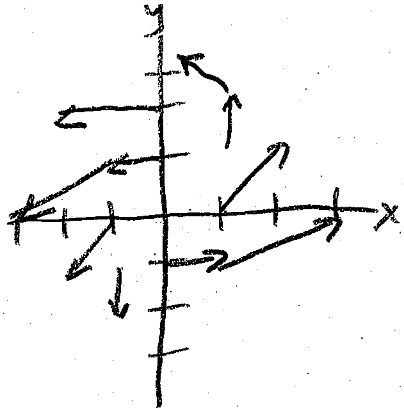
Calc III - Ch 16 - Extra Practice

16.1 and 16.2 day 1

#1b. Sketch the vector field for $\vec{F}(x,y)=\langle x-y,\ x\rangle$

(x,y)	$\vec{F}=\langle x-y, x\rangle$
$(0,0)$	$\langle 0,0\rangle$
$(1,1)$	$\langle 0,1\rangle$
$(1,-1)$	$\langle 2,1\rangle$
$(-1,1)$	$\langle -2,-1\rangle$
$(-1,-1)$	$\langle 0,-1\rangle$
$(1,0)$	$\langle 1,1\rangle$
$(0,1)$	$\langle -1,0\rangle$
$(-1,0)$	$\langle -1,-1\rangle$
$(0,-1)$	$\langle 1,0\rangle$
$(0,2)$	$\langle -2,0\rangle$
$(1,2)$	$\langle -1,1\rangle$



#2b. Evaluate the line integral, where C is the given curve: $\int_C xy\,ds \quad C: x=t^2,\ y=2t,\ 0\le t\le 1$

$\vec{r}=\langle t^2, 2t\rangle$

$\vec{r}\,'=\langle 2t, 2\rangle$

$|\vec{r}\,'|=\sqrt{4t^2+4}=\sqrt{4}\sqrt{t^2+1}=2\sqrt{t^2+1}$

$\int_0^1 (t^2)(2t)2\sqrt{t^2+1}\,dt$

$u=t^2+1,\ t^2=u-1$

$2\int_0^1 t^2\,2t\sqrt{t^2+1}\,dt \quad \frac{du}{dt}=2t \quad t=0\to u=1$

$du=2t\,dt \quad t=1\to u=2$

$2\int_1^2 (u-1)u^{1/2}\,du=2\int_1^2 (u^{3/2}-u^{1/2})\,du$

$=2\left[\frac{2}{5}u^{5/2}-\frac{2}{3}u^{3/2}\right]_1^2$

$=2\left[\left(\frac{2}{5}(2)^{5/2}-\frac{2}{3}(2)^{3/2}\right)-\left(\frac{2}{5}(1)^{5/2}-\frac{2}{3}(1)^{3/2}\right)\right]$

$=2\left[\frac{8\sqrt{2}}{5}-\frac{4\sqrt{2}}{3}\right]-\left(\frac{2}{5}-\frac{2}{3}\right)=\boxed{\frac{8\sqrt{2}+4}{15}}$

#3b. Evaluate the line integral, where C is the given curve: $\int_C x\sin y\,ds$ where C is the line segment from $(0,3)$ to $(4,6)$.

$\vec{r}=(1-t)\vec{r}_0+t\vec{r}_1=(1-t)\langle 0,3\rangle+t\langle 4,6\rangle$

$\vec{r}=\langle 0,3-3t\rangle+\langle 4t,6t\rangle \quad 0\le t\le 1$

$\vec{r}=\langle 4t,3+3t\rangle$

$\vec{r}\,'=\langle 4,3\rangle$

$|\vec{r}\,'|=\sqrt{4^2+3^2}=5$

$\int_0^1 (4t)\sin(3+3t)\,5\,dt$

u-sub first $\quad t=\frac{u-3}{3}$

$20\int_0^1 t\sin(3+3t)\,dt \quad u=3+3t \quad t=0\to u=3$

$\frac{du}{dt}=3 \quad t=1\to u=6$

$du=3\,dt$

$20\int_3^6 t\sin u\left(\frac{1}{3}du\right)$

changing letters:

$20\int_3^6 \frac{1}{3}(u-3)\sin u\left(\frac{1}{3}du\right)=\frac{20}{9}\int_3^6 (w-3)\sin w\,dw$

$=\frac{20}{9}\int_3^6 w\sin w\,dw-\frac{20}{3}\int_3^6 \sin w\,dw$

by parts: $\quad -\frac{20}{3}\left[-\cos w\right]_3^6$

$u=w \quad dv=\sin w\,dw$

$\frac{du}{dw}=1 \quad \int dv=\int \sin w\,dw$

$du=dw \quad v=-\cos w$

$uv-\int v\,du$

$-w\cos w+\int \cos w\,dw$

$\frac{20}{9}\left[-w\cos w+\sin w\right]_3^6-\frac{20}{3}\left[-\cos w\right]_3^6$

$\frac{20}{9}\left[(-6\cos(6)+\sin(6))-(-3\cos(3)+\sin(3))\right]$

$-\frac{20}{3}\left[-\cos(6)-(-\cos(3))\right]$

$=\boxed{\frac{20}{9}\left[-3\cos(6)+\sin(6)-\sin(3)\right]}$

#4b. Evaluate the line integral, where C is the given curve: $\int_C (x+yz)\,dx + 2x\,dy + xyz\,dz$ where C consists of line segments from $(1,0,1)$ to $(2,3,1)$ and from $(2,3,1)$ to $(2,5,2)$.

$\vec{r} = (1-t)\vec{r}_0 + t\vec{r}_1$

C_1 from $(1,0,1)$ to $(2,3,1)$

$\vec{r} = (1-t)\langle 1,0,1\rangle + t\langle 2,3,1\rangle$

$\vec{r} = \langle 1-t, 0, 1-t\rangle + \langle 2t, 3t, t\rangle$

$\vec{r} = \langle 1+t, 3t, 1\rangle$

$0 \le t \le 1$

$\frac{dx}{dt} = 1 \quad \frac{dy}{dt} = 3 \quad \frac{dz}{dt} = 0$

$dx = dt \quad dy = 3dt \quad dz = 0$

C_2 from $(2,3,1)$ to $(2,5,2)$

$\vec{r} = (1-t)\langle 2,3,1\rangle + t\langle 2,5,2\rangle$

$\vec{r} = \langle 2, 3+2t, 1+t\rangle$

$0 \le t \le 1$

$\frac{dx}{dt} = 0 \quad \frac{dy}{dt} = 2 \quad \frac{dz}{dt} = 1$

$dx = 0 \quad dy = 2dt \quad dz = dt$

$\int_{C_1} (x+yz)dx + 2x\,dy + xyz\,dz + \int_{C_2} (x+yz)dx + 2x\,dy + xyz\,dz$

$= \int_0^1 ((1+t)+(3t)(1))dt + 2(1+t)3dt + \cancel{(1+t)(3t)(1)(0)}$

$+ \int_0^1 \cancel{((2)+(3+2t)(1+t))(0)} + 2(2)2dt + (2)(3+2t)(1+t)\,dt$

$= \int_0^1 (1+4t)dt + (6+6t)dt + \int_0^1 8dt + (6+10t+4t^2)dt$

$= \int_0^1 (1+4t+6+6t+8+6+10t+4t^2)dt$

$= \int_0^1 (4t^2+20t+21)\,dt$

$= \left[\frac{4}{3}t^3 + 10t^2 + 21t\right]_0^1$

$= \left[\frac{4}{3}(1)^3 + 10(1)^2 + 21(1)\right] - [0]$

$= \boxed{\frac{97}{3}}$

#5b. Evaluate the line integral, where C is the given curve: $\int_C (2x+9z)\,ds$

$C: x=t,\ y=t^2,\ z=t^3,\ 0 \le t \le 1$

$\vec{r} = \langle t, t^2, t^3\rangle$

$\vec{r}' = \langle 1, 2t, 3t^2\rangle$

$|\vec{r}'| = \sqrt{(1)^2 + (2t)^2 + (3t^2)^2}$

$|\vec{r}'| = \sqrt{1+4t^2+9t^4}$

$\int_0^1 (2(t)+9(t^3))\sqrt{1+4t^2+9t^4}\,dt$

$\int_0^1 (2t+9t^3)\sqrt{1+4t^2+9t^4}\,dt$

$u = 1+4t^2+9t^4$

$\frac{du}{dt} = 8t+36t^3 = 4(2t+9t^3)$

$\frac{1}{4}du = (2t+9t^3)dt \quad t=0 \to u=1$

$t=1 \to u=14$

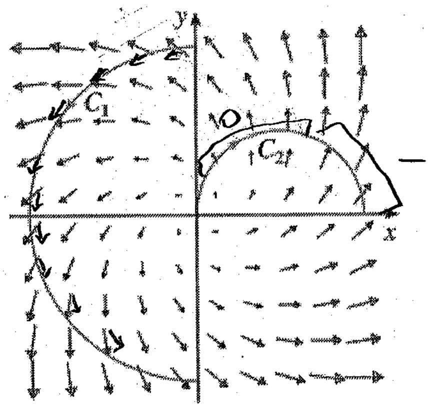
$\frac{1}{4}\int_1^{14} u^{1/2}du = \frac{1}{4}\left(\frac{2}{3}\right)\left[u^{3/2}\right]_1^{14}$

$= \frac{1}{6}\left[(14)^{3/2} - (1)^{3/2}\right]$

$= \boxed{\frac{1}{6}\{14\sqrt{14} - 1\}}$

16.2 day 2

#1b. The figure shows a vector field $\vec{F}$ and two curves C_1 and C_2. Are the line integrals of $\vec{F}$ over C_1 and C_2 $\left(\int_C \vec{F} \cdot d\vec{r}\right)$ positive, negative, or zero. Explain.



$\int_{C_1} \vec{F} \cdot d\vec{r}$ would be $\boxed{\text{positive}}$ because the path is in the direction of the force field arrows.

$\int_{C_2} \vec{F} \cdot d\vec{r}$ would be $\boxed{\text{negative}}$ because the path is in the opposite direction of the force field arrows (and at the beginning of the path, the path is perpendicular to the arrows (which would add zero).

#2b. Evaluate the line integral $\int_{C_1} \vec{F} \cdot d\vec{r}$ where C is given by the vector function $\vec{r}(t)$

$\vec{F}(x,y,z) = \langle \sin x,\ \cos y,\ xz \rangle$

$\vec{r}(t) = \langle t^3,\ -t^2,\ t \rangle \quad 0 \le t \le 1$

$\vec{r}' = \langle 3t^2, -2t, 1 \rangle$

$\vec{F}(r) = \langle \sin(t^3), \cos(-t^2), (t^3)(t) \rangle$

$= \langle \sin(t^3), \cos(-t^2), t^4 \rangle$

$\vec{F} \cdot d\vec{r} = \langle \sin(t^3), \cos(-t^2), t^4 \rangle \cdot \langle 3t^2, -2t, 1 \rangle$

$= 3t^2 \sin(t^3) + (-2t)\cos(-t^2) + t^4$

$\int_0^1 (3t^2 \sin(t^3))dt + \int_0^1 (-2t)\cos(-t^2)dt + \int_0^1 t^4 dt$

$u = t^3 \quad t=0 \to u=0$
$\frac{du}{dt} = 3t^2 \quad t=1 \to u=1$
$du = 3t^2 dt$

$u = -t^2 \quad t=0 \to u=0$
$\frac{du}{dt} = -2t \quad t=1 \to u=-1$
$du = -2t\,dt$

$\int_0^1 \sin u\, du + \int_0^{-1} \cos u\, du + \int_0^1 t^4 dt$

$-[\cos u]_0^1 + [\sin u]_0^{-1} + \frac{1}{5}[t^5]_0^1$

$-(\cos 1 - \cos 0) + (\sin(-1) - \sin 0) + \frac{1}{5}(1^5 - 0^5)$

$-\cos(1) + 1 + \sin(-1) - 0 + \frac{1}{5}$

$\boxed{\frac{6}{5} - \cos(1) + \sin(-1)}$

#3b. Find the work done by the force field $\vec{F}(x,y)=\langle y, -x\rangle$ on a particle that moves along the curve $y=2x^3$ from $(1,2)$ to $(3,54)$.

use x as parameter:

$\vec{r}=\langle t, 2t^3\rangle \quad 1\le t\le 3$

$\vec{r}'=\langle 1, 6t^2\rangle$

$\vec{F}(r)=\langle (2t^3), -(t)\rangle=\langle 2t^3, -t\rangle$

$\vec{F}\cdot\vec{r}'=\langle 2t^3,-t\rangle\cdot\langle 1,6t^2\rangle$

$=(2t^3)(1)+(-t)(6t^2)$

$=2t^3-6t^3$

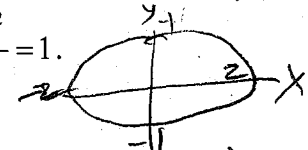
$=-4t^3$

$\int_1^3(-4t^3)dt=-4\left(\frac{1}{4}\right)[t^4]_1^3$

$-[(3)^4-(1)^4]$

$\boxed{-80}$

#4b. Show that a constant force field does zero work on a particle that moves once uniformly around the ellipse $\frac{x^2}{4}+\frac{y^2}{1}=1$.



parametrization for an ellipse:

$\vec{r}=\langle 2\cos t, \sin t\rangle$ (graph in PAR mode in calculator to check :))

$(0\le t\le 2\pi)$

$\vec{r}'=\langle -2\sin t, \cos t\rangle$

constant force: $\vec{F}=\langle a,b\rangle$ (a, b are constants)

$\vec{F}(r)=\langle a,b\rangle$

$\vec{F}\cdot\vec{r}'=\langle a,b\rangle\cdot\langle -2\sin t,\cos t\rangle$

$(a)(-2\sin t)+(b)(\cos t)$

$b\cos t-2a\sin t$

$\int_0^{2\pi}(b\cos t-2a\sin t)dt$

$b\int_0^{2\pi}\cos t\,dt-2a\int_0^{2\pi}\sin t\,dt$

$b[\sin t]_0^{2\pi}-2a[-\cos t]_0^{2\pi}$

$b(\sin 2\pi-\sin 0)+2a[\cos 2\pi-\cos 0]$

$b(0-0)+2a(1-1)$

$=\boxed{0}$

16.3

#1b. Determine whether or not $\vec{F}$ is a conservative vector field. If it is, find a function f such that $\vec{F} = \nabla f$.

