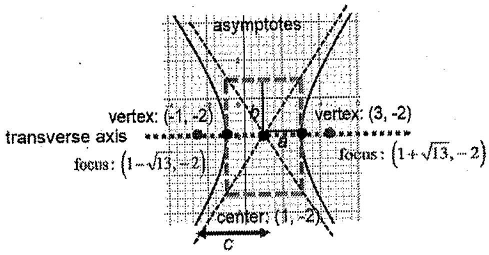
$(x+3)^2+(y-4)^2=5$ (circle) center: $(-3,4)$

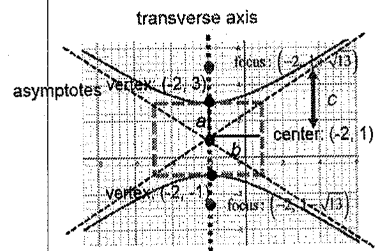
radius $=\sqrt{5}$



Hyperbolas







opens in direction of positive term

$$\frac{(x-1)^2}{4}-\frac{(y+2)^2}{9}=1 \qquad \frac{(y-1)^2}{4}-\frac{(x+2)^2}{9}=1$$

Standard form:

$c^2=a^2+b^2$

$c = \textit{distance from center to foci}$

$$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$$

horizontal transverse axis

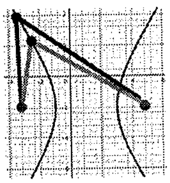
$asymptotes: (y-k)=\pm\frac{b}{a}(x-h)$

$$\frac{(y-1)^2}{4}-\frac{(x+2)^2}{9}=1$$

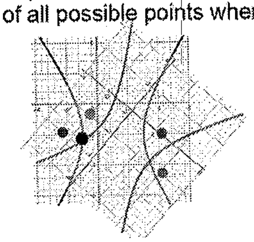
vertical transverse axis

$asymptotes: (y-k)=\pm\frac{a}{b}(x-h)$

Geometrically, the difference of the distances from a point on a hyperbola to each focus is constant...



...this is useful in location detection systems. If two points receive a signal and the delay between the times received is known, a hyperbola traces out the focus of all possible points where the emitting object might be.



If you have a 2nd set of two detectors, the location is at the intersection of the two hyperbolas from the two pairs of detectors.

Hyperbola examples

$c^2=a^2+b^2$

$c = \textit{distance from center to foci}$

$$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$$

horizontal transverse axis

$$\frac{(y-1)^2}{4}-\frac{(x+2)^2}{9}=1$$

vertical transverse axis

$asymptotes: (y-k)=\pm\frac{b}{a}(x-h)$

$$16x^2-4y^2+32x+16y-64=0$$

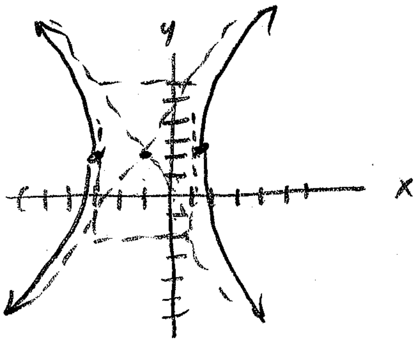
$(16x^2+32x \quad) + (-4y^2+16y \quad) = 64$

$16(x^2+2x+1) - 4(y^2-4y+4) = 64 + 16 - 16$

$16(x+1)^2 - 4(y-2)^2 = 64$

$\frac{16(x+1)^2}{64} - \frac{4(y-2)^2}{64} = 1$ center: (-1,2)

$\frac{(x+1)^2}{4} - \frac{(y-2)^2}{16} = 1$ $width_x = 2$, $width_y = 4$



$$9x^2-4y^2-18x-16y+29=0$$

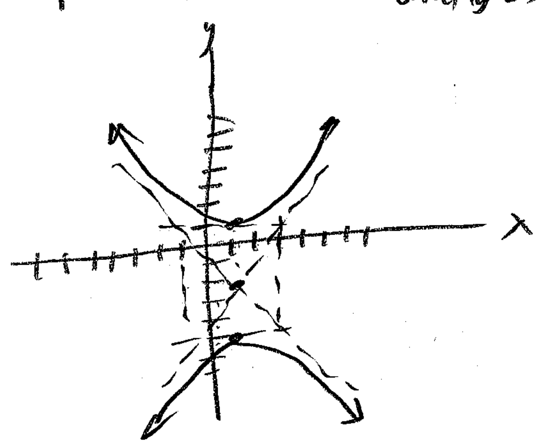
$(9x^2-18x \quad) + (-4y^2-16y \quad) = -29$

