# Calc 3 Final Exam Review — Solutions

(1) $(1,-2,2) \to (4,1,7)$ $\qquad \vec{r}(t) = (1-t)\vec{r_0} + t\,\vec{r_1}$

$$= (1-t)\langle 1,-2,2\rangle + t\langle 4,1,7\rangle$$

$$= \langle 1-t, -2+2t, 2-2t\rangle + \langle 4t, t, 7t\rangle$$

$$= \langle 1-t+4t, -2+2t+t, 2-2t+7t\rangle$$

$$= \boxed{\langle 1+3t, -2+3t, 2+5t\rangle \quad 0 \le t \le 1}$$

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(2) (a) $(0,1,1)\ (1,0,1)\ (1,1,0)$

$\vec{v_1} = \langle 0-1, 1-0, 1-1\rangle = \langle -1, 1, 0\rangle$

$\vec{v_2} = \langle 1-1, 0-1, 1-0\rangle = \langle 0, -1, 1\rangle$

$$\vec{n} = \vec{v_1} \times \vec{v_2} = \begin{vmatrix} + & - & + \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{vmatrix} = \langle 1-0, -(-1-0), 1-0\rangle = \langle \underset{a}{1}, \underset{b}{1}, \underset{c}{1}\rangle$$

$ax+by+cz = \vec{n}\cdot\vec{r_0}$

$(1)x+(1)y+(1)z = \langle 1,1,1\rangle \cdot \langle 0,1,1\rangle$ ← any point on plane

$= (1)(0)+(1)(1)+(1)(1)$

$$\boxed{x+y+z=2}$$

(b) $\parallel$ to $\underset{(a}{x} + \underset{b}{y} + \underset{c)}{z} = -2$ ← $\vec{n}$ for this plane is $\langle 1,1,1\rangle$ at $(-1,6,-5)$

Same for new plane:

$ax+by+cz = \vec{n}\cdot\vec{r_0}$

$(1)x+(1)y+(1)z = \langle 1,1,1\rangle \cdot \langle -1,6,-5\rangle$

$= (1)(-1)+(1)(6)+(1)(-5)$

$$\boxed{x+y+z=0}$$

(3) $\vec{r}(t) = \langle 4-3t, 2t, 3t-1 \rangle$

(a) $\vec{r}(0) = \langle 4-3(0), 2(0), 3(0)-1 \rangle = \langle 4, 0, -1 \rangle$ at $\boxed{(4,0,-1)}$

(b) velocity $= v = \vec{r}'(t) = \langle -3, 2, 3 \rangle$

speed $= |\vec{v}| = \sqrt{(-3)^2+(2)^2+(3)^2} = \boxed{\sqrt{22}}$

(c) $\langle 4-3t, 2t, 3t-1 \rangle = \langle -2, 4, 5 \rangle$

$2t = 4$ at $\boxed{t=2}$

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(4) $\vec{r}(t) = \langle 2\sin t, 5t, 2\cos t \rangle$

(a) $\vec{T} = \frac{\vec{r}'}{|\vec{r}'|}$ $\quad \vec{r}' = \langle 2\cos t, 5, -2\sin t \rangle$

$|\vec{r}'| = \sqrt{(2\cos t)^2 + (5)^2 + (-2\sin t)^2}$

$= \sqrt{4\cos^2 t + 4\sin^2 t + 25}$

$= \sqrt{4+25} = \sqrt{29}$

So $\vec{T} = \frac{\langle 2\cos t, 5, -2\sin t \rangle}{\sqrt{29}}$

$= \boxed{\left\langle \frac{2}{\sqrt{29}}\cos t, \frac{5}{\sqrt{29}}, \frac{-2}{\sqrt{29}}\sin t \right\rangle}$

(b) $\vec{N} = \frac{\vec{T}'}{|\vec{T}'|}$ $\quad \vec{T}' = \left\langle -\frac{2}{\sqrt{29}}\sin t, 0, \frac{-2}{\sqrt{29}}\cos t \right\rangle$