(i) $\vec{F}(x,y) = \langle \overset{P}{e^x \sin y},\ \overset{Q}{e^x \cos y} \rangle$

$\frac{\partial P}{\partial y} = e^x \cos y \quad \frac{\partial Q}{\partial x} = e^x \cos y = \underline{yes}$

$f_x = e^x \sin y$

$f = \int e^x \sin y\,dx = e^x \sin y + g(y)$

$f_y = e^x \cos y + g'(y) \overset{\text{must}}{=} e^x \cos y$

$g'(y) = 0 \quad g(y) = \int 0\,dy = C$

$\boxed{f(x,y) = e^x \sin y + C}$

(ii) $\vec{F}(x,y) = \left\langle \overset{P}{\ln y + 2xy^3},\ \overset{Q}{3x^2y^2 + \frac{x}{y}} \right\rangle$

$\frac{\partial P}{\partial y} = \frac{1}{y} + 6xy^2 \quad \frac{\partial Q}{\partial x} = 6xy^2 + \frac{1}{y}$

$=$ yes, conservative

$f_x = \ln y + 2xy^3$

$f = \int(\ln y + 2xy^3)dx = (\ln y)x + x^2y^3 + g(y)$

$f_y = \frac{1}{y}x + 3x^2y^2 + g'(y) \overset{\text{must}}{=} 3x^2y^2 + \frac{x}{y}$

$g'(y) = 0, \quad g(y) = \int 0\,dy = C$

$\boxed{f(x,y) = x\ln y + x^2y^3 + C}$

#2b. (hints) Check to see if the field is conservative. If it is, then the value computed for a line integral depends only upon the endpoints (independent of path), and you can also use the function f as the antiderivative to compute the value of the line path integral.

#3b. Find a function f such that $\vec{F} = \nabla f$ and use it to evaluate $\int_C \vec{F} \cdot d\vec{r}$ along the given curve C.

$\vec{F}(x, y, z) = \langle yz,\ xz,\ xy + 2z \rangle$

C is the line segment from $(1,0,-2)$ to $(4,6,3)$

w/3D just pick one to start...

$f_x = yz$ could be a function of either or both

$f = \int yz\,dx = xyz + g(y,z)$

$f_y = xz + \frac{\partial g}{\partial y} \overset{\text{must}}{=} xz$, so $\frac{\partial g}{\partial y} = 0$

$g(y,z) = \int \frac{\partial g}{\partial y}\,dy = \int 0\,dy = C = h(z)$ the last variable

so $f = xyz + h(z)$

now $f_z = xy + h'(z) \overset{\text{must}}{=} xy + 2z$

$h'(z) = 2z$, $h(z) = \int 2z\,dz = z^2 + c$

and $f(x,y,z) = xyz + z^2 + c$

since we could find f, $\vec{F}$ is conservative

$\int_C \vec{F} \cdot d\vec{r} = \left[xyz + z^2\right]_{(1,0,-2)}^{(4,6,3)}$

$\left[(4)(6)(3) + (3^2)\right] - \left[(1)(0)(-2) + (-2)^2\right]$

$81 - 4$

$= \boxed{77}$

#4b. Show that the line integral is independent of path and evaluate the integral.

$\int_C \left(1 - ye^{-x}\right) dx + e^{-x}\,dy$

C is any path from $(0,1)$ to $(1,2)$

$\frac{\partial P}{\partial y} = -e^{-x}$ $\frac{\partial Q}{\partial x} = -e^{-x}$ $=$, so conservative

can start with either... let's start w/ f_y:

$f_y = e^{-x}$ the other variable

$f = \int e^{-x}\,dy = e^{-x}y + g(x)$

$f_x = -e^{-x}y + g'(x) \overset{\text{must}}{=} 1 - ye^{-x}$

$g'(x) = 1$, $g(x) = \int 1\,dx = x + c$

$f(x,y) = ye^{-x} + x + c$

so $\int_C \vec{F} \cdot d\vec{r} = \left[ye^{-x} + x\right]_{(0,1)}^{(1,2)}$

$\left[(2)e^{-1} + (1)\right] - \left[(1)e^{0} + (0)\right]$

$\frac{2}{e} + 1 - 1$

$= \boxed{\frac{2}{e}}$

#5b. Find the work done by the force field $\vec{F}$ in moving an object from P to Q.

$\vec{F}(x,y)=\langle e^{-y},\ -xe^{-y}\rangle$

$P(0,1),\quad Q(2,0)$

$\vec{F}$ conservative? $\frac{\partial P}{\partial y}=-e^{-y}$ $\frac{\partial Q}{\partial x}=-e^{-y}$

yes

$f_x=e^{-y}$

$f=\int e^{-y}dx=xe^{-y}+g(y)$

$f_y=-xe^{-y}+g'(y)\overset{\text{must}}{=}-xe^{-y}$

$g'(y)=0,\ g(y)=\int 0\,dy=C$

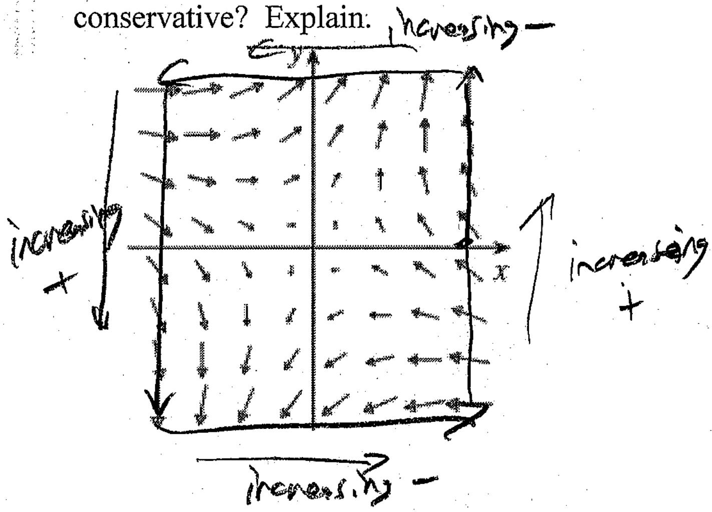
$f(x,y)=xe^{-y}+C$

$W=\int_C \vec{F}\cdot d\vec{r}=\left[xe^{-y}\right]_{(0,1)}^{(2,0)}$

$[(2)e^{0}]-[(0)e^{-1}]$

$=\boxed{2}$

#6b. Is the vector field shown in the figure conservative? Explain:



choosing a closed path,
it appears that the positive and negative contributions cancel to zero,
so it is plausible that
$\oint_C \vec{F}\cdot d\vec{r}=0$ for a closed path
which means
this field is conservative

16.4

#1b. Evaluate the line integral (i) directly and (ii) using Green's Theorem.

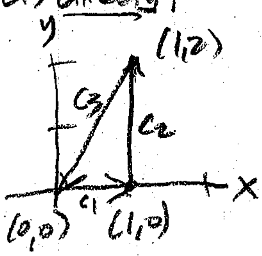
$\oint_C xy\,dx + x^2y^3\,dy$ $\quad \vec{F} = \langle xy, x^2y^3 \rangle$

C is the triangle with vertices (0,0), (1,0) and (1,2).

$\vec{F}$ conservative? $\frac{\partial P}{\partial y} = x$, $\frac{\partial Q}{\partial x} = 2xy^3$

$\neq$, no

(i) directly:



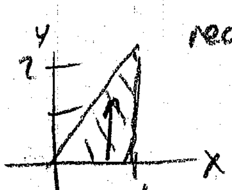
C_1: $\vec{r} = (1-t)\langle 0,0\rangle + t\langle 1,0\rangle$
$= \langle t, 0\rangle \ (0 \le t \le 1)$
$\vec{r}' = \langle 1, 0\rangle$
$\vec{F}(r) = \langle (t)(0), (t)^2(0)^3\rangle$
$= \langle 0,0\rangle$
$\vec{F}\cdot\vec{r}' = \langle 0,0\rangle\cdot\langle 1,0\rangle$
$= 0$
$\int_0^1 (0)\,dt = (0)$

C_2: $\vec{r} = (1-t)\langle 1,0\rangle + t\langle 1,2\rangle$
$= \langle 1, 2t\rangle \ (0 \le t \le 1)$
$\vec{r}' = \langle 0,2\rangle$
$\vec{F}(r) = \langle (1)(2t), (1)^2(2t)^3\rangle$
$= \langle 2t, 8t^3\rangle$
$\vec{F}\cdot\vec{r}' = \langle 2t, 8t^3\rangle\cdot\langle 0,2\rangle$
$= 16t^3$
$\int_0^1 16t^3\,dt = 4[t^4]_0^1 = 4 - 0 = (4)$

C_3: $\vec{r} = (1-t)\langle 1,2\rangle + t\langle 0,0\rangle$
$= \langle 1-t, 2-2t\rangle \ (0 \le t \le 1)$
squared cubed is complicated...
instead, how about use x as parameter for $y = 2x$?
$\vec{r} = \langle t, 2t\rangle$ (t from 1 to 0)
$\vec{r}' = \langle 1,2\rangle$
$\vec{F}(r) = \langle (t)(2t), (t)^2(2t)^3\rangle = \langle 2t^2, 8t^5\rangle$
$\vec{F}\cdot\vec{r}' = \langle 2t^2, 8t^5\rangle\cdot\langle 1,2\rangle = 2t^2 + 16t^5$
$\int_1^0 (2t^2+16t^5)\,dt = \left[\frac{2}{3}t^3 + \frac{16}{6}t^6\right]_1^0 = 0 - \frac{10}{3} = \left(-\frac{10}{3}\right)$

then add each curve's contribution: $0 + 4 + \left(-\frac{10}{3}\right) = \boxed{\frac{2}{3}}$

(ii) Green's $\iint\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dA$



rectangular:
$y = 0$ to $y = 2x$
$x = 0$ to $x = 1$

integrand: $2xy^3 - x$

$\int_0^1 \int_0^{2x} (2xy^3 - x)\,dy\,dx$

$\int_0^{2x} (2xy^3 - x)\,dy = \left[\frac{1}{2}xy^4 - xy\right]_0^{2x} = \left(\frac{1}{2}x(2x)^4 - x(2x)\right) - (0) = 8x^5 - 2x^2$

$\int_0^1 (8x^5 - 2x^2)\,dx = \left[\frac{8}{6}x^6 - \frac{2}{3}x^3\right]_0^1 = \left(\frac{8}{6} - \frac{2}{3}\right) - (0) = \boxed{\frac{2}{3}}$

#1c. Evaluate the line integral

$\oint_C x\,dx + y\,dy$ $\quad \vec{F} = \langle x, y\rangle$

C is the triangle with vertices (0,0), (1,0) and (1,2).

$\vec{F}$ conservative?
$\frac{\partial P}{\partial y} = 1 \quad \frac{\partial Q}{\partial x} = 1$
$=$, yes so $\oint \vec{F}\cdot d\vec{r} = \boxed{0}$
(closed path)

#2b. Use Green's Theorem to evaluate the line integral along the given positively oriented curve.

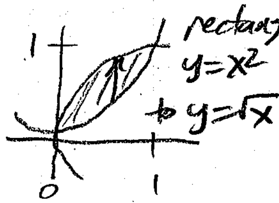
$$\oint_C \left(y+e^{\sqrt{x}}\right)dx+\left(2x+\cos y^2\right)dy$$

C is the boundary of the region enclosed by the parabolas $y=x^2$ and $x=y^2$.

$\vec{F}=\langle y+e^{\sqrt{x}}, 2x+\cos(y^2)\rangle$

conservative? $\frac{\partial P}{\partial y}=1$, $\frac{\partial Q}{\partial x}=2$ $\neq$, no

$\iint\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dA$



rectangular $y=x^2$ to $y=\sqrt{x}$

intersections:

$\begin{cases} y=x^2 \\ x=y^2 \end{cases}$ $y=(y^2)^2=y^4$

$y^4-y=0$

$y(y^3-1)=0$

$y=0 \quad y=1$

$x=0 \quad x=1$

integrand: $\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}=2-1=1$

$$\int_0^1\int_{x^2}^{\sqrt{x}}(1)\,dy\,dx$$

$$\int_{x^2}^{\sqrt{x}}(1)\,dy=(y)_{x^2}^{\sqrt{x}}=\sqrt{x}-x^2$$

$$\int_0^1\left(x^{1/2}-x^2\right)dx$$

$$=\left[\tfrac{2}{3}x^{3/2}-\tfrac{1}{3}x^3\right]_0^1$$

$$=\left(\tfrac{2}{3}(1)^{3/2}-\tfrac{1}{3}(1)^3\right)-(0)$$

$$=\boxed{\tfrac{1}{3}}$$

#3b. Use Green's Theorem to evaluate the line integral along the given positively oriented curve.

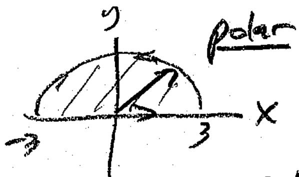
$$\oint_C y^3dx-x^2\,dy \qquad \vec{F}=\langle y^3, -x^2\rangle$$

C is the top half of the circle $x^2+y^2=9$.

$\vec{F}$ conservative? $\frac{\partial P}{\partial y}=3y^2$ $\frac{\partial Q}{\partial x}=-2x$

$\neq$, no

$\iint\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dA$



polar

$r=0$ to $r=3$, $\theta=0$ to $\theta=\pi$

integrand: $\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}=-2x-3y^2$

to polar: $-2(r\cos\theta)-3(r\sin\theta)^2$

$=-2r\cos\theta-3r^2\sin^2\theta$

$$\int_0^\pi\int_0^3\left(-2r\cos\theta-3r^2\sin^2\theta\right)r\,dr\,d\theta$$

$$\int_0^3\left(-2r^2\cos\theta-3r^3\sin^2\theta\right)dr$$

$$=\left[-\tfrac{2}{3}\cos\theta\, r^3-\tfrac{3}{4}\sin^2\theta\, r^4\right]_0^3$$

$$=\left(-\tfrac{2}{3}\cos\theta(3)^3-\tfrac{3}{4}\sin^2\theta(3)^4\right)-(0)=-18\cos\theta-\tfrac{243}{4}\sin^2\theta$$

$\sin^2\theta=\frac{1}{2}-\frac{1}{2}\cos(2\theta)$

$$-18\int_0^\pi\cos\theta\,d\theta-\tfrac{243}{4}\int_0^\pi\sin^2\theta\,d\theta$$

$$-\tfrac{243}{3}\int_0^\pi\left[\tfrac{1}{2}-\tfrac{1}{2}\cos(2\theta)\right]d\theta$$

$$-18[\theta]_0^\pi-\tfrac{243}{3}\cdot\tfrac{1}{2}[\theta]_0^\pi+\tfrac{243}{3}\cdot\tfrac{1}{2}\left[\tfrac{1}{2}\sin(2\theta)\right]_0^\pi$$

$$-18(\pi-0)-\tfrac{243}{6}(\pi-0)+\tfrac{243}{12}\underbrace{(\sin 2\pi-\sin 0)}_{0}$$

$$-18\pi-\tfrac{243}{6}\pi=\boxed{-\tfrac{117}{2}\pi}$$

"positively-oriented curve" means path is counter-clockwise

negative result means overall contributions of $\vec{F}\cdot\vec{r}'$ were in opposite direction of field arrows for more of the path than in direction of field arrows.