$9(x^2-2x+1) - 4(y^2+4y+4) = -29 + 9 - 16$

$9(x-1)^2 - 4(y+2)^2 = -36$

$\frac{9(x-1)^2}{-36} - \frac{4(y+2)^2}{-36} = 1$ center: (1,-2)

$\frac{(y+2)^2}{9} - \frac{(x-1)^2}{4} = 1$ $width_x = 2$, $width_y = 3$



Quickly recognizing which conic section from the equation

$x^2-6x-8y-7=0$	One squared term = parabola
$9x^2+4y^2-36x+8y+4=0$	Two squared terms, same sign = ellipse
$9x^2-4y^2-18x-16y+29=0$	Two squared terms, different signs = hyperbola
$4x-y^2-2y-9=0$	One squared term = parabola
$16x^2-4y^2+32x+16y-64=0$	Two squared terms, different signs = hyperbola
$2x^2+2y^2+12x-16y+40=0$	Two squared terms, same sign = ellipse, but coefficients of squared terms are same too, so circle

Parabola

$$(x-h)^2=4p(y-k)$$

$$(y-k)^2=4p(x-h)$$

x^2 like $y=x^2$
y^2 'other one'

p = dist. vertex to focus and dist. vertex to directrix

Ellipse

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$$

$$\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1$$

a is always bigger,
a under term of major axis

$$c^2=a^2-b^2$$

Hyperbola

$$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$$

$$\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$$

a not always bigger,
a always under first term
first term is transverse axis

$$c^2=a^2+b^2$$

a = dist. center to vertex
c = dist. center to focus

b = dist. center to point on minor axis

b = dist. to 'other side of box'

asymptotes from center through corners of box:

$$(y-k)=\pm\frac{b}{a}(x-h)$$

$$(y-k)=\pm\frac{a}{b}(x-h)$$

(look at box to see which)

eccentricity $e=\dfrac{c}{a}$

12.6: Cylinders and Quadric Surfaces

A convenient web-based 3d graph application...
https://www.geogebra.org/3d?lang=en

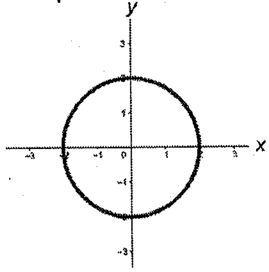
Cylinders

in $\mathbb{R}^2$...

In 2D, this equation...

$$x^2 + y^2 = 4$$

...represents a circle:

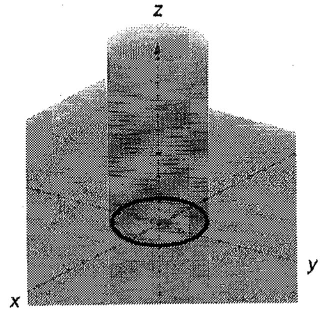


in $\mathbb{R}^3$...

In 3D, this equation...

$$x^2 + y^2 = 4$$

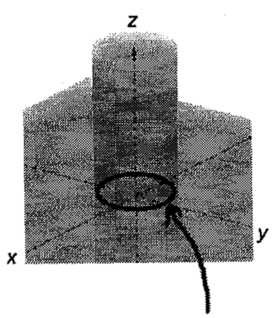
...represents a circular cylinder...

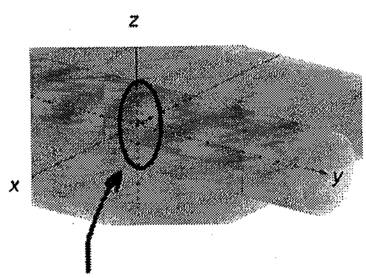


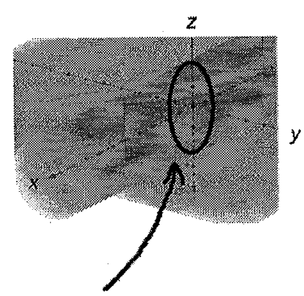
...because z is a variable, and it is unspecified, it can be anything so the circle in the x-y plane is extended up and down in the z direction.

Depending upon which variable is not included, the circular cross-section may be extended in any variable axis:

$$x^2 + y^2 = 4 \qquad x^2 + z^2 = 4 \qquad y^2 + z^2 = 4$$





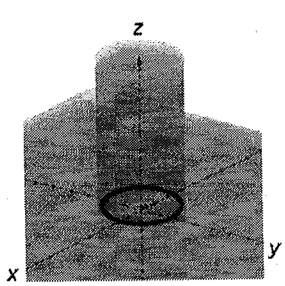


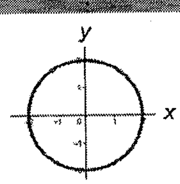
These 'cross-sections' of the intersection of the surface with the x-y, x-z, or y-z planes are called **traces**.

