Calc III - Ch 16 - Required Practice **Name:** Solutions

Ch16 Test Review (for test day 1)

#1. Find $div\ \vec{F}$ and $curl\ \vec{F}$

(a) $\vec{F}(x,y,z) = \langle x^2yz,\ xy^2z,\ xyz^2 \rangle$

$div\ \vec{F} = \frac{\partial}{\partial x}[x^2yz] + \frac{\partial}{\partial y}[xy^2z] + \frac{\partial}{\partial z}[xyz^2]$

$= 2xyz + 2xyz + 2xyz$

$= \boxed{6xyz}$

$curl\ \vec{F} = \begin{vmatrix} + & - & + \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2yz & xy^2z & xyz^2 \end{vmatrix}$

$= \boxed{\langle xz^2 - xy^2,\ -(yz^2 - x^2y),\ y^2z - x^2z \rangle}$

(b) $\vec{F}(x,y,z) = \langle 1,\ x+yz,\ xy - \sqrt{z} \rangle$

$div\ \vec{F} = \frac{\partial}{\partial x}[1] + \frac{\partial}{\partial y}[x+yz] + \frac{\partial}{\partial z}[xy - z^{1/2}]$

$= 0 + z + \left(-\frac{1}{2}z^{-1/2}\right)$

$= \boxed{z - \frac{1}{2\sqrt{z}}}$

$curl\ \vec{F} = \begin{vmatrix} + & - & + \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 1 & x+yz & xy - z^{1/2} \end{vmatrix}$

$= \langle x - y,\ -(y - 0),\ 1 - 0 \rangle$

$= \boxed{\langle x-y,\ -y,\ 1 \rangle}$

#2. Determine whether or not the field is conservative. If it is, find the potential function such that $\nabla f = \vec{F}$.

(a) $\vec{F}(x,y) = \langle \overset{P}{e^x \cos y}, \overset{Q}{e^x \sin y} \rangle$ conservative? $\frac{\partial P}{\partial y} = -e^x \sin y$, $\frac{\partial Q}{\partial x} = e^x \sin y$

not conservative (even $-$ sign is enough)

so no potential function f.

(b) $\vec{F}(x,y) = \langle \overset{P}{e^x \sin y}, \overset{Q}{e^x \cos y} \rangle$ conservative? $\frac{\partial P}{\partial y} = e^x \cos y$, $\frac{\partial Q}{\partial x} = e^x \cos y$ $=$ yes

$f_x = e^x \sin y$

$f = \int e^x \sin y \, dx = e^x \sin y + g(y)$

$f_y = e^x \cos y + g'(y) \overset{\text{must}}{=} e^x \cos y \rightarrow g'(y) = 0$

$g(y) = \int 0 \, dy = C$

so $\boxed{f = e^x \sin y + C}$ (or you can leave off the constant if you wish)

(c) $\vec{F}(x,y) = \langle \overset{P}{3x^2 + 2y^2}, \overset{Q}{4xy + 3} \rangle$ conservative? $\frac{\partial P}{\partial y} = 4y$, $\frac{\partial Q}{\partial x} = 4y$ $=$ yes

$f_x = 3x^2 + 2y^2$

$f = \int (3x^2 + 2y^2) \, dx = x^3 + 2y^2 x + g(y)$

$f_y = 0 + 4yx + g'(y) \overset{\text{must}}{=} 4xy + 3 \Rightarrow g'(y) = 3$

$g(y) = \int 3 \, dy = 3y + C$

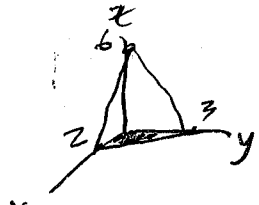
so $\boxed{f = x^3 + 2xy^2 + 3y + C}$

#3. Set up the integral to find the surface area (do not evaluate the integral). (integrand = 1)

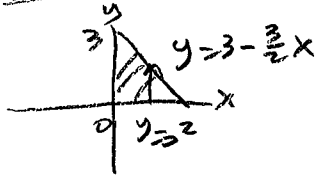
(a) the part of the plane $3x+2y+z=6$ that lies in the first octant.

$z = 6-3x-2y$

$A = \iint_D f(x,y,z)\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA$



Parameter domain:



$$\int_0^2 \int_0^{3-\frac{3}{2}x} (1)\sqrt{1+(-3)^2+(-2)^2}\,dy\,dx$$

$\vec{r}(x,y) = \langle x, y, 6-3x-2y\rangle$

$\frac{\partial z}{\partial x} = -3, \quad \frac{\partial z}{\partial y} = -2$

(b) the part of the plane with vector equation $\vec{r}(u,v) = \langle 1+v,\ u-2v,\ 3-5u+v\rangle$ that is given by $0 \le u \le 1,\ \ 0 \le v \le 1$.