#4b. Use Green's Theorem to evaluate $\int_C \vec{F} \cdot d\vec{r}$.

(Check the orientation of the curve before applying the theorem)

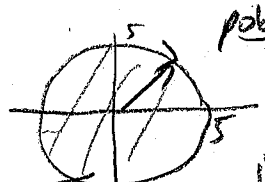
$\vec{F}(x,y) = \langle e^x + x^2 y,\ e^y - xy^2 \rangle$

C is the circle $x^2 + y^2 = 25$ oriented clockwise.

$\vec{F} = \langle e^x + x^2 y, e^y - xy^2 \rangle$

conservative? $\frac{\partial P}{\partial y} = x^2$ $\quad \frac{\partial Q}{\partial x} = -y^2$ (no)

Green's: $\iint \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$



polar: $r = 0$ to 5

$\theta = 0$ to 2π

integrand:

(this is − path orientation) $-y^2 - x^2$

to polar:

$-(r\sin\theta)^2 - (r\cos\theta)^2$

$-r^2(\sin^2\theta + \cos^2\theta)$

$-r^2$

$\int_0^{2\pi}\int_0^5 (-r^2)\, r\, dr\, d\theta$

$-\int_0^5 r^3\, dr = -\frac{1}{4}(r^4)\Big|_0^5$

$= -\frac{1}{4}(5^4 - 0^4) = -\frac{625}{4}$

$\int_0^{2\pi}\left(-\frac{625}{4}\right) d\theta = -\frac{625}{4}\left[\theta\right]_0^{2\pi}$

$= -\frac{625}{2}\pi$

but − oriented path so...

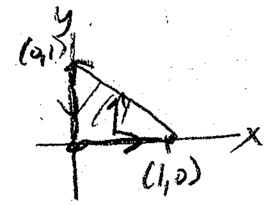
$= -\left(-\frac{625}{2}\pi\right) = \boxed{\frac{625\pi}{2}}$

#5b. Use Green's Theorem to find the work done by the force $\vec{F}(x,y) = \langle x(x+y),\ xy^2 \rangle$ in moving a particle from the origin along the x-axis to $(1,0)$, then along the line segment to $(0,1)$, and then back to the origin along the y-axis.

$x^2 + xy$

$\vec{F}$ conservative? $\frac{\partial P}{\partial y} = x$ $\quad \frac{\partial Q}{\partial x} = y^2$ (no)

Green's $\iint \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$



rectangular

$y = 0$ to $y = 1 - x$

$x = 0$ to 1

integrand: $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = y^2 - x$

$\int_0^1 \int_0^{1-x} (y^2 - x)\, dy\, dx$

$\int_0^{1-x} (y^2 - x)\, dy = \left[\frac{1}{3}y^3 - xy\right]_0^{1-x}$

$\left[\frac{1}{3}(1-x)^3 - x(1-x)\right] - [0]$

$(1-x)^3$ by binomial theorem

1
1 1
1 2 1
1 3 3 1

$1(1)^3(-x)^0 + 3(1)^2(-x)^1 + 3(1)^1(-x)^2 + 1(1)^0(-x)^3$

$1 - 3x + 3x^2 - x^3$

$\frac{1}{3}[1 - 3x + 3x^2 - x^3] - x + x^2$

$\frac{1}{3} - x + x^2 - \frac{1}{3}x^3 - x + x^2$

$\int_0^1 \left(\frac{1}{3} - 2x + 2x^2 - \frac{1}{3}x^3\right) dx$

$\left[\frac{1}{3}x - x^2 + \frac{2}{3}x^3 - \frac{1}{12}x^4\right]_0^1$

$\frac{1}{3} - 1 + \frac{2}{3} - \frac{1}{12} - (0)$

$= \boxed{-\frac{1}{12}}$

16.5

#1b. Find (i) the curl and (ii) the divergence of the vector field $\vec{F}(x,y,z)=\langle 1,\ x+yz,\ xy-\sqrt{z}\rangle$

(i) $\operatorname{curl}\vec{F}=\nabla\times\vec{F}=\begin{vmatrix} \overset{+}{\partial/\partial x} & \overset{-}{\partial/\partial y} & \overset{+}{\partial/\partial z} \\ 1 & x+yz & xy-\sqrt{z}\end{vmatrix}$

$=\langle \frac{\partial}{\partial y}[xy-\sqrt{z}]-\frac{\partial}{\partial z}[x+yz],\ -(\frac{\partial}{\partial x}[xy-\sqrt{z}]-\frac{\partial}{\partial z}[1]),$
$\frac{\partial}{\partial x}[x+yz]-\frac{\partial}{\partial y}[1]\rangle$ ⟸ (I don't usually write this line)

$=\langle x-y,-(y-0),1-0\rangle$

$=\boxed{\langle x-y,-y,1\rangle}$

(ii) $\operatorname{div}\vec{F}=\nabla\cdot\vec{F}$

$=\langle \frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\rangle\cdot\langle 1,x+yz,xy-\sqrt{z}\rangle$
(I don't usually write this line)

$=\frac{\partial}{\partial x}[1]+\frac{\partial}{\partial y}[x+yz]+\frac{\partial}{\partial z}[xy-z^{1/2}]$

$=0+z+(-\tfrac{1}{2}z^{-1/2})$

$=\boxed{z-\frac{1}{2\sqrt{z}}}$

#2b. Find (i) the curl and (ii) the divergence of the vector field $\vec{F}(x,y,z)=\frac{1}{\sqrt{x^2+y^2+z^2}}\langle x,\ y,\ z\rangle$

(i) $\operatorname{curl}\vec{F}=\nabla\times\vec{F}=\begin{vmatrix} \overset{+}{\partial/\partial x} & \overset{-}{\partial/\partial y} & \overset{+}{\partial/\partial z} \\ \frac{x}{\sqrt{x^2+y^2+z^2}} & \frac{y}{\sqrt{x^2+y^2+z^2}} & \frac{z}{\sqrt{x^2+y^2+z^2}}\end{vmatrix}$

first (x) component:

$\frac{\partial}{\partial y}\left[\frac{z}{\sqrt{x^2+y^2+z^2}}\right]-\frac{\partial}{\partial z}\left[\frac{y}{\sqrt{x^2+y^2+z^2}}\right]$

$\frac{\sqrt{x^2+y^2+z^2}(0)-z(\frac{1}{2}(x^2+y^2+z^2)^{-1/2}(2y))}{(\sqrt{x^2+y^2+z^2})^2}$

$-\frac{\sqrt{x^2+y^2+z^2}(0)-y(\frac{1}{2}(x^2+y^2+z^2)^{-1/2}(2z))}{(\sqrt{x^2+y^2+z^2})^2}$

$=\frac{\left(\frac{-yz}{\sqrt{x^2+y^2+z^2}}\right)}{x^2+y^2+z^2}-\frac{\left(\frac{yz}{\sqrt{x^2+y^2+z^2}}\right)}{x^2+y^2+z^2}=0$

similarly other components are zero, so

$\boxed{\operatorname{curl}\vec{F}=\langle 0,0,0\rangle=\vec{0}}$

(ii) $\operatorname{div}\vec{F}=\nabla\cdot\vec{F}$

$=\frac{\partial}{\partial x}\left[\frac{x}{\sqrt{x^2+y^2+z^2}}\right]+\frac{\partial}{\partial y}\left(\frac{y}{\sqrt{x^2+y^2+z^2}}\right)+\frac{\partial}{\partial z}\left[\frac{z}{\sqrt{x^2+y^2+z^2}}\right]$

x component: $\frac{\sqrt{x^2+y^2+z^2}(1)-x(\frac{1}{2}(x^2+y^2+z^2)^{-1/2}(2x))}{(\sqrt{x^2+y^2+z^2})^2}$

$=\frac{\sqrt{x^2+y^2+z^2}-\frac{x^2}{\sqrt{x^2+y^2+z^2}}}{x^2+y^2+z^2}$

$=\frac{\frac{\sqrt{x^2+y^2+z^2}}{1}\left(\frac{\sqrt{x^2+y^2+z^2}}{\sqrt{x^2+y^2+z^2}}\right)-\frac{x^2}{\sqrt{x^2+y^2+z^2}}}{x^2+y^2+z^2}$

$=\frac{\frac{x^2+y^2+z^2-x^2}{\sqrt{x^2+y^2+z^2}}}{x^2+y^2+z^2}=\frac{y^2+z^2}{(x^2+y^2+z^2)^{3/2}}$

similarly for y & z:

$=\frac{y^2+z^2}{(x^2+y^2+z^2)^{3/2}}+\frac{x^2+z^2}{(x^2+y^2+z^2)^{3/2}}+\frac{x^2+y^2}{(x^2+y^2+z^2)^{3/2}}$

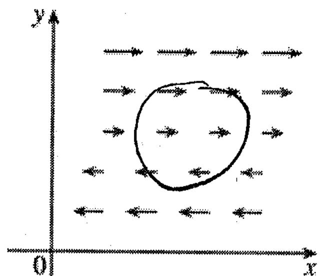
$=\frac{2(x^2+y^2+z^2)}{\sqrt{x^2+y^2+z^2}(x^2+y^2+z^2)}=\boxed{\frac{2}{\sqrt{x^2+y^2+z^2}}=\operatorname{div}\vec{F}}$

(problem difficulty can vary widely :))

#3b. The vector field $\vec{F}$ is shown in the xy-plane and looks the same in all other horizontal planes (its z-component is zero).

(i) Is $div\,\vec{F}$ positive, negative, or zero? Explain.

(ii) Determine whether $curl\,\vec{F} = \vec{0}$. If not, in which direction does $curl\,\vec{F}$ point?



same amount entering and leaving so $\boxed{div\,\vec{F} = 0}$

there is a tendency to rotate in this direction ↺

so $curl\,\vec{F}$ is non-zero

and by RH rule into paper (in $-z$ direction)

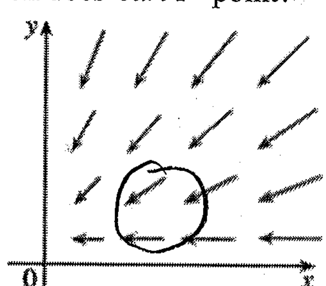
a function

$\boxed{\text{so } curl\,\vec{F} = \langle 0, 0, -R\rangle}$

#4b. The vector field $\vec{F}$ is shown in the xy-plane and looks the same in all other horizontal planes (its z-component is zero).

(i) Is $div\,\vec{F}$ positive, negative, or zero? Explain.

(ii) Determine whether $curl\,\vec{F} = \vec{0}$. If not, in which direction does $curl\,\vec{F}$ point?



More entering than leaving, so acting like a sink

$\boxed{div\,\vec{F} < 0}$

No tendency to rotate

so $\boxed{curl\,\vec{F} = \vec{0}}$

#5b. Let f be a scalar field and $\vec{F}$ a vector field. State whether each expression is meaningful. If not, explain why. If so, state whether it is a scalar field or a vector field.

(i) $grad\ f = \nabla f = \langle f_x, f_y, f_z\rangle$

another way to write gradient!

$= \boxed{\text{a vector}}$

(ii) $curl\left(grad\ f\right)$

$curl(\nabla f)$

$curl(\text{vector}) = \boxed{\text{a vector}}$

(iii) $\nabla\left(div\,\vec{F}\right)$

$\nabla\ \text{scalar} = \boxed{\text{vector}}$

(iv) $\nabla\left(div\ f\right)$ $\boxed{\text{not meaningful}}$

can't take div of scalar

(v) $div\left(div\,\vec{F}\right)$ $\boxed{\text{not meaningful}}$

$div(\text{scalar})$

can't take div of scalar

(vi) $div\left(curl\left(\nabla f\right)\right)$

$curl(\text{vector})$

$div(\text{vector}) = \boxed{\text{a scalar}}$

16.6 day 1

#1b. Determine whether the points P and Q lie on the given surface.

$\vec{r}(u,v) = \langle u+v,\ u^2 - v,\ u+v^2 \rangle$

$P(3,-1,5)$, $Q(-1,3,4)$

$\underline{P}$: $\begin{cases} u+v=3 \\ u^2-v=-1 \\ u+v^2=5 \end{cases}$ try to solve system...

$u+v=3 \rightarrow v=3-u$

$u^2-(3-u)=-1$ $\quad u+(3-u)^2=5$

$u^2+u-3=-1$

$u^2+u-2=0$

$(u+2)(u-1)=0$

$u=-2$ or $u=1$

try in other eqn: $u+(3-u)^2=5$

$u=-2$? $(-2)+(3-(-2))^2 \stackrel{?}{=} 5$

$-2+25 \neq 5$

$u=1$? $(1)+(3-1)^2 \stackrel{?}{=} 5$

$5=5$ ✓

$v=3-(1)=2$

Since $\vec{r}(1,2) = \langle 3,-1,5 \rangle$

$\boxed{P \text{ is on this surface}}$

$\underline{Q}$: $\begin{cases} u+v=-1 \rightarrow v=-1-u \\ u^2-v=3 \\ u+v^2=4 \end{cases}$

$u^2-(-1-u)=3$ $\quad u+(-1-u)^2=4$

$u^2+u+1=3$

$u^2+u-2=0$

$(u+2)(u-1)=0$

$u=-2$ or $u=1$

try in $u+(-1-u)^2=4$

$u=-2$? $(-2)+(-1-(-2))^2 \stackrel{?}{=} 4$

$-1 \neq 4$

$u=1$? $(1)+(-1-1)^2 \stackrel{?}{=} 4$

$5 \neq 4$

no u,v work

so $\boxed{Q \text{ is not on this surface}}$

#2b. Identify the surface with the given vector equation.