Cylinders are named for the type of curve which defined their cross-section:

Circular Cylinder

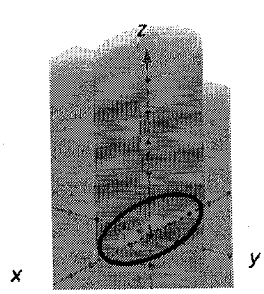
$$x^2 + y^2 = 4$$

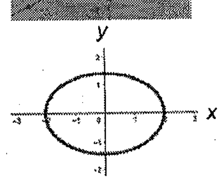




Elliptical Cylinder

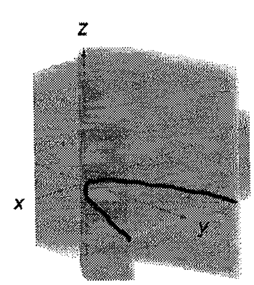
$$x^2 + 2y^2 = 4$$

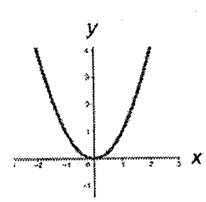




Parabolic Cylinder

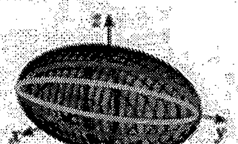
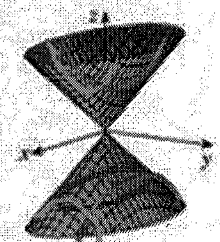
$$y = x^2$$





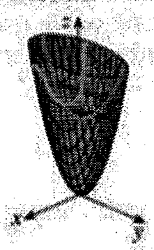
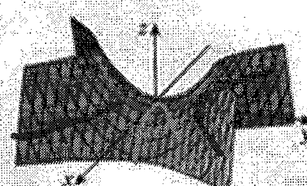
Quadric Surfaces

A quadric surface is the graph of a 2nd-degree equation in 3 variables (3D version of conic sections), and are named for the trace curves:

Surface	Equation	Surface	Equation
Ellipsoid 	$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ All traces are ellipses. If $a=b=c$, the ellipsoid is a sphere.	Cone 	$\frac{z^2}{c^2}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$ Horizontal traces are ellip… Vertical traces in the plan… $x=k$ and $y=k$ are hyperbolas if $k\neq 0$ but ar… pairs of lines if $k=0$.

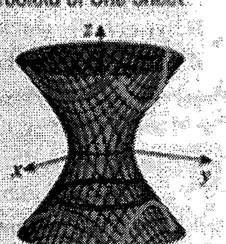
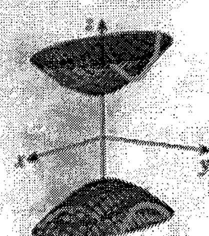
Ellipsoid: 3 squared terms on left side, all positive, constant on right

Cone: 3 squared terms, all positive, but no constant and one term on other side of equation.

Surface	Equation	Surface	Equation
Elliptic Paraboloid 	$\frac{z}{c}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces are parabolas. The variable raised to the first power indicates the axis of the paraboloid.	Hyperbolic Paraboloid 	$\frac{z}{c}=\frac{x^2}{a^2}-\frac{y^2}{b^2}$ Horizontal traces are hyperbolas. Vertical traces are parabolas. The case where $c<0$ is illustrated.

Elliptic Paraboloid: 2 squared terms (both positive), term on other side is not squared. The non-squared side is the direction in which it 'opens'.

Hyperbolic Paraboloid: 2 squared terms (one negative), term on other side is not squared. The plane with the squared variables (here x-y) is the plane of the hyperbolic traces.