$A = \iint_D f(\vec{r}(u,v))\,|\vec{r}_u \times \vec{r}_v|\,dA$

$\vec{r}_u = \langle 0, 1, -5\rangle, \quad \vec{r}_v = \langle 1, -2, 1\rangle$

$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} + & - & + \\ 0 & 1 & -5 \\ 1 & -2 & 1 \end{vmatrix} = \langle 1-10, -(0+5), 0-1\rangle = \langle -9, -5, -1\rangle$

$|\vec{r}_u \times \vec{r}_v| = \sqrt{9^2+5^2+1^2} = \sqrt{107}$

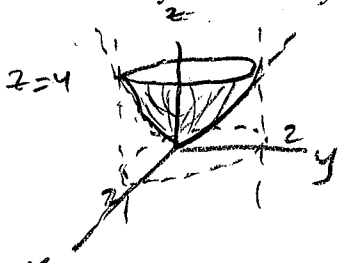
$$\int_0^1 \int_0^1 (1)\sqrt{107}\,du\,dv$$

(or $dv\,du$)

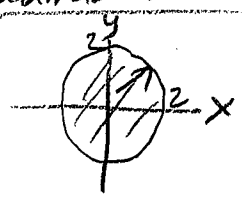
#3 (continued). Set up the integral to find the surface area (do not evaluate the integral).

(c) the part of the paraboloid $z = x^2 + y^2$ that lies inside the cylinder $x^2 + y^2 = 4$.

$$\iint_D f(x,y,z)\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA$$



parameter domain:



$\vec{r}(x,y) = \langle x, y, x^2+y^2 \rangle$

$\frac{\partial z}{\partial x} = 2x, \quad \frac{\partial z}{\partial y} = 2y$

use polar:

$r=0$ to $r=2$

$\theta=0$ to $\theta=2\pi$

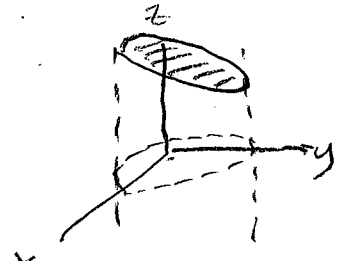
$$\int_0^{2\pi}\int_0^2 (1)\sqrt{1+(2x)^2+(2y)^2}\,r\,dr\,d\theta$$

to polar: $4x^2+4y^2$, $4(x^2+y^2)$, $4r^2$

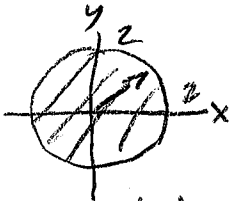
$$\boxed{\int_0^{2\pi}\int_0^2 (1)\sqrt{1+4r^2}\,r\,dr\,d\theta}$$

(d) the part of the plane $z = 4 - y$ that lies inside the cylinder $x^2 + y^2 = 4$.

$$\iint_D f(x,y,z)\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA$$



parameter domain:



$\vec{r}(x,y) = \langle x, y, 4-y \rangle$

$\frac{\partial z}{\partial x} = 0, \quad \frac{\partial z}{\partial y} = -1$

use polar:

$r=0$ to $r=2$

$\theta=0$ to $\theta=2\pi$

$$\boxed{\int_0^{2\pi}\int_0^2 (1)\sqrt{1+(0)^2+(-1)^2}\,r\,dr\,d\theta \quad \text{or} \quad \int_0^{2\pi}\int_0^2 (1)\sqrt{2}\,r\,dr\,d\theta}$$

#4. Set up (but do not evaluate) the line integral.

$\int f(\vec{r})\,|\vec{r}'|\,dt$

(a) $\int_C y^3\,ds$ if C is defined by:

$x = t^3,\ y = t$ *for* $0 \le t \le 2$.

$\vec{r}(t) = \langle t^3, t \rangle$

$\vec{r}'(t) = \langle 3t^2, 1 \rangle$

$|\vec{r}'(t)| = \sqrt{(3t^2)^2 + (1)^2} = \sqrt{9t^4+1}$

$f(\vec{r}) = (t)^3 = t^3$

$\int_0^2 t^3\sqrt{9t^4+1}\,dt$

(b) $\int_C xy\,ds$ if C is defined by:

$\int f(\vec{r})\,|\vec{r}'|\,dt$

$x = t^2,\ y = 2t$ *for* $0 \le t \le 1$.