$\vec{r}(s,t) = \langle s,\ t,\ t^2 - s^2 \rangle$

$x=s$

$y=t$

$z=t^2-s^2=y^2-x^2$

$z=y^2-x^2$ is a $\boxed{\text{hyperbolic paraboloid}}$

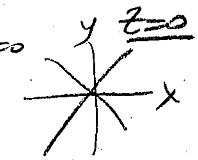
more detail?...

y is major axis.

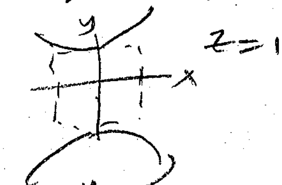
level curves: $z=0$ $\quad y^2-x^2=0$

$y^2=x^2$

$y=\pm x$

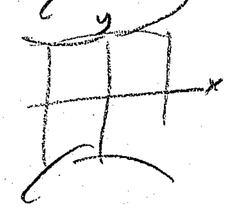


$z=1$: $\frac{y^2}{1}-\frac{x^2}{1}=1$



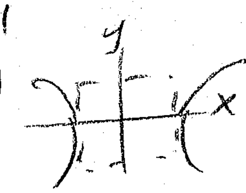
$z=4$: $y^2-x^2=4$

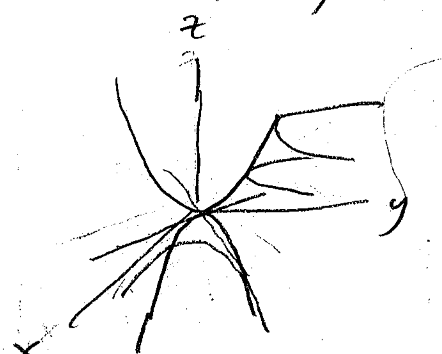
$\frac{y^2}{(2)^2}+\frac{x^2}{(2)^2}=1$



$z=-1$: $y^2-x^2=-1$

$x^2-y^2=1$





(use 3D software if you want to see the shape of $z=y^2-x^2$)

parameter domain: $D=\{(x,y) \mid -\infty \le x \le \infty, -\infty \le y \le \infty\}$

#3b. Find a parametric representation for the surface: the plane that passes through the point $(2, 4, 6)$ and contains the vectors $\langle 2,1,-1\rangle$ and $\langle 1,-1,2\rangle$.

equation of plane:

$$\vec{n} = \langle 2,1,-1\rangle \times \langle 1,-1,2\rangle = \begin{vmatrix} + & - & + \\ 2 & 1 & -1 \\ 1 & -1 & 2 \end{vmatrix}$$

$$\vec{n} = \langle 2-1, -(4+1), -2-1\rangle = \langle 1,-5,-3\rangle$$

$$\vec{r_0} = \langle 2,4,6\rangle$$

$$ax+by+cz = \vec{n}\cdot\vec{r_0}$$

$$1x-5y-3z = \langle 1,-5,-3\rangle \cdot \langle 2,4,6\rangle$$

$$= (1)(2)+(-5)(4)+(-3)(6)$$

$$x-5y-3z = -36,$$

$$3z = x-5y+36$$

$$z = \tfrac{1}{3}x - \tfrac{5}{3}y + 12$$

to parametrize, use x, y as the parameters:

$x=u$

$y=v$

$z = \tfrac{1}{3}u - \tfrac{5}{3}v + 12$

can write as:

$$\boxed{\vec{r}(u,v) = \langle u, v, \tfrac{1}{3}u - \tfrac{5}{3}v + 12\rangle \quad \text{or} \quad \vec{r}(x,y) = \langle x, y, \tfrac{1}{3}x - \tfrac{5}{3}y + 12\rangle}$$

entire plane, so parameter domain is:

$$\boxed{D = \{(x,y) \mid -\infty < x < \infty, -\infty < y < \infty\}}$$

#4b. Find a parametric representation for the surface: the part of the hyperboloid $x^2 + y^2 - z^2 = 1$ that lies to the right of the xz-plane.

suggests using x, z as parameters,

Solve for y:

$$x^2+y^2-z^2=1$$

$$y^2 = 1-x^2+z^2$$

$$y = \pm\sqrt{1-x^2+z^2}$$

"to the right of xz plane" = positive y

$$y = \sqrt{1-x^2+z^2}$$

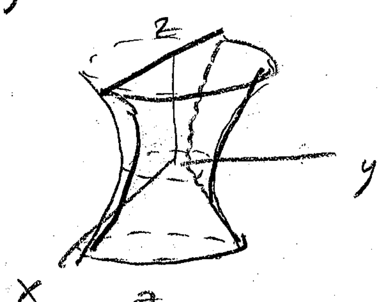
$x=u$

$z=v$

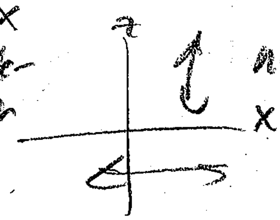
$y=\sqrt{1-u^2+v^2}$

$$\boxed{\vec{r}(u,v) = \langle u, \sqrt{1-u^2+v^2}, v\rangle \quad \text{or} \quad \vec{r}(x,z) = \langle x, \sqrt{1-x^2+z^2}, z\rangle}$$

to get ranges for parametric surfaces you really need a 3D sketch... (using software): $x^2+y^2-z^2=1$



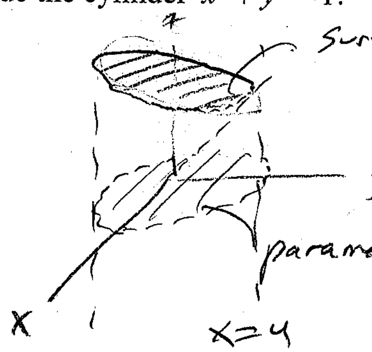
parameter domain



no restrictions and there is surface in y over the entire xz plane so...

$$\boxed{D = \{(x,z) \mid -\infty < x < \infty, -\infty < y < \infty\}}$$

#5b. Find a parametric representation for the surface: the part of the plane $z = x + 3$ that lies inside the cylinder $x^2 + y^2 = 1$.



surface is $z = x+3$

parameter domain (make x, y the parameters)

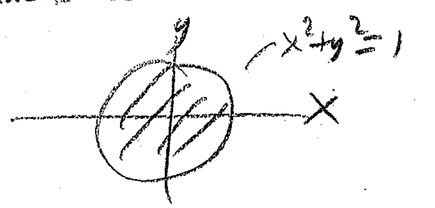
$x = u$

$y = v$

$z = (u) + 3$

$$\boxed{\vec{r}(u,v) = \langle u, v, u+3 \rangle \quad \text{or} \quad \vec{r}(x,y) = \langle x, y, x+3 \rangle}$$

parameter domain:



$$\boxed{D = \{(x,y) \mid x^2+y^2 \leq 1\}}$$

16.6 day 2

#1b. Find an equation of the tangent plane to the given parametric surface at the specified point.

$\vec{r}(u,v)=\langle u^2,\ 2u\sin v,\ u\cos v\rangle$

(1,0,1)

$u^2=1 \longrightarrow u=\pm 1$

$2u\sin v=0$

$u\cos v=1 \rightarrow u=\frac{1}{\cos v}$

$2\left(\frac{1}{\cos v}\right)\sin v=0$, $2\tan v=0$

$\tan v=0$

$u=\frac{1}{\cos v} \leftarrow v=0$ (or π)

$v=0:\ u=\frac{1}{1}=1$ $\quad$ $v=\pi:\ u=\frac{1}{\cos\pi}=-1$

$\boxed{(u,v)=(1,0)}$ $\quad$ $\cancel{(-1,0)}$

$(1,0,1)$ $\quad$ $\cancel{(-1,0,-1)}$

$\vec{r}_u=\langle 2u, 2\sin v, \cos v\rangle = \langle 2,0,1\rangle$

$\vec{r}_v=\langle 0, 2u\cos v, -u\sin v\rangle = \langle 0,2,0\rangle$ at $(1,0)$

$\vec{n}=\vec{r}_u\times\vec{r}_v=\begin{vmatrix} + & - & + \\ 2 & 0 & 1 \\ 0 & 2 & 0 \end{vmatrix}$

$=\langle 0-2, -(0-0), 4-0\rangle = \langle -2,0,4\rangle$

$\vec{r}_0=\langle 1,0,1\rangle$

$ax+by+cz=\vec{n}\cdot\vec{r}_0$

$-2x+0y+4z=\langle -2,0,4\rangle\cdot\langle 1,0,1\rangle$

$=(-2)(1)+(0)(0)+(4)(1)$

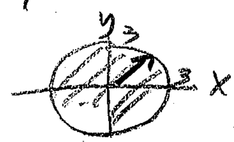
$-2x+4z=2$

$\boxed{-x+2z=1}$

#2b. Find the area of the surface: the part of the plane $2x+5y+z=10$ that lies inside the cylinder $x^2+y^2=9$.

surface: $\vec{r}=\langle x, y, 10-2x-5y\rangle$

parameter domain:



polar

$r=0$ to $r=3$

$\theta=0$ to $\theta=2\pi$

using $\iint|\vec{r}_u\times\vec{r}_v|\,dA$:

$\vec{r}_u=\vec{r}_x=\langle 1,0,-2\rangle$, $\vec{r}_v=\vec{r}_y=\langle 0,1,-5\rangle$

$\vec{r}_x\times\vec{r}_y=\begin{vmatrix} + & - & + \\ 1 & 0 & -2 \\ 0 & 1 & -5 \end{vmatrix}=\langle 0+2, -(-5-0), 1-0\rangle=\langle 2,5,1\rangle$

$|\vec{r}_x\times\vec{r}_y|=\sqrt{2^2+5^2+1^2}=\sqrt{30}$

$\int_0^{2\pi}\int_0^3\sqrt{30}\,r\,dr\,d\theta$

—or—

using $\iint\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA$

$\frac{\partial z}{\partial x}=-2$ $\quad$ $\frac{\partial z}{\partial y}=-5$ $\quad$ $\sqrt{1+(-2)^2+(-5)^2}=\sqrt{30}$

$\int_0^{2\pi}\int_0^3\sqrt{30}\,r\,dr\,d\theta$

$\int_0^3\sqrt{30}\,r\,dr=\sqrt{30}\,\frac{1}{2}\left[r^2\right]_0^3=\frac{\sqrt{30}}{2}(3^2-0)$

$\int_0^{2\pi}\frac{9\sqrt{30}}{2}\,d\theta=\frac{9\sqrt{30}}{2}\left[\theta\right]_0^{2\pi}$

$=\boxed{9\sqrt{30}\,\pi}$

#3b. Find the area of the surface: the part of the surface $z = 1 + 3x + 2y^2$ that lies above the triangle with vertices (0,0), (0,1), and (2,1).

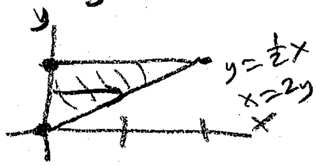
Surface: $\vec{r} = \langle x, y, 1+3x+2y^2 \rangle$

parameter domain:

rectangular

$x=0$ to $x=2y$

$y=0$ to $y=1$



using $\iint |\vec{r}_u \times \vec{r}_v|\,dA$ $\quad \vec{r}_u = \vec{r}_x = \langle 1, 0, 3 \rangle$

$\vec{r}_v = \vec{r}_y = \langle 0, 1, 4y \rangle$

$$\vec{r}_x \times \vec{r}_y = \begin{vmatrix} + & - & + \\ 1 & 0 & 3 \\ 0 & 1 & 4y \end{vmatrix} = \langle 0-3, -(4y-0), 1-0 \rangle = \langle -3, -4y, 1 \rangle$$

$$|\vec{r}_x \times \vec{r}_y| = \sqrt{(3)^2 + (4y)^2 + (1)^2} = \sqrt{16y^2+10}$$

$$\int_0^1 \int_0^{2y} \sqrt{16y^2+10}\,dx\,dy$$

—or—

using $\iint \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\,dA$

$\frac{\partial z}{\partial x} = 3$, $\frac{\partial z}{\partial y} = 4y$

integrand: $\sqrt{1 + 3^2 + (4y)^2} = \sqrt{16y^2+10}$

$$\int_0^1 \int_0^{2y} \sqrt{16y^2+10}\,dx\,dy$$

$$\int_0^{2y} \sqrt{16y^2+10}\,dx = \sqrt{16y^2+10}\,[x]_0^{2y}$$

$$= \sqrt{16y^2+10}\,(2y)$$

$\int_0^1 \sqrt{16y^2+10}\,(2y)\,dy$ $\quad u = 16y^2+10$, $\frac{du}{dy} = 32y$, $2y = \frac{1}{16}du$ $\quad y=0 \to u=10$, $y=1 \to u=26$

$$\frac{1}{16}\int_{10}^{26} u^{1/2}\,du = \frac{1}{16}\left(\frac{2}{3}\right)\left[u^{3/2}\right]_{10}^{26}$$

$$= \boxed{\frac{1}{24}\left(26\sqrt{26} - 10\sqrt{10}\right)}$$

#4b. Find the area of the surface: the part of the paraboloid $x = y^2 + z^2$ that lies inside the cylinder $y^2 + z^2 = 9$.

parameterized

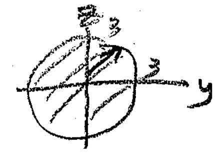
Surface: $\vec{r} = \langle y^2+z^2, y, z \rangle$

parameter domain:

polar!