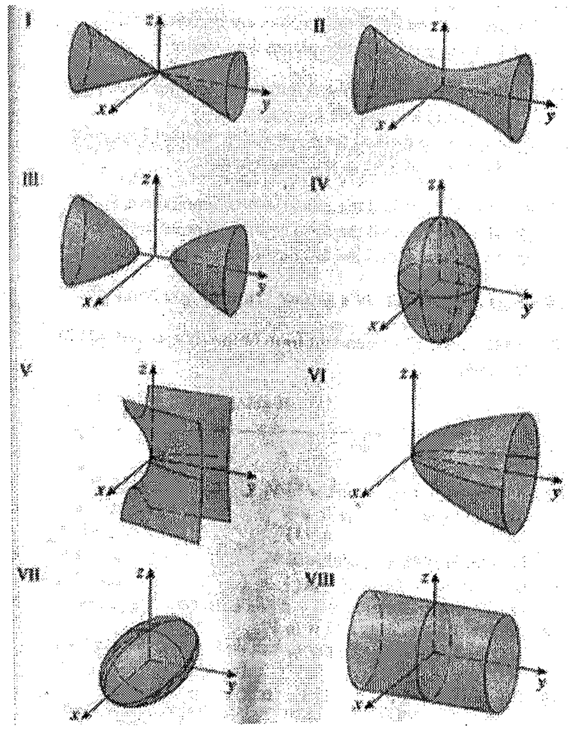
Surface	Equation	Surface	Equation
Hyperboloid of One Sheet 	$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ Horizontal traces are ellip… Vertical traces are hyperbo… The axis of symmetry corresponds to the variable whose coefficient is negativ…	Hyperboloid of Two Sheets 	$-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ Horizontal traces in $z=k$ ar… ellipses if $k>c$ or $k<-$… Vertical traces are hyperbo… The two minus signs indica… two sheets.

Hyperboloid of One Sheet: 3 squared terms with constant on other side, but one term is negative. The plane with the positive terms (here x-y) is the plane of the elliptical traces.

Hyperboloid of Two Sheet: 3 squared terms with constant on other side, but twos terms are negative. The plane with the negative terms (here x-y) is the plane of the elliptical traces.

Cylinders and Quadric Surfaces

Match and name each:



#21. $x^2+4y^2+9z^2=1$ ellipsoid VII

#22. $9x^2+4y^2+z^2=1$ ellipsoid IV

#23. $x^2-y^2+z^2=1$ hyperboloid of one sheet II

#24. $-x^2+y^2-z^2=1$ hyperboloid of two sheets III

#25. $y=2x^2+z^2$ elliptical paraboloid VI

#26. $y^2=x^2+z^2$ cone I

#27. $x^2+2z^2=1$ cylinder VIII

#28. $y=x^2-z^2$ hyperbolic paraboloid V

Name each:

#11. $x=y^2+4z^2$ elliptical paraboloid

#12. $9x^2-y^2+z^2=0$ elliptical cone (move y^2 to other side)

#13. $x^2=y^2+4z^2$ elliptical cone

#14. $25x^2+4y^2+z^2=100$ ellipsoid

#15. $-x^2+4y^2-z^2=4$ hyperboloid of two sheets

#16. $4x^2+9y^2+z=0$ elliptical paraboloid

#17. $36x^2+y^2+36z^2=36$ ellipsoid

#18. $4x^2-16y^2+z^2=16$ hyperboloid of one shee

#19. $y=z^2-x^2$ hyperbolic paraboloid

#20. $x=y^2-z^2$ hyperbolic paraboloid

Sketching Quadric Surfaces

To sketch, draw traces in the xy, yz, and xz planes...

ex: $x^2+2z^2-6x-y+10=0$

First group variables and complete the square:

$$(x^2-6x)-(y)+2(z)^2=-10$$

$$(x^2-6x+\underline{\underline{9}})-(y)+2(z)^2=-10+\underline{\underline{9}}$$

$$(x-3)^2-(y)+2(z)^2=-1$$

$$(x-3)^2+2(z)^2=(y-1)$$

2 squared terms, both positive, other term on other side not squared, should be an elliptic paraboloid.

Now consider each plane and let the other variable equal constant, k, to draw traces:

<u>xy (let z=k)</u>

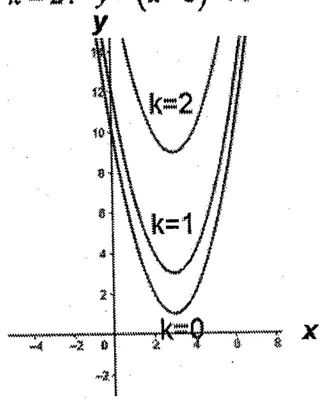
$(x-3)^2+2(k)^2=(y-1)$

$y=(x-3)^2+2k^2+1$

$k=0:\ \ y=(x-3)^2+1$

$k=1:\ \ y=(x-3)^2+3$

$k=2:\ \ y=(x-3)^2+9$



<u>xz (y=k)</u>

$(x-3)^2-(k)+2(z)^2=-1$

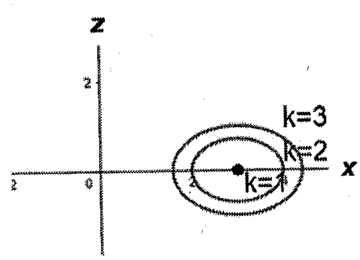
$(x-3)^2+2(z)^2=k-1$

$k=0:\ \ (x-3)^2+2(z-0)^2=-1$ (*no curve*)