$\vec{r}(t) = \langle t^2, 2t \rangle$

$\vec{r}'(t) = \langle 2t, 2 \rangle$

$|\vec{r}'(t)| = \sqrt{(2t)^2 + (2)^2} = \sqrt{4t^2+4}$

$f(\vec{r}) = (t^2)(2t) = 2t^3$

$\int_0^1 2t^3\sqrt{4t^2+4}\,dt$

#5. Set up (**and evaluate**) the line integral. (hint: check to see if the field is conservative)

(a) $\int_C \vec{F}\cdot\vec{dr}$ for $\vec{F}=\langle \overset{P}{x^2},\ \overset{Q}{y^2}\rangle$ along curve C defined by the arc of the parabola $y=2x^2$ from $(-1,2)$ to $(2,8)$.

$\vec{F}$ conservative? $\frac{\partial P}{\partial y}=0,\ \frac{\partial Q}{\partial x}=0 \Rightarrow$ yes

So $\int_C \vec{F}\cdot d\vec{r} = f(\text{end})-f(\text{start})$

$f_x = x^2$

$f=\int x^2\,dx = \frac{1}{3}x^3+g(y)$

$f_y = 0+g'(y) \overset{\text{must}}{=} y^2 \rightarrow g'(y)=y^2$

$g(y)=\int y^2\,dy = \frac{1}{3}y^3 (+c)$

$f=\frac{1}{3}x^3+\frac{1}{3}y^3$

So $\int_C \vec{F}\cdot d\vec{r} = f(2,8)-f(-1,2) = \left[\frac{1}{3}(2)^3+\frac{1}{3}(8)^3\right]-\left[\frac{1}{3}(-1)^3+\frac{1}{3}(2)^3\right]$

$\frac{8}{3}+\frac{512}{3}+\frac{1}{3}-\frac{8}{3} = \boxed{\frac{513}{3}}$

(don't need to use the parts)

(b) $\int_C \vec{F}\cdot\vec{dr}$ for $\vec{F}=\langle \overset{P}{xy^2},\ \overset{Q}{x^2y}\rangle$ along curve C defined by $\vec{r}(t)=\left\langle t+\sin\left(\frac{1}{2}\pi t\right),\ t+\cos\left(\frac{1}{2}\pi t\right)\right\rangle$

$\vec{F}$ conservative? $\frac{\partial P}{\partial y}=2xy,\ \frac{\partial Q}{\partial x}=2xy \Rightarrow$ yes

So $\int_C \vec{F}\cdot d\vec{r} = f(\text{end})-f(\text{start})$

$0\le t\le 1$

$f_x = xy^2$

$f=\int xy^2\,dx = \frac{1}{2}x^2y^2+g(y)$

$f_y = x^2y+g'(y) \overset{\text{must}}{=} x^2y \rightarrow g'(y)=0$

$g(y)=\int 0\,dy = 0 (+c)$

$f=\frac{1}{2}x^2y^2$

So $\int_C \vec{F}\cdot d\vec{r} = f(\text{end})-f(\text{start})$

$= f(2,1)-f(0,1)$

$= \frac{1}{2}(2)^2(1)^2-\frac{1}{2}(0)^2(1)^2$

$= 2-0$

$= \boxed{2}$

end: $t=1$

$\vec{r}(1)=\left\langle 1+\sin\frac{\pi}{2},\ 1+\cos\frac{\pi}{2}\right\rangle$

$=\langle 1+1,\ 1+0\rangle = \langle 2,1\rangle$

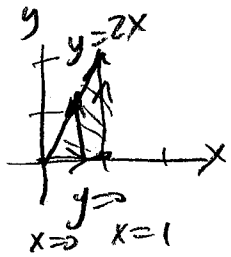
start: $t=0$

$\vec{r}(0)=\langle 0+\sin 0,\ 0+\cos 0\rangle$

$=\langle 0+0,\ 0+1\rangle = \langle 0,1\rangle$

#6. Set up (but do not evaluate) the line integral. (hint: use Green's Theorem)

(a) $\int_C \underset{P}{(xy)}\,dx + \underset{Q}{\left(x^2y^3\right)}\,dy$ along C which traces the path around a triangle from $(0,0)$ to $(1,0)$ to $(1,2)$ and back to $(0,0)$.