$r=0$ to $r=3$

$\theta=0$ to $\theta=2\pi$



using $\iint |\vec{r}_u \times \vec{r}_v|\,dA$ $\quad \vec{r}_u = \vec{r}_y = \langle 2y, 1, 0 \rangle$

$\vec{r}_v = \vec{r}_z = \langle 2z, 0, 1 \rangle$

$$\vec{r}_y \times \vec{r}_z = \begin{vmatrix} + & - & + \\ 2y & 1 & 0 \\ 2z & 0 & 1 \end{vmatrix} = \langle 1-0, -(2y-0), 0-2z \rangle = \langle 1, -2y, -2z \rangle$$

$$|\vec{r}_y \times \vec{r}_z| = \sqrt{1^2 + 4y^2 + 4z^2} = \sqrt{1+4r^2}$$

$$\int_0^{2\pi} \int_0^3 \sqrt{1+4r^2}\,r\,dr\,d\theta$$

—or—

using $\iint \sqrt{1 + \left(\frac{\partial x}{\partial y}\right)^2 + \left(\frac{\partial x}{\partial z}\right)^2}\,dA$

$\frac{\partial x}{\partial y} = 2y$, $\frac{\partial x}{\partial z} = 2z$

integrand: $\sqrt{1 + (2y)^2 + (2z)^2} = \sqrt{1+4y^2+4z^2} = \sqrt{1+4r^2}$

$$\int_0^{2\pi} \int_0^3 \sqrt{1+4r^2}\,r\,dr\,d\theta$$

$\int_0^3 \sqrt{1+4r^2}\,r\,dr$ $\quad u = 1+4r^2$, $\frac{du}{dr} = 8r$, $r\,dr = \frac{1}{8}du$ $\quad r=0 \to u=1$, $r=3 \to u=37$

$$\frac{1}{8}\int_1^{37} u^{1/2}\,du = \frac{1}{8}\left(\frac{2}{3}\right)\left[u^{3/2}\right]_1^{37} = \frac{1}{12}\left[37^{3/2} - 1^{3/2}\right]$$

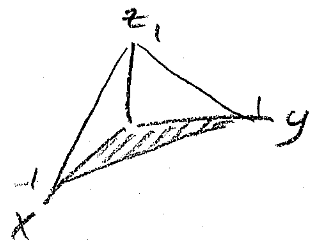
$$\int_0^{2\pi} \frac{1}{12}\left(37\sqrt{37} - 1\right)d\theta = \frac{1}{12}\left(37\sqrt{37}-1\right)[\theta]_0^{2\pi}$$

$$\boxed{\frac{\pi}{6}\left(37\sqrt{37} - 1\right)}$$

16.7 day 1

#1b. Evaluate the surface integral $\iint_S yz\, dS$ (scalar)

S is the part of the plane $x+y+z=1$ that lies in the first octant.



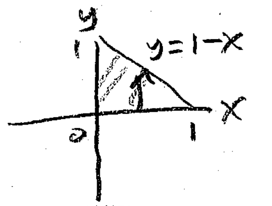
surface: $\vec{r} = \langle x, y, 1-x-y \rangle$

parameter domain: need to sketch:

rectangular

$y=0$ to $y=1-x$

$x=0$ to $x=1$



choosing $\iint f(x,y,z)\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA$ $\quad \frac{\partial z}{\partial x}=-1, \ \frac{\partial z}{\partial y}=-1$

$$\int_0^1\int_0^{1-x} y(1-x-y)\sqrt{1+(-1)^2+(-1)^2}\,dy\,dx = \sqrt{3}\int_0^1\int_0^{1-x}(y-xy-y^2)\,dy\,dx$$

$(1-x)^3$ by binomial theorem:

$1(1)^3(-x)^0+3(1)^2(-x)^1+3(1)^1(-x)^2+1(1)^0(-x)^3$

$1-3x+3x^2-x^3$

$$\int_0^{1-x}(y-xy-y^2)\,dy = \left[\tfrac{1}{2}y^2-\tfrac{1}{2}xy^2-\tfrac{1}{3}y^3\right]_0^{1-x}$$

$$= \tfrac{1}{2}(1-x)^2-\tfrac{1}{2}x(1-x)^2-\tfrac{1}{3}(1-x)^3$$

$$= \tfrac{1}{2}(1-2x+x^2)-\tfrac{1}{2}x(1-2x+x^2)-\tfrac{1}{3}(1-3x+3x^2-x^3)$$

$$= \tfrac{1}{2}-x+\tfrac{1}{2}x^2-\tfrac{1}{2}x+x^2-\tfrac{1}{2}x^3-\tfrac{1}{3}+x-x^2+\tfrac{1}{3}x^3$$

$$= \tfrac{1}{6}-\tfrac{1}{2}x+\tfrac{1}{2}x^2-\tfrac{1}{6}x^3$$

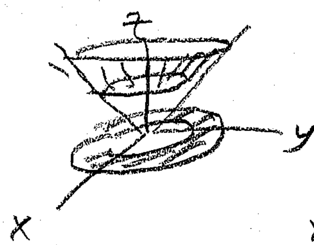
$$\sqrt{3}\int_0^1\left(\tfrac{1}{6}-\tfrac{1}{2}x+\tfrac{1}{2}x^2-\tfrac{1}{6}x^3\right)dx = \sqrt{3}\left[\tfrac{1}{6}x-\tfrac{1}{4}x^2+\tfrac{1}{6}x^3-\tfrac{1}{24}x^4\right]_0^1$$

$$= \sqrt{3}\left(\tfrac{1}{6}-\tfrac{1}{4}+\tfrac{1}{6}-\tfrac{1}{24}\right)-(0) = \boxed{\frac{\sqrt{3}}{24}}$$

#2b. Evaluate the surface integral $\iint_S x^2 z^2 \, dS$ (scalar)

S is the part of the cone $z^2 = x^2 + y^2$ that lies between the planes $z = 1$ and $z = 3$.

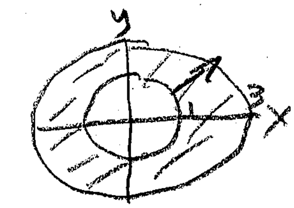
surface: $z = \sqrt{x^2+y^2}$



intersections

$\begin{cases} z=1 \\ z^2=x^2+y^2 \end{cases}$ $x^2+y^2=1$ $\begin{cases} z=3 \\ z^2=x^2+y^2 \end{cases}$ $x^2+y^2=9$

parameter domain: need to sketch:



polar

$r=1$ to $r=3$

$\theta=0$ to $\theta=2\pi$

although we could use $\iint f(x,y,z)\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA$, I'll choose $\iint f(x,y,z)\,|\vec{r}_u \times \vec{r}_v|\,dA$ (more general, works in all cases)

$\vec{r} = \langle x, y, \sqrt{x^2+y^2}\rangle$

$\vec{r}_u = \vec{r}_x = \langle 1, 0, \frac{1}{2}(x^2+y^2)^{-1/2}(2x)\rangle = \langle 1, 0, \frac{x}{\sqrt{x^2+y^2}}\rangle$

$\vec{r}_v = \vec{r}_y = \langle 0, 1, \frac{1}{2}(x^2+y^2)^{-1/2}(2y)\rangle = \langle 0, 1, \frac{y}{\sqrt{x^2+y^2}}\rangle$

$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} + & - & + \\ 1 & 0 & \frac{x}{\sqrt{x^2+y^2}} \\ 0 & 1 & \frac{y}{\sqrt{x^2+y^2}} \end{vmatrix} = \left\langle 0 - \frac{x}{\sqrt{x^2+y^2}}, -\left(\frac{y}{\sqrt{x^2+y^2}} - 0\right), 1-0 \right\rangle = \left\langle \frac{-x}{\sqrt{x^2+y^2}}, \frac{-y}{\sqrt{x^2+y^2}}, 1 \right\rangle$

$|\vec{r}_u \times \vec{r}_v| = \sqrt{\left(\frac{-x}{\sqrt{x^2+y^2}}\right)^2 + \left(\frac{-y}{\sqrt{x^2+y^2}}\right)^2 + (1)^2} = \sqrt{\frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2} + 1} = \sqrt{\frac{x^2+y^2}{x^2+y^2}+1} = \sqrt{1+1} = \sqrt{2}$

$\int_0^{2\pi}\int_1^3 \underbrace{x^2(x^2+y^2)\sqrt{2}}_{\text{convert to polar}}\, r\,dr\,d\theta \rightarrow \int_0^{2\pi}\int_0^3 (r\cos\theta)^2 r^2 \sqrt{2}\, r\,dr\,d\theta$

$= \sqrt{2}\int_0^{2\pi} \cos^2\theta\,d\theta \int_1^3 r^5\,dr = \sqrt{2}\int_0^{2\pi}\left(\frac{1}{2}+\frac{1}{2}\cos(2\theta)\right)d\theta \int_1^3 r^5\,d\theta$

$= \sqrt{2}\left[\frac{1}{2}\theta + \frac{1}{4}\sin(2\theta)\right]_0^{2\pi}\left[\frac{1}{6}r^6\right]_1^3 = \sqrt{2}\left(\frac{1}{2}(2\pi + \frac{1}{4}\sin(4\pi)) - \frac{1}{2}(0) + \frac{1}{4}\sin 0\right)\left(\frac{1}{6}(3)^6 - \frac{1}{6}(1)^6\right)$

$= \sqrt{2}(\pi)\left(\frac{729}{6} - \frac{1}{6}\right) = \boxed{\frac{364\sqrt{2}\,\pi}{3}}$

#3b. Evaluate the surface integral $\iint_S (z + x^2 y)\, dS$

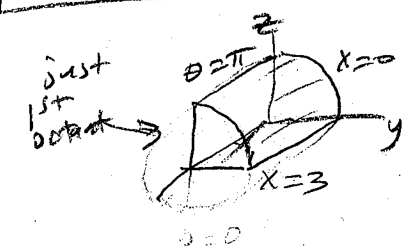
S is the part of the cylinder $y^2 + z^2 = 1$ that lies between the planes $x = 0$ and $x = 3$ in the first octant.

surface: when the surface is a cylinder, sphere, or parts of these (hemisphere), it is often good to take advantage of that by using cylindrical or spherical coordinates. Here, we can use cylindrical coordinates (except that the 'height' direction is x):

normally, $x = r\cos\theta$, $y = r\sin\theta$, $z = z$ so have we will use: $x = x$, $y = r\cos\theta$, $z = r\sin\theta$ for $y^2+z^2=1$, $r=1$ so $x = x$, $y = \cos\theta$, $z = \sin\theta$

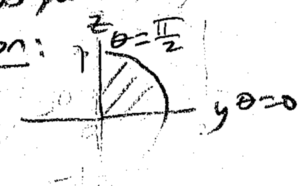
or $\vec{r}(u,v) = \vec{r}(x,\theta) = \langle x, \cos\theta, \sin\theta \rangle$

parameter domain: sketch in 3D:



Since we're not using cylindrical to find volume, we just have two parameters as variables: x and θ:

projection:



parameter domain:

rectangular
$\theta = 0$ to $\pi/2$
$x = 0$ to 3

must use $\iint f(x,y,z) |\vec{r_u} \times \vec{r_v}|\, dA$ $\quad \vec{r_u} = \vec{r_x} = \langle 1, 0, 0 \rangle$ $\quad \vec{r_v} = \vec{r_\theta} = \langle 0, -\sin\theta, \cos\theta \rangle$

$$\vec{r_u} \times \vec{r_v} = \begin{vmatrix} + & - & + \\ 1 & 0 & 0 \\ 0 & -\sin\theta & \cos\theta \end{vmatrix} = \langle 0-0, -(\cos\theta - 0), -\sin\theta - 0 \rangle = \langle 0, -\cos\theta, -\sin\theta \rangle$$

$$|\vec{r_u} \times \vec{r_v}| = \sqrt{\cos^2\theta + \sin^2\theta} = 1, \quad \int_0^3 \int_0^{\pi/2} \left[(\sin\theta) + (x)^2(\cos\theta) \right] (1)\, d\theta\, dx$$

$$\int_0^{\pi/2} \sin d\theta + x^2 \int_0^{\pi/2} \cos\theta\, d\theta = -\cos\theta \Big]_0^{\pi/2} + x^2 \left[\sin\theta\right]_0^{\pi/2} = -(\cos\tfrac{\pi}{2} - \cos 0) + x^2(\sin\tfrac{\pi}{2} - \sin 0)$$

$$= -(0-1) + x^2(1-0) = 1 + x^2$$

$$\int_0^3 (1+x^2)\, dx = \left[x + \tfrac{1}{3}x^3\right]_0^3 = \left((3) + \tfrac{1}{3}(3)^3\right) - [0] = \boxed{12}$$

(on required practice #3, you should be using spherical coordinates)

16.7 day 2

#1b. Evaluate the surface integral $\iint_S \vec{F} \cdot dS$ (vector field)

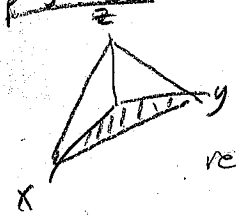
(find the flux of $\vec{F}$ across S):

$\vec{F}(x,y,z) = \langle \overset{P}{xze^y}, \ \overset{Q}{-xze^y}, \ \overset{R}{z} \rangle$ $\quad z = 1-x-y$

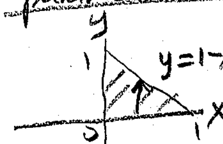
S is the part of the plane $x+y+z=1$ in the first octant and has downward orientation.

surface: $\vec{r}(x,y) = \langle x, y, 1-x-y \rangle$

projection:



parameter domain:



rectangular! $y=0$ to $y=1-x$
$x=0$ to $x=1$

Two ways to get the integrand...

① $-P\frac{\partial z}{\partial x} - Q\frac{\partial z}{\partial y} + R$

$\frac{\partial z}{\partial x} = -1, \quad \frac{\partial z}{\partial y} = -1$

$-(xze^y)(-1) - (-xze^y)(-1) + (z)$

$xe^y(1-x-y) - xe^y(1-x-y) + (1-x-y)$

$1-x-y$

② $\vec{F}\cdot(\vec{r_u}\times\vec{r_v})$ $\quad \vec{r_u} = \vec{r_x} = \langle 1, 0, -1 \rangle$

$\vec{r_v} = \vec{r_y} = \langle 0, 1, -1 \rangle$

$\vec{r_u}\times\vec{r_v} = \begin{vmatrix} + & - & + \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} = \langle 0+1, -(-1-0), 1-0 \rangle$

$= \langle 1, 1, 1 \rangle$

$\vec{F}\cdot(\vec{r_u}\times\vec{r_v}) = \langle xze^y, -xze^y, z \rangle \cdot \langle 1,1,1 \rangle$

$= xze^y(1) + (-xze^y)(1) + (z)(1)$

$= xze^y - xze^y + z = z = 1-x-y$

Now, the integral...

$$\int_0^1 \int_0^{1-x} (1-x-y)\,dy\,dx$$

$$\int_0^{1-x} (1-x-y)\,dy = \left[y - xy - \tfrac{1}{2}y^2 \right]_0^{1-x}$$

$= \left((1-x) - x(1-x) - \tfrac{1}{2}(1-x)^2\right) - [0]$

$= 1 - x - x + x^2 - \tfrac{1}{2}(1-2x+x^2)$

$= 1 - 2x + x^2 - \tfrac{1}{2} + x - \tfrac{1}{2}x^2 = \tfrac{1}{2} - x + \tfrac{1}{2}x^2$

$$\int_0^1 \left(\tfrac{1}{2} - x + \tfrac{1}{2}x^2\right)dx = \left[\tfrac{1}{2}x - \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3\right]_0^1$$

$= \left[\tfrac{1}{2}(1) - \tfrac{1}{2}(1)^2 + \tfrac{1}{6}(1)^3\right) - [0]$

$= \tfrac{1}{2} - \tfrac{1}{2} + \tfrac{1}{6} = \tfrac{1}{6}$ but... "has downward orientation" is reversed from + orientation for planes.