$k=1:\ \ (x-3)^2+2(z-0)^2=0$ (*a point*)

$k=2:\ \ (x-3)^2+2(z-0)^2=1$

$k=3:\ \ (x-3)^2+2(z-0)^2=2$



<u>yz (x=k)</u>

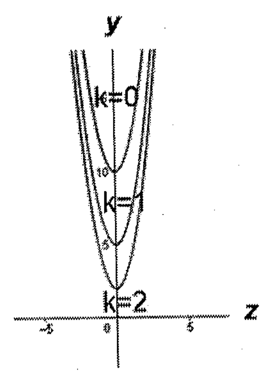
$(k-3)^2-(y)+2(z)^2=-1$

$y=2(z)^2+(k-3)^2+1$

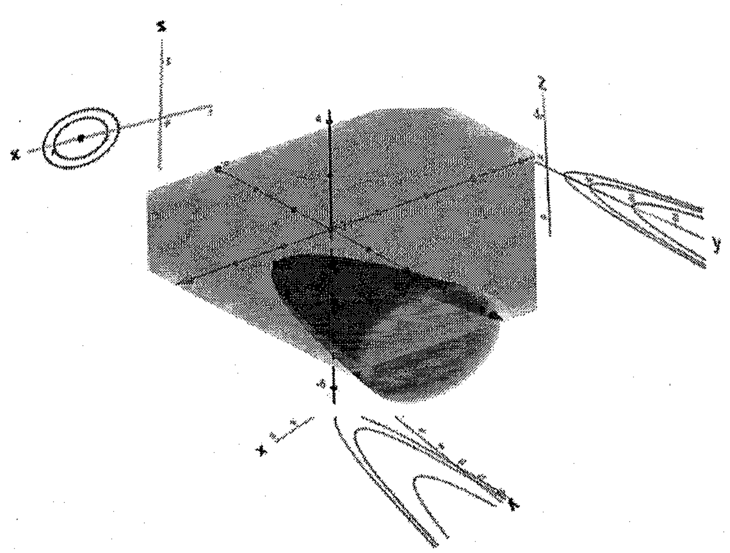
$k=0:\ \ y=2(z)^2+10$

$k=1:\ \ y=2(z)^2+5$

$k=2:\ \ y=2(z)^2+2$



Now try to put these together to make a 3D object:



(this is very difficult to do by hand...most people use 3D graphing software)

Sketching Quadric Surfaces

You try one...draw traces for each plane and try to sketch the 3D object:

$$4x^2 - 16y^2 + z^2 = 16$$

xy (z=k)

$z=k=0$: $4x^2-16y^2=16$

$\frac{x^2}{4}-\frac{y^2}{1}=1$

$\frac{x^2}{(2)^2}-\frac{(y)^2}{(1)^2}=1$

$z=k=4$: $4x^2-16y^2=0$

$16y^2=4x^2$

$y^2=\frac{1}{4}x^2$

$y=\pm\frac{1}{2}x$

$z=k=8$: $4x^2-16y^2=-48$

$16y^2-4x^2=48$

$\frac{y^2}{3}-\frac{x^2}{12}=1$

$\frac{y^2}{(\sqrt{3})^2}-\frac{x^2}{(\sqrt{12})^2}=1$

yz (x=k)

$x=k=0$: $-16y^2+z^2=16$

$\frac{z^2}{16}-\frac{y^2}{1}=1$

$\frac{z^2}{(4)^2}-\frac{y^2}{(1)^2}=1$

$x=k=2$: $-16y^2+z^2=0$

$16y^2=z^2$

$z=\pm 4y$

$x=k=4$: $-16y^2+z^2=-48$

$16y^2-z^2=48$

$\frac{y^2}{3}-\frac{z^2}{48}=1$

$\frac{y^2}{(\sqrt{3})^2}-\frac{z^2}{(\sqrt{48})^2}=1$

xz (y=k)

$y=k=0$: $4x^2+z^2=16$

$\frac{x^2}{4}+\frac{z^2}{16}=1$

$(2)^2 \quad (4)^2$

$y=k=1$: $4x^2+z^2=32$

$\frac{x^2}{8}+\frac{z^2}{32}=1$

$(\sqrt{8})^2 \quad (\sqrt{32})^2$

$y=k=2$: $4x^2+z^2=80$

$\frac{x^2}{20}+\frac{z^2}{80}=1$

$(\sqrt{20})^2 \quad (\sqrt{80})^2$