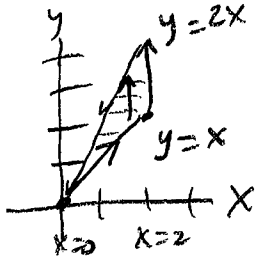


$$\oint \vec{F}\cdot d\vec{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$\frac{\partial Q}{\partial x} = 2xy^3 \qquad \frac{\partial P}{\partial y} = x$$

$$\int_0^1 \int_0^{2x} (2xy^3 - x)\,dy\,dx$$

(b) $\int_C \underset{P}{\left(xy^2\right)}\,dx + \underset{Q}{\left(2x^2y\right)}\,dy$ along C which traces the path around a triangle from $(0,0)$ to $(2,2)$ to $(2,4)$ and back to $(0,0)$.



$$\oint \vec{F}\cdot d\vec{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$\frac{\partial Q}{\partial x} = 4xy \qquad \frac{\partial P}{\partial y} = 2xy$$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 4xy - 2xy = 2xy$$

$$\int_0^2 \int_x^{2x} 2xy\,dy\,dx$$

#7. Find the equation of the tangent plane to the parametric surface.

(a) $x = u+v,\ y = 3u^2,\ z = u-v$ at $(2,3,0)$.

at $(2,3,0)$: $u+v=2$, $3u^2=3 \to u=\pm 1$, $u-v=0$

$u+v=2$
$+\ u-v=0$
$2u=2$ $\quad u=1$ $\quad v=1$

$\vec{r} = \langle u+v, 3u^2, u-v\rangle$

$\vec{r}_u = \langle 1, 6u, 1\rangle$ $\quad \vec{r}_v = \langle 1, 0, -1\rangle$

$$\vec{n} = \vec{r}_u \times \vec{r}_v = \begin{vmatrix} + & - & + \\ 1 & 6u & 1 \\ 1 & 0 & -1 \end{vmatrix} = \langle -6u-0, -(-1-1), 0-6u\rangle = \langle -6u, 2, -6u\rangle$$

at $(u,v)=(1,1)$, $\vec{n} = \langle -6(1), 2, -6(1)\rangle = \langle -6, 2, -6\rangle$

$ax+by+cz = \vec{n}\cdot\vec{r}_0$

$-6x+2y-6z = \langle -6,2,-6\rangle \cdot \langle 2,3,0\rangle = (-6)(2)+(2)(3)+(-6)(0) = -6$

$$\boxed{\begin{array}{c} -6x+2y-6z=-6 \\ \text{or} \\ -3x+y-3z=-3 \\ \text{or} \\ 3x-y+3z=3 \end{array}}$$

(b) $x = u^2,\ y = v^2,\ z = uv$ at $u=1,\ v=1$.

at $u=1, v=1$:

$\vec{r}(1,1) = \langle (1)^2, (1)^2, (1)(1)\rangle = \langle 1,1,1\rangle$ $(\vec{r}_0)$

$\vec{r} = \langle u^2, v^2, uv\rangle$

$\vec{r}_u = \langle 2u, 0, v\rangle$ $\quad \vec{r}_v = \langle 0, 2v, u\rangle$

$$\vec{n} = \vec{r}_u \times \vec{r}_v = \begin{vmatrix} + & - & + \\ 2u & 0 & v \\ 0 & 2v & u \end{vmatrix} = \langle 0-2v^2, -(2u^2-0), 4uv-0\rangle = \langle -2v^2, -2u^2, 4uv\rangle$$

at $(u,v)=(1,1)$, $\vec{n} = \langle -2(1)^2, -2(1)^2, 4(1)(1)\rangle = \langle -2,-2,4\rangle$

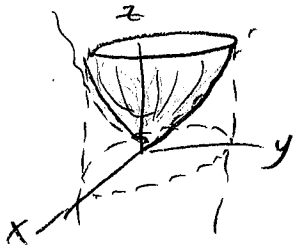
$ax+by+cz = \vec{n}\cdot\vec{r}_0$

$-2x-2y+4z = \langle -2,-2,4\rangle \cdot \langle 1,1,1\rangle = (-2)(1)+(-2)(1)+(4)(1) = 0$

$$\boxed{\begin{array}{c} -2x-2y+4z=0 \\ \text{or} \\ x+y-2z=0 \end{array}}$$

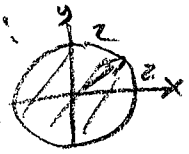
#8. Set up (but do not evaluate) the surface integral.