So $\boxed{-\tfrac{1}{6}}$

#2b

#. Evaluate the surface integral $\iint_S \vec{F} \cdot dS$ (vector field)

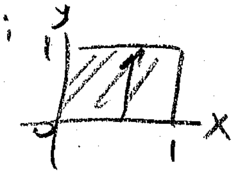
(find the flux of $\vec{F}$ across S):

$\vec{F}(x, y, z) = \langle \overset{P}{xy},\ \overset{Q}{4x^2},\ \overset{R}{yz} \rangle$

S is the surface $z = xe^y$, $0 \le x \le 1$, $0 \le y \le 1$ with upward orientation.

surface: $\vec{r}(x,y) = \langle x, y, xe^y \rangle$

parameter domain:



rectangular
$y=0$ to $y=1$
$x=0$ to $x=1$

integrand... using $-P\frac{\partial z}{\partial x} - Q\frac{\partial z}{\partial y} + R$

$\frac{\partial z}{\partial x} = e^y$, $\frac{\partial z}{\partial y} = xe^y$ $\quad -(xy)(e^y) - (4x^2)(xe^y) + (y\underbrace{z}_{=xe^y})$

$\underline{-xye^y} - 4x^3e^y + \underline{xye^y} = -4x^3e^y$

integral... $\int_0^1 \int_0^1 (-4x^3e^y)\,dy\,dx = -4\int_0^1 x^3\,dx \int_0^1 e^y\,dy$

$= -4\left[\frac{1}{4}x^4\right]_0^1 \left[e^y\right]_0^1$

$= -4\left(\frac{1}{4}(1)^4 - 0\right)(e^1 - e^0)$

$= -4\left(\frac{1}{4}\right)(e-1)$

$= -(e-1)$

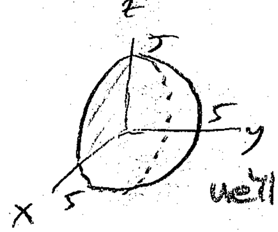
$= \boxed{1-e}$

#3b
~~#26~~. Evaluate the surface integral $\iint_S \vec{F} \cdot dS$

(find the flux of $\vec{F}$ across S):

$\vec{F}(x,y,z) = \langle xz,\ x,\ y \rangle$

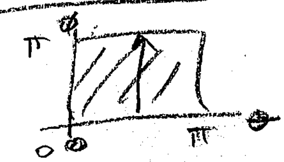
S is the hemisphere $x^2+y^2+z^2=25,\quad y \ge 0$ oriented in the direction of the positive y-axis.



we could use x, z as parameters or spherical coordinates.
we'll do spherical here (probably easier)...

$\rho = 5$
$\phi = 0$ to $\phi = \pi$
$\theta = 0$ to $\theta = \pi$

parameter domain:



surface: $\vec{r}(\phi,\theta) = \langle 5\sin\phi\cos\theta, 5\sin\phi\sin\theta, 5\cos\phi \rangle$

for integrand, must use $\vec{F}\cdot(\vec{r_u}\times\vec{r_v})$

$\vec{r_u} = \vec{r_\phi} = \langle 5\cos\phi\cos\theta, 5\cos\phi\sin\theta, -5\sin\phi \rangle$, $\vec{r_v} = \vec{r_\theta} = \langle -5\sin\phi\sin\theta, 5\sin\phi\cos\theta, 0 \rangle$

$$\vec{r_u}\times\vec{r_v} = \begin{vmatrix} + & - & + \\ 5\cos\phi\cos\theta & 5\cos\phi\sin\theta & -5\sin\phi \\ -5\sin\phi\sin\theta & 5\sin\phi\cos\theta & 0 \end{vmatrix}$$

$= \langle 0 + 25\sin^2\phi\cos\theta, -(0 - 25\sin^2\phi\sin\theta), 25\sin\phi\cos\phi\underline{\cos^2\theta} + 25\sin\phi\cos\phi\underline{\sin^2\theta} \rangle$

$= \langle 25\sin^2\phi\cos\theta, 25\sin^2\phi\sin\theta, 25\sin\phi\cos\phi \rangle$

$\vec{F}(\vec{r}) = \langle (5\sin\phi\cos\theta)(5\cos\phi), 5\sin\phi\cos\theta, 5\sin\phi\sin\theta \rangle \cdot \langle 25\sin^2\phi\cos\theta, 25\sin^2\phi\sin\theta, 25\sin\phi\cos\phi \rangle$

$\vec{F}\cdot(\vec{r_u}\times\vec{r_v}) = \langle 25\sin\phi\cos\phi\cos\theta, 5\sin\phi\cos\theta, 5\sin\phi\sin\theta \rangle$

$= 625\sin^3\phi\cos\phi\cos^2\theta + 125\sin^3\phi\sin\theta\cos\theta + 125\sin^2\phi\cos\phi\sin\theta$

$\int_0^\pi\int_0^\pi (625\sin^3\phi\cos\phi\cos^2\theta + 125\sin^3\phi\sin\theta\cos\theta + 125\sin^2\phi\cos\phi\sin\theta)\,d\phi\,d\theta$

$(625\cos^2\theta)\underbrace{\int_0^\pi \sin^3\phi\cos\phi\,d\phi}$ $+ 125\sin\theta\cos\theta\left[\int_0^\pi \sin^3\phi\,d\phi\right] + 125\sin\theta\underbrace{\int_0^\pi \sin^2\phi\cos\phi\,d\phi}$

$u = \sin\phi \quad \phi = 0 \to u = 0$
$\frac{du}{d\phi} = \cos\phi \quad \phi = \pi \to u = 0$
$\cos\phi\,d\phi = du \quad \int_0^0 u^3\,du$
$(=0)$

$\int_0^\pi \sin^2\phi\sin\phi\,d\phi$

$\int_0^\pi (1-\cos^2\phi)\sin\phi\,d\phi$

$\int_0^\pi \sin\phi\,d\phi - \int_0^\pi \cos^2\phi\sin\phi\,d\phi$

$\left[-\cos\phi\right]_0^\pi + \int_1^{-1} u^2\,du$

$\left[-\cos\pi + \cos 0\right] + \left[\frac{1}{3}u^3\right]_1^{-1}$

$-(-1)+(1)$

$2 + \left(\frac{1}{3}(-1) - \frac{1}{3}(1)\right)$

$125\sin\theta\cos\theta\left(\frac{4}{3}\right)$

$\int_0^0 u^2\,du$
$(=0)$

$u = \sin\phi \quad \phi = 0 \to u = 0$
$\frac{du}{d\phi} = \cos\phi \quad \phi = \pi \to u = 0$
$\cos\phi\,d\phi = du$

$u = \cos\phi \quad \phi = 0 \to u = 1$
$\frac{du}{d\phi} = -\sin\phi \quad \phi = \pi \to u = -1$
$\sin\phi\,d\phi = -du$

→ continued

#3b (continued)...

outer integral: $\int_0^{\pi} \left(125\sin\theta\cos\theta\left(\frac{4}{3}\right)\right) d\theta$

$\frac{500}{3}\int_0^{\pi} \sin\theta\cos\theta\, d\theta$ $\quad u=\sin\theta \quad \frac{du}{d\theta}=\cos\theta \quad \cos\theta\, d\theta = du$ $\quad \theta=0 \Rightarrow u=0 \quad \theta=\pi \Rightarrow u=0$

$\frac{500}{3}\int_0^{0} u\, du$

$= \boxed{0}$

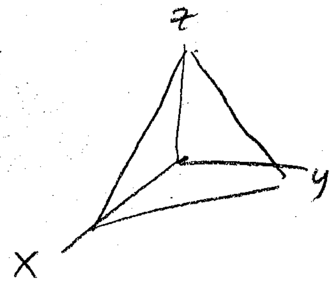
(using x, z as parameters is even more difficult)

#4b. Evaluate the surface integral $\iint_S \vec{F} \cdot dS$

(find the flux of $\vec{F}$ across S):

$\vec{F}(x,y,z) = \langle y,\ z-y,\ x \rangle$

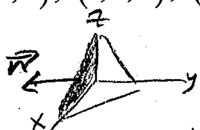
S is the surface of the tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$.



4 surfaces

for each, we will find a normal, $\vec{n}$, then $\vec{F}\cdot(\vec{r}_u \times \vec{r}_v)$ becomes $\vec{F}\cdot\vec{n}$ (this must reflect $\vec{F}(\vec{r})$ for that surface)

(1)



$\vec{n} = \langle 0,-1,0\rangle$ here, $y=0$

so $\vec{F}(\vec{r}) = \langle 0, z, x\rangle$

$\vec{F}\cdot\vec{n} = \langle 0,z,x\rangle \cdot \langle 0,-1,0\rangle = -z$

$y=1-x$

$\int_0^1 \int_0^{1-x} (-z)\,dz\,dx$

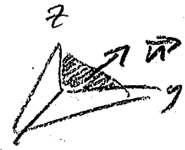
$\int_0^{1-x} (-z)\,dz = -\frac{1}{2}[z^2]_0^{1-x} = -\frac{1}{2}(1-x)^2$

$-\frac{1}{2} + x - \frac{1}{2}x^2$

$\int_0^1 (-\frac{1}{2}+x-\frac{1}{2}x^2)\,dx = [-\frac{1}{2}x + \frac{1}{2}x^2 - \frac{1}{6}x^3]_0^1$

$= -\frac{1}{2}+\frac{1}{2}-\frac{1}{6} = (-\frac{1}{6})$

(2)



$\vec{n} = \langle -1,0,0\rangle$ $(x=0)$

$\vec{F} = \langle y, z-y, 0\rangle$

$\vec{F}\cdot\vec{n} = \langle y, z-y, 0\rangle \cdot \langle -1,0,0\rangle$

$= -y$

$z=1-y$

$\int_0^1 \int_0^{1-y} (-y)\,dz\,dy$

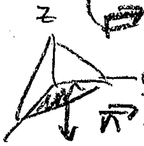
$\int_0^{1-y} -y[z]_0^{1-y} = -y(1-y)$

$= -y + y^2$

$\int_0^1 (-y+y^2)\,dy = [-\frac{1}{2}y^2 + \frac{1}{3}y^3]_0^1$

$= -\frac{1}{2} + \frac{1}{3} = (-\frac{1}{6})$

(3)



$\vec{n} = \langle 0,0,-1\rangle$ $(z=0)$

$\vec{F} = \langle y, -y, x\rangle$

$\vec{F}\cdot\vec{n} = \langle y,-y,x\rangle \cdot \langle 0,0,-1\rangle$

$= -x$

$y=1-x$

$\int_0^1 \int_0^{1-x} (-x)\,dy\,dx$

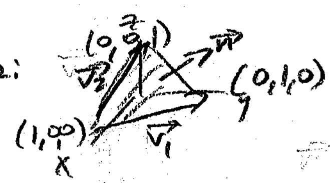
$\int_0^{1-x} (-x)\,dy = -x[y]_0^{1-x} = -x(1-x)$

$= -x + x^2$

$\int_0^1 (-x+x^2)\,dx = [-\frac{1}{2}x^2 + \frac{1}{3}x^3]_0^1$

$= -\frac{1}{2} + \frac{1}{3} = (-\frac{1}{6})$

(4) for this one, need 2 vectors in plane:



$\vec{v}_1 = \langle 0-1, 1-0, 0-0\rangle = \langle -1,1,0\rangle$

$\vec{v}_2 = \langle 0-1, 0-0, 1-0\rangle = \langle -1,0,1\rangle$

$\vec{n} = \vec{v}_1 \times \vec{v}_2 = \begin{vmatrix} + & - & + \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix}$

$\vec{n} = \langle 1-0, -(-1-0), 0+1\rangle = \langle 1,1,1\rangle$

...and use to parametrize:

make an equation for the surface plane:

$ax+by+cz = \vec{n}\cdot\vec{r}_3$

$x+y+z = \langle 1,1,1\rangle \cdot \langle 1,0,0\rangle = 1$

parameter domain:

$y=1-x$

$\vec{r}(x,y) = \langle x, y, 1-x-y\rangle$

now, $\vec{F}(\vec{r}) = \langle y, (1-x-y)-y, x\rangle = \langle y, 1-x-2y, x\rangle$

$\vec{F}\cdot\vec{n} = \langle y, 1-x-2y, x\rangle \cdot \langle 1,1,1\rangle = y+1-x-2y+x = 1-y$

$\int_0^1 \int_0^{1-x} (1-y)\,dy\,dx$, $\int_0^{1-x} (1-y)\,dy = [y-\frac{1}{2}y^2]_0^{1-x} = (1-x) - \frac{1}{2}(1-x)^2$ $(1-2x+x^2)$ $= 1-x-\frac{1}{2}+x-\frac{1}{2}x^2 = \frac{1}{2}-\frac{1}{2}x^2$

$\int_0^1 (\frac{1}{2}-\frac{1}{2}x^2)\,dx = [\frac{1}{2}x - \frac{1}{6}x^3]_0^1 = \frac{1}{2}-\frac{1}{6} = (\frac{1}{3})$

Sum of outward fluxes $= (-\frac{1}{6}) + (-\frac{1}{6}) + (-\frac{1}{6}) + (\frac{1}{3}) = \boxed{-\frac{1}{6}}$

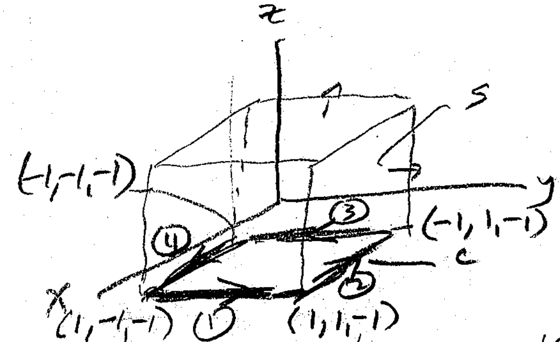
16.8

#1b. Using Stokes' Theorem, write out and evaluate the single-integral which is equivalent to the surface integral which calculates

$\iint_S \left(curl\, \vec{F} \right) \cdot dS$ where

$\vec{F}(x,y,z) = \langle xyz,\ xy,\ x^2yz \rangle$

S consists of the top and the four sides (but not the bottom) of the cube with vertices $(\pm1,\ \pm1,\ \pm1)$ oriented outward.