$|\vec{T}'| = \sqrt{\left(\frac{-2}{\sqrt{29}}\sin t\right)^2 + 0^2 + \left(\frac{-2}{\sqrt{29}}\cos t\right)^2}$

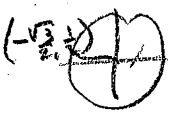
$= \sqrt{\frac{4}{29}\sin^2 t + \frac{4}{29}\cos^2 t} = \sqrt{\frac{4}{29}} = \frac{2}{\sqrt{29}}$

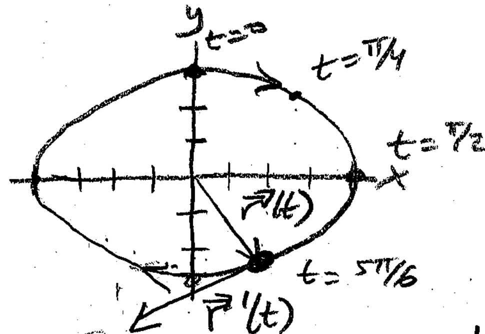
so $\vec{N} = \frac{\left\langle -\frac{2}{\sqrt{29}}\sin t, 0, \frac{-2}{\sqrt{29}}\cos t \right\rangle}{\frac{2}{\sqrt{29}}} = \boxed{\langle -\sin t, 0, -\cos t \rangle}$

(c) $K = \frac{|\vec{T}'|}{|\vec{r}'|} = \frac{\left(\frac{2}{\sqrt{29}}\right)}{\sqrt{29}} = \boxed{\frac{2}{29}}$

(5) $\vec{r}(t) = \langle 4\sin t, 3\cos t \rangle$

| $t$ | $\langle 4\sin t, 3\cos t \rangle$ |
|---|---|
| $0$ | $\langle 0, 3 \rangle$ |
| $\frac{\pi}{4}$ | $\langle 4(\frac{\sqrt{2}}{2}), 3(\frac{\sqrt{2}}{2}) \rangle$ <br> 2,828   2,121 |
| $\frac{\pi}{2}$ | $\langle 4, 0 \rangle$ |
| $\frac{5\pi}{6}$ | $\langle 4(\frac{1}{2}), 3(-\frac{\sqrt{3}}{2}) \rangle$ <br> 2   $-2{,}598$ |




$\vec{r}(\frac{5\pi}{6}) = \langle 2, -\frac{3\sqrt{3}}{2} \rangle$

2   $-2{,}598$

$\vec{r}'(t) = \langle 4\cos t, -3\sin t \rangle$

$\vec{r}'(\frac{5\pi}{6}) = \langle 4(-\frac{\sqrt{3}}{2}), 3(\frac{1}{2}) \rangle$

$-3{,}464, \; 1{,}5$

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(6) $\langle y^2z^3, xy, x^2z \rangle \cdot \langle y^3, z^4, xz \rangle$

$\boxed{(y^2z^3)(y^3) + (xy)(z^4) + (x^2z)(xz)}$

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(7) $\langle 3, 1, -1 \rangle \times \langle -2, 5, 0 \rangle$

$$\begin{vmatrix} + & - & + \\ 3 & 1 & -1 \\ -2 & 5 & 0 \end{vmatrix} = \langle (1)(0) - (-1)(5), -[(3)(0) - (-1)(-2)], (3)(5) - (1)(-2) \rangle$$

$$= \langle 0 + 5, -(0 - 2), 15 + 2 \rangle$$

$$= \boxed{\langle 5, 2, 17 \rangle}$$

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(8) $f = xy + ye^x + xe^y$

$\boxed{f_x = y + ye^x + e^y}$   $\boxed{f_y = x + e^x + xe^y}$

$\boxed{f_{xx} = ye^x}$   $\boxed{f_{xy} = 1 + e^x + e^y}$   $\boxed{f_{yy} = xe^y}$