(a) $\iint_S (z)\,dS$ where S is the part of the paraboloid $z = x^2 + y^2$ that lies inside the cylinder $x^2 + y^2 = 4$.



surface: $\vec{r}(x,y) = \langle x, y, x^2+y^2 \rangle$

parameter domain:



polar:
$r=0 \to r=2$
$\theta=0 \to \theta=2\pi$

$\iint f(x,y,z)\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA$

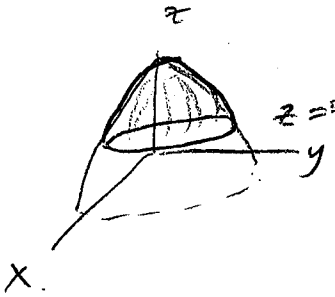
$\frac{\partial z}{\partial x} = 2x \quad \frac{\partial z}{\partial y} = 2y$

$\int_0^{2\pi}\int_0^2 (x^2+y^2)\sqrt{1+(2x)^2+(2y)^2}\,r\,dr\,d\theta$

↳ to polar: $r^2\sqrt{1+4(x^2+y^2)} = r^2\sqrt{1+4r^2}$

$$\int_0^{2\pi}\int_0^2 r^2\sqrt{1+4r^2}\;r\,dr\,d\theta$$

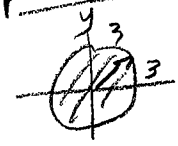
(b) $\iint_S (xz)\,dS$ where S is the part of the paraboloid $z = 16 - x^2 - y^2$ that lies above the plane $z = 7$.



intersection:
$\begin{cases} z = 16-x^2-y^2 \\ z = 7 \end{cases}$
$7 = 16-x^2-y^2$
$x^2+y^2 = 9$

surface:
$\vec{r}(x,y) = \langle x, y, 16-x^2-y^2 \rangle$

parameter domain:



polar:
$r=0 \to r=3$
$\theta=0 \to \theta=2\pi$

$\iint f(x,y,z)\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA$

$\frac{\partial z}{\partial x} = -2x \quad \frac{\partial z}{\partial y} = -2y$

$\int_0^{2\pi}\int_0^3 x(16-x^2-y^2)\sqrt{1+(-2x)^2+(-2y)^2}\,r\,dr\,d\theta$

↳ to polar:

$(r\cos\theta)(16-r^2)\sqrt{1+4r^2}$

$$\int_0^{2\pi}\int_0^3 (r\cos\theta)(16-r^2)\sqrt{1+4r^2}\;r\,dr\,d\theta$$

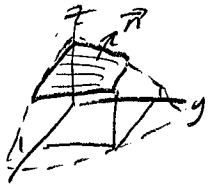
Ch16 Test Review (for test day 2)

#9. Set up (but do not evaluate) the integral to evaluate the flux of $\vec{F}$ over S

(a) $\vec{F}(x,y,z)=\langle xy,\ yz,\ zx\rangle$

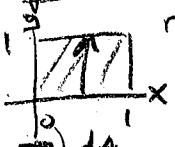
S is the part of the paraboloid

$z=4-x^2-y^2$ that lies above the square $0\le x\le 1,\ 0\le y\le 1$ and has upward orientation.



Surface: $\vec{r}(x,y)=\langle x, y, 4-x^2-y^2\rangle$

parameter domain: rectangular: $y=0$ to $y=1$, $x=0$ to $x=1$



$\iint \vec{F}\cdot(\vec{r}_u\times\vec{r}_v)\,dA$

$\vec{r}_u=\vec{r}_x=\langle 1,0,-2x\rangle$ $\vec{r}_v=\vec{r}_y=\langle 0,1,-2y\rangle$

$\vec{n}=\vec{r}_u\times\vec{r}_v=\begin{vmatrix} + & - & + \\ 1 & 0 & -2x \\ 0 & 1 & -2y \end{vmatrix}$

$=\langle 0+2x,-(-2y-0),1-0\rangle=\langle 2x,2y,1\rangle$

$\vec{F}(\vec{r})=\langle xy,\ y(4-x^2-y^2),\ (4-x^2-y^2)x\rangle$

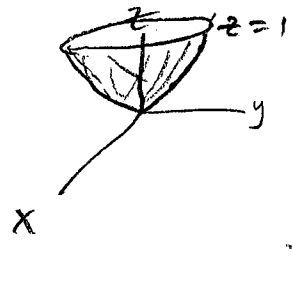
$\vec{F}\cdot\vec{n}=\langle xy,\ y(4-x^2-y^2),\ (4-x^2-y^2)x\rangle\cdot\langle 2x,2y,1\rangle$

$=2x^2y+2y^2(4-x^2-y^2)+(4-x^2-y^2)x$

$$\int_0^1\int_0^1 \left(2x^2y+2y^2(4-x^2-y^2)+(4-x^2-y^2)x\right)dy\,dx$$

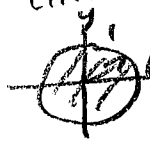
(b) $\vec{F}(x,y,z)=\langle x^2,\ xy,\ z\rangle$

S is the part of the paraboloid $z=x^2+y^2$ that lies below the plane $z=1$ with upward orientation.