C consists of 4 line segments so we will sum $\oint \vec{F}\cdot d\vec{r} = \int \vec{F}\cdot\vec{r}\,'dt$ for each segment...

① $\vec{r} = (1-t)\vec{r}_0 + t\vec{r}_1 = (1-t)\langle 1,-1,-1\rangle + t\langle 1,1,-1\rangle$

$\vec{r} = \langle 1, -1+2t, -1\rangle \quad (0 \le t \le 1)$

$\vec{r}\,' = \langle 0, 2, 0\rangle$

$\vec{F}(\vec{r}) = \langle (1)(-1+2t)(1), (1)(-1+2t), (1)^2(-1+2t)(1)\rangle$

$= \langle -1+2t, -1+2t, -1+2t\rangle$

$\vec{F}\cdot\vec{r}\,' = \langle -1+2t, -1+2t, -1+2t\rangle \cdot \langle 0,2,0\rangle$

$= -2+4t$

$\int_0^1 (-2+4t)\,dt = [-2t+2t^2]_0^1 = -2+2 = (0)$

② $\vec{r} = (1-t)\langle 1,1,-1\rangle + t\langle -1,1,-1\rangle$

$\vec{r} = \langle 1-2t, 1, -1\rangle$

$\vec{r}\,' = \langle -2, 0, 0\rangle$

$\vec{F}(\vec{r}) = \langle (1-2t)(1)(-1), (1-2t)(1), (1-2t)^2(1)(-1)\rangle$

$= \langle -1+2t, 1-2t, -(1-2t)^2\rangle$

$\vec{F}\cdot\vec{r}\,' = \langle -1+2t, 1-2t, -(1-2t)^2\rangle \cdot \langle -2,0,0\rangle$

$= 2-4t$

$\int_0^1 (2-4t)\,dt = [2t-2t^2]_0^1 = 2-2 = (0)$

③ $\vec{r} = (1-t)\langle -1,1,-1\rangle + t\langle -1,-1,-1\rangle$

$\vec{r} = \langle -1, 1-2t, -1\rangle$

$\vec{r}\,' = \langle 0, -2, 0\rangle$

$\vec{F}(\vec{r}) = \langle (-1)(1-2t)(-1), (-1)(1-2t), (-1)^2(1-2t)(-1)\rangle$

$= \langle 1-2t, -1+2t, -1+2t\rangle$

$\vec{F}\cdot\vec{r}\,' = \langle 1-2t, -1+2t, -1+2t\rangle \cdot \langle 0,-2,0\rangle$

$= 2-4t$

$\int_0^1 (2-4t)\,dt = [2t-2t^2]_0^1 = 2-2 = (0)$

④ $\vec{r} = (1-t)\langle -1,-1,-1\rangle + t\langle 1,-1,-1\rangle$

$\vec{r} = \langle -1+2t, -1, -1\rangle$

$\vec{r}\,' = \langle 2, 0, 0\rangle$

$\vec{F}(\vec{r}) = \langle (-1+2t)(-1)(-1), (-1+2t)(-1), (-1+2t)^2(-1)(-1)\rangle$

$= \langle -1+2t, 1-2t, (-1+2t)^2\rangle$

$\vec{F}\cdot\vec{r}\,' = \langle -1+2t, 1-2t, (-1+2t)^2\rangle \cdot \langle 2,0,0\rangle$

$= -2+4t$

$\int_0^1 (-2+4t)\,dt = [-2t+2t^2]_0^1 = -2+2 = (0)$

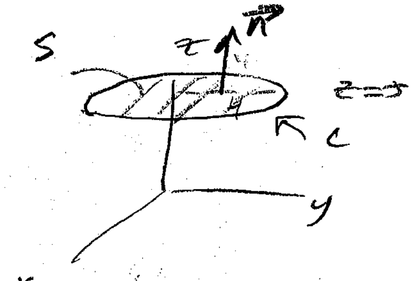
sum of contributions $= 0+0+0+0 = \boxed{0}$

Note: for #1, there is only 1 curve (a circle) which can be parametrized using $\vec{r}(t) = \langle r\cos\theta, r\sin\theta, z\rangle$ where r & z are specific values

#2b. Using Stokes' Theorem, write out and evaluate the double-integral which is equivalent to the line integral $\int_C \vec{F} \cdot d\vec{r}$ which sums the contributions of the field $\vec{F}$ along path C

$\vec{F}(x,y,z) = \langle yz,\ 2xz,\ e^{xy} \rangle$

C is the circle $x^2 + y^2 = 16$, $z = 5$.



S can be any surface with this as boundary, so just use the disk (which is part of plane z=5)

we just need $\vec{n}$ for the surface which is $\vec{n} = \langle 0,0,1 \rangle$

$$\iint_S (\text{curl}\,\vec{F}) \cdot d\vec{S} = \iint_S (\text{curl}\,\vec{F}) \cdot \vec{n}\, dA$$

$$\text{curl}\,\vec{F} = \begin{vmatrix} + & - & + \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ yz & 2xz & e^{xy} \end{vmatrix}$$

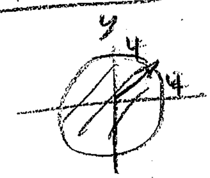
$$= \langle xe^{xy} - 2x,\ -(ye^{xy} - y),\ 2z - z \rangle$$

$$= \langle xe^{xy} - 2x,\ y - ye^{xy},\ z \rangle \quad z=5$$

$$= \langle xe^{xy} - 2x,\ y - ye^{xy},\ 5 \rangle$$

$$\text{curl}\,\vec{F} \cdot \vec{n} = \langle xe^{xy} - 2x,\ y - ye^{xy},\ 5 \rangle \cdot \langle 0,0,1 \rangle = 5$$

parameter domain:



polar: $r = 0$ to $r = 4$, $\theta = 0$ to $\theta = 2\pi$

$$\int_0^{2\pi}\int_0^4 (5)\, r\,dr\,d\theta$$

$$\int_0^4 5r\,dr = \frac{5}{2}\left[r^2\right]_0^4 = \frac{5}{2}(4^2 - 0) = 40$$

$$\int_0^{2\pi} (40)\,d\theta = 40\left(\theta\right)_0^{2\pi} = 40(2\pi) = \boxed{80\pi}$$

Note: try to make the surface bounded by the curve part of a plane if possible :)

#3b. Verify that Stokes' Theorem is true for the given vector field $\vec{F}$ and surface S by writing out and evaluating integrals for both sides of the Stokes' Theorem equation.

$\vec{F}(x,y,z)=\langle y,\ z,\ x\rangle$

S is the hemisphere $x^2+y^2+z^2=1,\ y\geq 0$ oriented in the direction of the positive y-axis.

$$\iint_S (\text{curl}\,\vec{F})\cdot d\vec{S}=\iint_S (\text{curl}\,\vec{F})\cdot\vec{n}\,dA$$

$$\text{curl}\,\vec{F}=\begin{array}{ccc} + & - & + \end{array}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix}$$

$$=\langle 0-1, -(1-0), 0-1\rangle$$
$$=\langle -1,-1,-1\rangle$$

need $\vec{n}$ for surface, here that is just $\vec{n}=\langle 0,1,0\rangle$

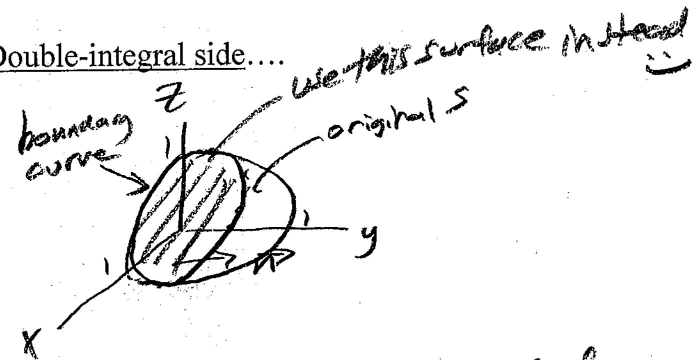
so $\text{curl}\,\vec{F}\cdot\vec{n}=\langle -1,-1,-1\rangle\cdot\langle 0,1,0\rangle$
$$=-1$$

$$\int_0^{2\pi}\int_0^1 (-1)\,r\,dr\,d\theta$$

$$\int_0^1 (-r)\,dr=-\tfrac{1}{2}\left[r^2\right]_0^1=-\tfrac{1}{2}(1^2-0^2)=-\tfrac{1}{2}$$

$$\int_0^{2\pi}\left(-\tfrac{1}{2}\right)d\theta=-\tfrac{1}{2}\left[\theta\right]_0^{2\pi}=\boxed{-\pi}$$

Double-integral side....

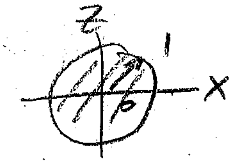


but Stokes' says we can use any surface with this boundary curve, so we'll just use the part of plane $y=0$

parametrize alternate surface:

$\vec{r}(x,z)=\langle x,0,z\rangle$

parameter domain:



polar:
$r=0$ to $r=1$
$\theta=0$ to $\theta=2\pi$

#3b. Verify that Stokes' Theorem is true for the given vector field $\vec{F}$ and surface S by writing out and evaluating integrals for both sides of the Stokes' Theorem equation.

$\vec{F}(x,y,z)=\langle y,\ z,\ x\rangle$

S is the hemisphere $x^2+y^2+z^2=1,\ y\ge 0$ oriented in the direction of the positive y-axis.

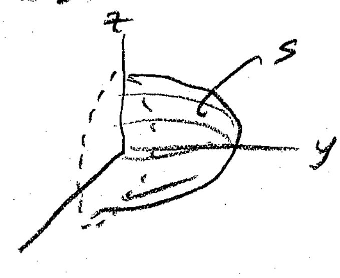
Double-integral side....

If you don't think to try a simple surface, here is the solution for the original surface..

parameter domain

$\phi=0$ to $\phi=\pi$

$\theta=0$ to $\theta=\pi$



easiest to use spherical coordinates: w/ $\rho=1$

$\vec{r}(\phi,\theta)=\langle \sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi\rangle$

curl $\vec{F}$ is unaffected by choice of surface, so still $=\langle -1,-1,-1\rangle$

now to get $\vec{n}$, need $\vec{r_u}\times\vec{r_v}$:

$\vec{r_u}=\vec{r_\phi}=\langle \cos\phi\cos\theta, \cos\phi\sin\theta, -\sin\phi\rangle$

$\vec{r_v}=\vec{r_\theta}=\langle -\sin\phi\sin\theta, \sin\phi\cos\theta, 0\rangle$

$$\vec{n}=\vec{r_u}\times\vec{r_v}=\begin{vmatrix} + & - & + \\ \cos\phi\cos\theta & \cos\phi\sin\theta & -\sin\phi \\ -\sin\phi\sin\theta & \sin\phi\cos\theta & 0\end{vmatrix}=\langle 0-\sin^2\phi\cos\theta, -(0-\sin^2\phi\sin\theta), \sin\phi\cos\phi\cos^2\theta+\sin\phi\cos\phi\sin^2\theta\rangle$$

$$\vec{n}=\langle -\sin^2\phi\cos\theta, \sin^2\phi\sin\theta, \sin\phi\cos\phi\rangle$$

$$(\text{curl}\vec{F})\cdot\vec{n}=\langle -1,-1,-1\rangle\cdot\langle -\sin^2\phi\cos\theta, \sin^2\phi\sin\theta, \sin\phi\cos\phi\rangle$$

$$=\sin^2\phi\cos\theta-\sin^2\phi\sin\theta-\sin\phi\cos\phi$$

$$\int_0^\pi\int_0^\pi(\sin^2\phi\cos\theta-\sin^2\phi\sin\theta-\sin\phi\cos\phi)\,d\phi\,d\theta$$

$$\cos\theta\int_0^\pi\sin^2\phi\,d\phi-\sin\theta\int_0^\pi\sin^2\phi\,d\phi-\int\sin\phi\cos\phi\,d\phi$$

$u=\sin\phi$ $\quad\phi=0\Rightarrow u=0$

$\frac{du}{d\phi}=\cos\phi$ $\quad\phi=\pi\Rightarrow u=0$

$\cos\phi\,d\phi=du$

$$(\cos\theta-\sin\theta)\int_0^\pi\left(\tfrac{1}{2}-\tfrac{1}{2}\cos(2\phi)\right)d\phi-\int_0^0 u\,du$$

$$(\cos\theta-\sin\theta)\left[\tfrac{1}{2}\theta-\tfrac{1}{4}\sin(2\theta)\right]_0^\pi-0$$

$$(\cos\theta-\sin\theta)\left[\tfrac{\pi}{2}-0-0+0\right]=\tfrac{\pi}{2}(\cos\theta-\sin\theta)$$

$$\tfrac{\pi}{2}\int_0^\pi(\cos\theta-\sin\theta)\,d\theta=\tfrac{\pi}{2}(\sin\theta+\cos\theta)_0^\pi=\tfrac{\pi}{2}(\sin\pi-\sin0+\cos\pi-\cos0)$$

$$\qquad 0-0+(-1)-(1)$$

$$=\tfrac{\pi}{2}(-2)=\boxed{-\pi}$$

Note: do not do this on the test (you will run out of time) always look for a simpler (planar) surface instead.

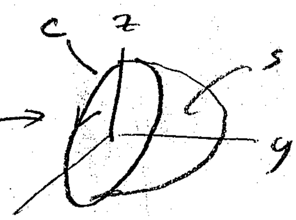
#3b. Verify that Stokes' Theorem is true for the given vector field $\vec{F}$ and surface S by writing out and evaluating integrals for both sides of the Stokes' Theorem equation.

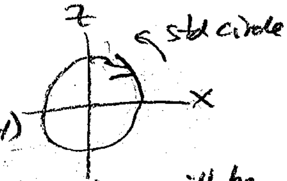
$\vec{F}(x,y,z) = \langle y,\ z,\ x\rangle$

S is the hemisphere $x^2+y^2+z^2=1,\ y\geq 0$ oriented in the direction of the positive y-axis.