(9) $z - z_0 = f_x(x-x_0) + f_y(y-y_0)$ $\quad f = 4x^2 - y^2 + 2y$ at $(-1,2,4)$

$\boxed{z - 4 = -8(x+1) - 2(y-2)}$

$f_x = 8x \quad f_y = -2y+2$

at $(-1,2,4)$

$f_x = 8(-1) = -8 \quad f_y = -2(2)+2 = -2$

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(10) $z = \frac{y}{x^2} = yx^{-2}$ $\quad f_x = \frac{\partial z}{\partial x} = -2yx^{-3} = \frac{-2y}{x^3}$ $\quad f_y = \frac{\partial z}{\partial y} = x^{-2} = \frac{1}{x^2}$

so $\boxed{\nabla z = \left\langle \frac{-2y}{x^3}, \frac{1}{x^2} \right\rangle}$

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(11) $f = 1 + 2x\sqrt{y} = 1 + 2xy^{1/2}$ $\quad D_u f = \nabla f \cdot \vec{u}$ ← unit vector in direction

$\nabla f = \langle f_x, f_y \rangle = \langle 2\sqrt{y}, xy^{-1/2} \rangle = \langle 2\sqrt{y}, \frac{x}{\sqrt{y}} \rangle$

don't evaluation at a point ("from any point")

but $\langle 4, -3 \rangle$ establishes direction — must convert to unit vector!

$|\langle 4, -3 \rangle| = \sqrt{4^2 + (-3)^2} = 5$

So $\vec{u} = \langle \frac{4}{5}, \frac{-3}{5} \rangle$

and $D_u f = \langle 2\sqrt{y}, \frac{x}{\sqrt{y}} \rangle \cdot \langle \frac{4}{5}, \frac{-3}{5} \rangle$

$= (2\sqrt{y})(\frac{4}{5}) + (\frac{x}{\sqrt{y}})(-\frac{3}{5})$

$= \boxed{\frac{8}{5}\sqrt{y} - \frac{3}{5}\frac{x}{\sqrt{y}}}$

(12) $f = x^3 - 12xy + 8y^3$

extrema / saddle when $f_x = 0$, $f_y = 0$

$$\begin{cases} f_x = 3x^2 - 12y = 0 \\ f_y = -12x + 24y^2 = 0 \end{cases}$$

Solve system by hand using substitution (no calculator solvers allowed on test)

$3x^2 - 12y = 0$

$12y = 3x^2$

$y = \frac{x^2}{4}$

$-12x + 24y^2 = 0$

$-12x + 24\left(\frac{x^2}{4}\right)^2 = 0$

$-12x + 24\frac{x^4}{16} = 0$

$-12x + \frac{3}{2}x^4 = 0$

$-24x + 3x^4 = 0$

$x(-24 + 3x^3) = 0$

$\underline{x = 0}$ or $-24 + 3x^3 = 0$

$y = \frac{x^2}{4} = 0$

$(0,0)$

$3x^3 = 24$

$x^3 = 8$

$\underline{x = 2}$

$y = \frac{x^2}{4} = \frac{2^2}{4} = 1$

$(2,1)$

$D = f_{xx}f_{yy} - f_{xy}^2$

$f_{xx} = 6x$

$f_{yy} = 48y$

$f_{xy} = -12$

$D = (6x)(48y) - (-12)^2 = 288xy - 144$

try each point:

| $(x,y)$ | $D$ | |
|---|---|---|
| $(0,0)$ | $288(0)(0) - 144 = -144 < 0$ | Saddle point at $(0,0)$ |
| $(2,1)$ | $288(2)(1) - 144 = 432 > 0$ max or min | local min at $(2,1)$ |

use concavity:

$f_{xx} = 6x = 6(2)$

$= 12 > 0$

concave up

so min