Surface: $\vec{r}(x,y)=\langle x, y, x^2+y^2\rangle$

parameter domain: (intersection $z=x^2+y^2=1$)



polar: $r=0$ to $r=1$, $\theta=0$ to $\theta=2\pi$

$\iint \vec{F}\cdot(\vec{r}_u\times\vec{r}_v)\,dA$

$\vec{r}_u=\vec{r}_x=\langle 1,0,2x\rangle$ $\vec{r}_v=\vec{r}_y=\langle 0,1,2y\rangle$

$\vec{n}=\vec{r}_u\times\vec{r}_v=\begin{vmatrix} + & - & + \\ 1 & 0 & 2x \\ 0 & 1 & 2y \end{vmatrix}$

$=\langle 0-2x,-(2y-0),1-0\rangle=\langle -2x,-2y,1\rangle$

$\vec{F}(\vec{r})=\langle x^2,\ xy,\ x^2+y^2\rangle$

$\vec{F}\cdot\vec{n}=\langle x^2,xy,x^2+y^2\rangle\cdot\langle -2x,-2y,1\rangle$

$=-2x^3-2xy^2+x^2+y^2$

to polar...

$-2x(x^2+y^2)+(x^2+y^2)=-2xr^2+r^2$

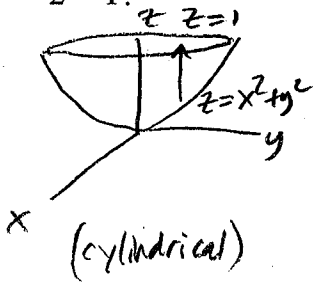
$=r^2(-2x+1)=r^2(-2r\cos\theta+1)$

$$\int_0^{2\pi}\int_0^1 r^2(-2r\cos\theta+1)\,r\,dr\,d\theta$$

#10. Using the Divergence Theorem, write the **triple-integral** (but do not evaluate) which is equivalent to the surface integral $\iint_S \vec{F} \cdot d\vec{S}$ which $= \iiint div\,\vec{F}\, dV$

calculates the flux of $\vec{F}$ across S.

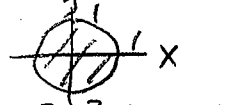
(a) $\vec{F} = \langle x^2, xy, z \rangle$ and S is the part of the paraboloid $z = x^2 + y^2$ that lies below the plane $z = 1$.



(cylindrical)

intersection:

$\begin{cases} z = x^2+y^2 \\ z = 1 \end{cases}$ $x^2+y^2=1$



$z = x^2+y^2 \to z = 1$

$r = 0 \to r = 1$

$\theta = 0 \to \theta = 2\pi$

$div\,\vec{F} = \frac{\partial}{\partial x}[x^2] + \frac{\partial}{\partial y}[xy] + \frac{\partial}{\partial z}[z]$

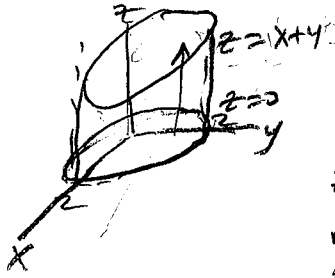
$= 2x + x + 1 = 3x + 1$

$\to$ cylindrical $= 3(r\cos\theta) + 1$

$\int_0^{2\pi} \int_0^1 \int_{x^2+y^2}^1 (3r\cos\theta + 1)\, r\, dz\, dr\, d\theta$

$\int_0^{2\pi} \int_0^1 \int_{r^2}^1 (3r\cos\theta + 1)\, r\, dz\, dr\, d\theta$

(b) $\vec{F} = \langle x^2, -x^2yz, 2xy^2 \rangle$ and S is the cylinder $x^2 + y^2 = 4$ between the planes $z = x + 4$ and $z = 0$.



(cylindrical)



$z = 0 \to z = x + 4$

$r = 0 \to r = 2$

$\theta = 0 \to \theta = 2\pi$

$div\,\vec{F} = \frac{\partial}{\partial x}[x^2] + \frac{\partial}{\partial y}[-x^2yz] + \frac{\partial}{\partial z}[2xy^2]$

$= 2x + (-x^2 z) + 0$

$= 2x - x^2 z \to$ cylindrical

$= 2(r\cos\theta) - (r\cos\theta)^2 z$

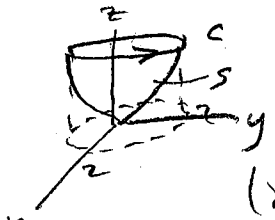
$\int_0^{2\pi} \int_0^2 \int_0^{x+4} (2r\cos\theta - (r\cos\theta)^2 z)\, r\, dz\, dr\, d\theta$

$\int_0^{2\pi} \int_0^2 \int_0^{r\cos\theta+4} (2r\cos\theta - (r\cos\theta)^2 z)\, r\, dz\, dr\, d\theta$

#11. Using Stokes' Theorem, write the **single-integral** (but do not evaluate) which calculates

$$\iint_S \left(curl\,\vec{F}\right)\cdot d\vec{S} = \int \vec{F}\cdot d\vec{r} = \int \vec{F}(\vec{r})\cdot\vec{r}\,'\,dt$$

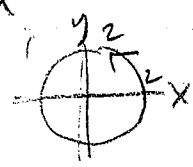
(a) $\vec{F} = \langle x^2z^2,\ y^2z^2,\ xyz\rangle$ and S is the part of the paraboloid $z = x^2 + y^2$ that inside the cylinder $x^2 + y^2 = 4$ and is oriented upward.



Intersection:

$\begin{cases} z = x^2+y^2 \\ x^2+y^2 = 4 \end{cases}$

$(x^2+y^2 \text{ w/ } z=4)$



$\vec{r}(t) = \langle 2\cos t, 2\sin t, 4\rangle$

$(0 \le t \le 2\pi)$

$\vec{r}\,' = \langle -2\sin t, 2\cos t, 0\rangle$

$\vec{F}(\vec{r}) = \langle (2\cos t)^2(4)^2, (2\sin t)^2(4)^2, (2\cos t)(2\sin t)(4)\rangle$

$= \langle 64\cos^2 t, 64\sin^2 t, 16\cos t\sin t\rangle$

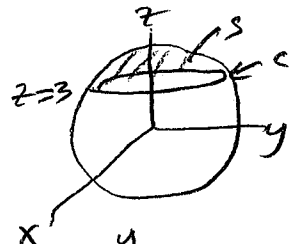
$\vec{F}\cdot\vec{r}\,' =$

$\langle 64\cos^2 t, 64\sin^2 t, 16\cos t\sin t\rangle \cdot \langle -2\sin t, 2\cos t, 0\rangle$

$= -128\cos^2 t\sin t + 128\sin^2 t\cos t$

$$\int_0^{2\pi} (-128\cos^2 t\sin t + 128\sin^2 t\cos t)\,dt$$

(b) $\vec{F} = \langle 2x^2,\ -2y^2,\ x^2z\rangle$ and S is the part of the sphere $x^2 + y^2 + z^2 = 13$ that lies above the plane $z = 3$ and is oriented upward.

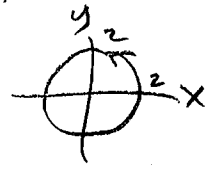


Intersection

$\begin{cases} x^2+y^2+z^2 = 13 \\ z = 3 \end{cases}$

$x^2 + y^2 + (3)^2 = 13$

$x^2 + y^2 = 4$



$\vec{r}(t) = \langle 2\cos t, 2\sin t, 3\rangle$

$(0 \le t \le 2\pi)$

$\vec{r}\,' = \langle -2\sin t, 2\cos t, 0\rangle$

$\vec{F}(\vec{r}) = \langle 2(2\cos t)^2, -2(2\sin t)^2, (2\cos t)^2(3)\rangle$

$= \langle 8\cos^2 t, -8\sin^2 t, 12\cos^2 t\rangle$

$\vec{F}\cdot\vec{r}\,' =$

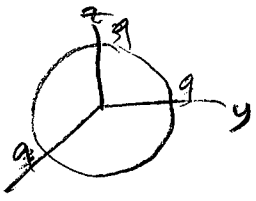
$\langle 8\cos^2 t, -8\sin^2 t, 12\cos^2 t\rangle \cdot \langle -2\sin t, 2\cos t, 0\rangle$

$= -16\cos^2 t\sin t - 16\sin^2 t\cos t$

$$\int_0^{2\pi} (-16\cos^2 t\sin t - 16\sin^2 t\cos t)\,dt$$

#12. Using the Divergence Theorem, write (but do not evaluate) the **triple-integral** which is equivalent to the surface integral $\iint_S \vec{F} \cdot \vec{dS}$ which $= \iiint \text{div}\,\vec{F}\, dV$

calculates the flux of $\vec{F}$ across S if $\vec{F} = \langle 3x^2,\ xyz,\ z^3 \rangle$ and S is the surface of a sphere of radius 9 centered at the origin.



$$\text{div}\,\vec{F} = \frac{\partial}{\partial x}[3x^2] + \frac{\partial}{\partial y}[xyz] + \frac{\partial}{\partial z}[z^3]$$

$$= 6x + xz + 3z^2$$

→ to spherical...

$$= 6(\rho\sin\phi\cos\theta) + (\rho\sin\phi\cos\theta)(\rho\cos\phi) + 3(\rho\cos\phi)^2$$

(spherical)

$\rho = 0$ to $\rho = 3$

$\phi = 0$ to $\phi = \pi$

$\theta = 0$ to $\theta = 2\pi$

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{9} \left[6(\rho\sin\phi\cos\theta) + (\rho\sin\phi\cos\theta)(\rho\cos\phi) + 3(\rho\cos\phi)^2\right] \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

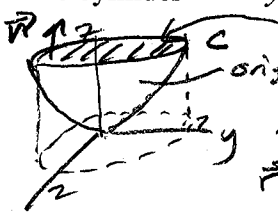
#13. Using Stokes' Theorem, write the **double-integral** (but do not evaluate) which is equivalent to the line integral $\int_C \vec{F} \cdot d\vec{r}$ which sums the contributions of the given field along the given path.

$= \iint (\text{curl}\vec{F}) \cdot \vec{n}\, dA$

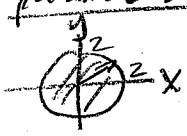
Hints: You can assume the path is oriented such that the normal vector to the enclosed surface is in the 'positive' direction.

Remember: you can choose any surface that is bounded by the curve of the given surface – you don't have to use the original surface.

(a) $\vec{F} = \langle x^2z^2,\ x^2,\ xyz \rangle$ along the path which is the intersection of the paraboloid $z = x^2 + y^2$ and the cylinder $x^2 + y^2 = 4$.



parameter domain



polar: r=0 to r=2
θ=0 to θ=2π

$\text{curl}\vec{F} = \begin{vmatrix} + & - & + \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2z^2 & x^2 & xyz \end{vmatrix}$

$= \langle xz-0, -(yz-2x^2z), 2x-0 \rangle$

$= \langle xz,\ 2x^2z-yz,\ 2x \rangle$

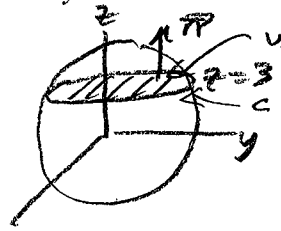
$\text{curl}\vec{F} \cdot \vec{n} = \langle xz,\ 2x^2z-yz,\ 2x \rangle \cdot \langle 0,0,1 \rangle$

$= 2x$

to polar $\rightarrow 2r\cos\theta$

$$\int_0^{2\pi} \int_0^2 (2r\cos\theta)\, r\, dr\, d\theta$$

(b) $\vec{F} = \langle x+y^2,\ y+z^2,\ z+x^2 \rangle$ along the path which is the intersection of the sphere $x^2 + y^2 + z^2 = 13$ and the plane $z = 3$.



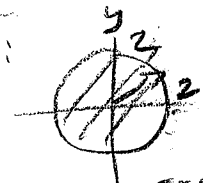
parameter domain

set by intersection:

$\begin{cases} x^2+y^2+z^2 = 13 \\ z = 3 \end{cases}$

$x^2+y^2+(3)^2 = 13$

$x^2+y^2 = 4$



polar: r=0 to r=2
θ=0 to θ=2π

$\text{curl}\vec{F} = \begin{vmatrix} + & - & + \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x+y^2 & y+z^2 & z+x^2 \end{vmatrix}$

$= \langle 0-2z, -(2x-0), 0-2y \rangle$

$= \langle -2z, -2x, -2y \rangle$

$\text{curl}\vec{F} \cdot \vec{n} = \langle -2z, -2x, -2y \rangle \cdot \langle 0,0,1 \rangle$

$= -2y$ to polar...

$= -2r\sin\theta$

$$\int_0^{2\pi} \int_0^2 (-2r\sin\theta)\, r\, dr\, d\theta$$