Single-integral side....

this direction for RH rule to produce +y normal for surface





parametrize the curve:

standard circle is... (radius=1) $y=0$

$\vec r(t) = \langle 1\cos t,\ 0,\ 1\sin t\rangle$ but this will be opposite needed direction, so swap x, z

$\vec r(t) = \langle \sin t, 0, \cos t\rangle$

parameter-domain:

$0 \leq t \leq 2\pi$ (to go around once)

$$\oint \vec F\cdot d\vec r = \oint \vec F\cdot \vec r\,'\,dt$$

$\vec r\,' = \langle \cos t, 0, -\sin t\rangle$

$\vec F(\vec r) = \langle (0), (\cos t), (\sin t)\rangle$

$= \langle 0, \cos t, \sin t\rangle$

$\vec F\cdot \vec r\,' = \langle 0, \cos t, \sin t\rangle \cdot \langle \cos t, 0, -\sin t\rangle$

$= -\sin^2 t$

$$\int_0^{2\pi}(-\sin^2 t)\,dt = -\int_0^{2\pi}\left(\tfrac12 - \tfrac12\cos(2t)\right)dt$$

$$= -\left[\tfrac12 t - \tfrac14\sin(2t)\right]_0^{2\pi}$$

$$= -\left[\left(\tfrac12(2\pi) - \tfrac14\sin 2\pi\right) - \left(0 - \tfrac14\sin 0\right)\right]$$

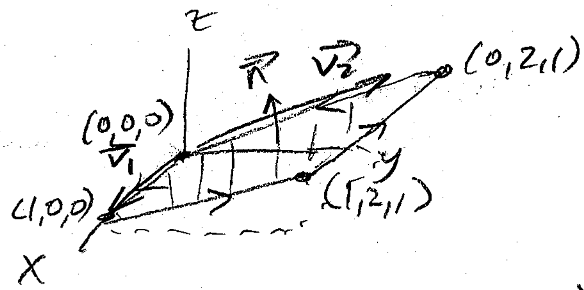
$$-(\pi - 0 - 0 + 0)$$

$$= \boxed{-\pi} \text{ verified } \checkmark$$

Extra #4. A particle moves along line segments from the origin to the points (1,0,0), (1,2,1), (0,2,1), and back to the origin under the influence of the force field:

$\vec{F}(x,y,z) = \langle z^2,\ 2xy,\ 4y^2 \rangle$

Find the work done.



$$\text{work} = \oint_C \vec{F} \cdot d\vec{r} \overset{\text{by Stokes'}}{=} \iint_S (\text{curl}\vec{F}) \cdot \vec{n}\, dA$$

$$\text{curl}\vec{F} = \begin{vmatrix} + & - & + \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ z^2 & 2xy & 4y^2 \end{vmatrix}$$

$= \langle 8y - 0, -(0 - 2z), 2y - 0 \rangle$

$= \langle 8y, 2z, 2y \rangle$

$= \langle 8y, 2(\frac{1}{2}y), 2y \rangle = \langle 8y, y, 2y \rangle$

$\text{curl}\vec{F} \cdot \vec{n}$: for work, really should have a unit vector here:

$$\vec{n} = \frac{\langle 0,-1,2 \rangle}{\sqrt{0^2+1^2+2^2}} = \frac{1}{\sqrt{5}}\langle 0,-1,2 \rangle$$

(if $\vec{n}$ is arbitrarily scaled, it changes our numerical result value)

$\frac{1}{\sqrt{5}}\langle 8y, y, 2y \rangle \cdot \langle 0,-1,2 \rangle$

$\frac{1}{\sqrt{5}}(0 - y + 4y) = \frac{3}{\sqrt{5}} y$

$$\int_0^1 \int_0^2 \frac{3}{\sqrt{5}} y \, dy\, dx$$

$\frac{3}{\sqrt{5}} \int_0^2 y\, dy = \frac{3}{2\sqrt{5}} \left(y^2 \right)_0^2 = \frac{3}{2\sqrt{5}}(4 - 0) = \frac{6}{\sqrt{5}}$

$\int_0^1 \frac{6}{\sqrt{5}} dx = \frac{6}{\sqrt{5}} (x)_0^1 = \frac{6}{\sqrt{5}}(1-0)$

$= \boxed{\frac{6}{\sqrt{5}}}$

parametrize planar curve:

2 vectors in plane to give right direction to match path:

$\vec{v_1} = \langle 1,0,0 \rangle$, $\vec{v_2} = \langle 0,2,1 \rangle$

$$\vec{n} = \vec{v_1} \times \vec{v_2} = \begin{vmatrix} + & - & + \\ 1 & 0 & 0 \\ 0 & 2 & 1 \end{vmatrix}$$

$= \langle 0 - 0, -(1-0), 2 - 0 \rangle$

$\vec{n} = \langle 0,-1,2 \rangle$

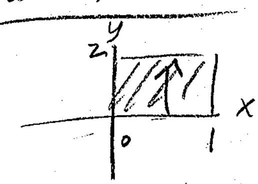
plane equation: $ax + by + cz = \vec{n} \cdot \vec{r_0}$

$0x - y + 2z = \langle 0,-1,2 \rangle \cdot \langle 0,0,0 \rangle$

$-y + 2z = 0$

$z = \frac{1}{2} y$ (need to replace z in $\text{curl}\vec{F} \cdot \vec{n}$)

parameter domain



$y = 0$ to $y = 2$

$x = 0$ to 1

technically, $\vec{n}$ should always be a unit normal vector, but it clutters the results with constants, so we'll ignore this unless we are using the numerical value, for example, in a physics calculation.

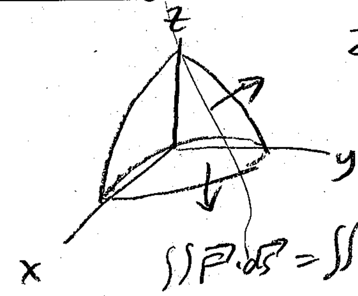
16.9

#1b Verify that the Divergence Theorem is true for the given vector field $\vec{F}$ on the region E by writing out and evaluating integrals for both sides of the Divergence Theorem equation.

$\vec{F}(x,y,z) = \langle x^2,\ xy,\ z\rangle$

E is the solid bounded by the paraboloid $z = 4 - x^2 - y^2$ and the xy-plane.

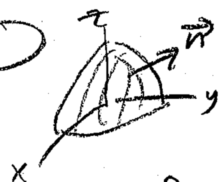
Double-integral side....



2 sides

$\iint \vec{F}\cdot d\vec{S} = \iint \vec{F}\cdot\vec{n}\,dA$ for each side

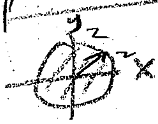
Side 1



parametrize surface:

$\vec{r}(x,y) = \langle x, y, 4-x^2-y^2\rangle$

parameter domain:



polar: $r=0$ to $r=2$

$\theta = 0$ to $\theta = 2\pi$

$\vec{n} = \vec{r_u}\times\vec{r_v}$, $\vec{r_u} = \vec{r_x} = \langle 1, 0, -2x\rangle$

$\vec{r_v} = \vec{r_y} = \langle 0, 1, -2y\rangle$

$\vec{n} = \vec{r_u}\times\vec{r_v} = \begin{vmatrix} + & - & + \\ 1 & 0 & -2x \\ 0 & 1 & -2y \end{vmatrix}$

$\vec{n} = \langle 0+2x, -(-2y-0), 1-0\rangle = \langle 2x, 2y, 1\rangle$

$\vec{F}(\vec{r}) = \langle (x)^2, (x)(y), (4-x^2-y^2)\rangle = \langle x^2, xy, 4-x^2-y^2\rangle$

$\vec{F}\cdot\vec{n} = \langle x^2, xy, 4-x^2-y^2\rangle\langle 2x, 2y, 1\rangle$ to polar

$= 2x^3 + 2xy^2 + 4 - x^2 - y^2$

$= 2x(x^2+y^2) + 4 - (x^2+y^2) = 2(r\cos\theta)r^2 + 4 - r^2$

$\int_0^{2\pi}\int_0^2 (2r^3\cos\theta + 4 - r^2)\,r\,dr\,d\theta$

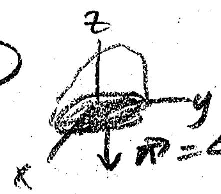
$2\cos\theta\int_0^2 r^4\,dr + 4\int_0^2 r\,dr - \int_0^2 r^3\,dr = 2\cos\theta\left(\frac{1}{5}r^5\right]_0^2 + 4\left[\frac{1}{2}r^2\right]_0^2 - \left[\frac{1}{4}r^4\right]_0^2$

$= \frac{64}{5}\cos\theta + 8 - 4 = \frac{64}{5}\cos\theta + 4$

$\frac{64}{5}\int_0^{2\pi}\cos\theta\,d\theta + 4\int_0^{2\pi} 1\,d\theta = \frac{64}{5}\left[\sin\theta\right]_0^{2\pi} + 4\left[\theta\right]_0^{2\pi}$

$= \frac{64}{5}(\underset{0}{\sin 2\pi} - \underset{0}{\sin 0}) + 4(2\pi) = 8\pi$

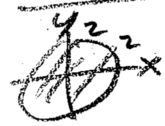
Side 2



surface is $z=0$

$\vec{n} = \langle 0, 0, -1\rangle$ $\vec{r}(x,y) = \langle x, y, 0\rangle$

parameter domain:



polar: $r = 0 \to 2$

$\theta = 0 \to 2\pi$

$\vec{F}(\vec{r}) = \langle x^2, xy, 0\rangle$

$\vec{F}\cdot\vec{n} = \langle x^2, xy, 0\rangle\langle 0, 0, -1\rangle$

$= 0$ so $\iint 0\,r\,dr\,d\theta = 0$)

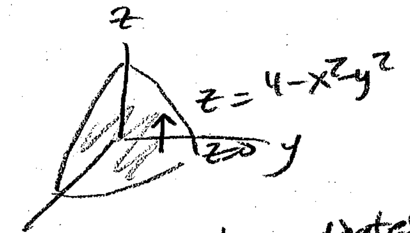
Sum of sides' contributions $= 8\pi + 0 = \boxed{8\pi}$

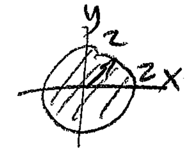
#1b (continued) Verify that the Divergence Theorem is true for the given vector field $\vec{F}$ on the region E by writing out and evaluating integrals for both sides of the Divergence Theorem equation.

$\vec{F}(x,y,z) = \langle x^2,\ xy,\ z \rangle$

E is the solid bounded by the paraboloid $z = 4 - x^2 - y^2$ and the xy-plane.

Triple-integral side....





* cylindrical coordinates

$z=0$ to $z = 4-x^2-y^2 = 4-r^2$

$r=0$ to $r=2$

$\theta=0$ to $\theta=2\pi$

$\iiint \operatorname{div} \vec{F}\, dV$

$\operatorname{div}\vec{F} = \frac{\partial}{\partial x}[x^2] + \frac{\partial}{\partial y}[xy] + \frac{\partial}{\partial z}[z]$

$= 2x + x + 1 = 3x + 1$

$\hookrightarrow$ to cylindrical

$= 3(r\cos\theta) + 1$

$$\int_0^{2\pi}\int_0^2\int_0^{4-r^2} (3r\cos\theta + 1)\, r\, dz\, dr\, d\theta$$

$(3r^2\cos\theta + r)\int_0^{4-r^2} 1\, dz = (3r^2\cos\theta + r)\left(z\right]_0^{4-r^2} = (3r^2\cos\theta + r)(4-r^2)$

$= 12r^2\cos\theta + 4r - 3r^4\cos\theta - r^3$

$12\cos\theta\int_0^2 r^2\,dr + 4\int_0^2 r\,dr - 3\cos\theta\int_0^2 r^4\,dr - \int_0^2 r^3\,dr$

$12\cos\theta\left[\frac{1}{3}r^3\right]_0^2 + 4\left[\frac{1}{2}r^2\right]_0^2 - 3\cos\theta\left[\frac{1}{5}r^5\right]_0^2 - \left[\frac{1}{4}r^4\right]_0^2$

$32\cos\theta + 8 - \frac{96}{5}\cos\theta - 4 = \frac{64}{5}\cos\theta + 4$

$\frac{64}{5}\int_0^{2\pi}\cos\theta\, d\theta + 4\int_0^{2\pi} 1\, d\theta = \frac{64}{5}\left(\sin\theta\right]_0^{2\pi} + 4\left(\theta\right]_0^{2\pi}$

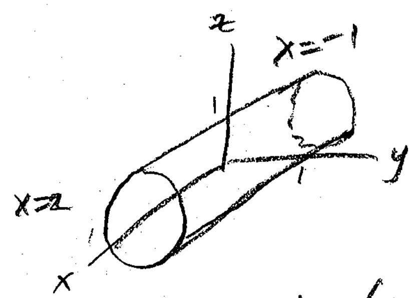
$= \frac{64}{5}(\sin 2\pi - \sin 0) + 4(2\pi) = \boxed{8\pi}$ verified ✓

$0 - 0$

#2b. Using the Divergence Theorem, write out and evaluate the triple-integral which is equivalent to the surface integral $\iint_S \vec{F} \cdot dS$ which calculates the flux of $\vec{F}$ across S if

$\vec{F}(x,y,z) = \langle 3xy^2,\ xe^z,\ z^3 \rangle$

S is the surface of the solid bounded by the cylinder $y^2 + z^2 = 1$ and the planes $x = -1$ and $x = 2$.



cylindrical in yz plane (+x)

$(r=1)$

$x=-1$ to $x=2$ $\quad x=x$

$r=0$ to $r=1$ $\quad y=\cos\theta$

$\theta=0$ to $\theta=2\pi$ $\quad z=\sin\theta$

$\iiint \operatorname{div} \vec{F}\, dv$

$\operatorname{div}\vec{F} = \frac{\partial}{\partial x}[3xy^2] + \frac{\partial}{\partial y}[xe^z] + \frac{\partial}{\partial z}[z^3]$

$= 3y^2 + 0 + 3z^2 = 3(y^2+z^2)$

to cylindrical $= 3r^2$

$\int_0^{2\pi}\int_0^1\int_{-1}^2 (3r^2)\, r\, dx\, dr\, d\theta$

$3r^3 \int_{-1}^2 1\, dx = 3r^3[x]_{-1}^2 = 3r^3(2-(-1)) = 9r^3$

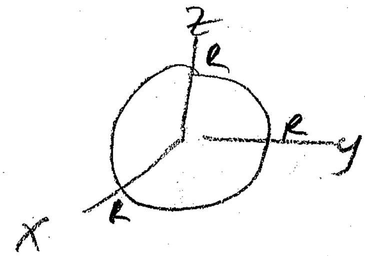
$9\int_0^1 r^3\, dr = 9\left(\frac{1}{4}r^4\right]_0^1 = \frac{9}{4}(1-0) = \frac{9}{4}$

$\int_0^{2\pi} \frac{9}{4}\, d\theta = \frac{9}{4}[\theta]_0^{2\pi} = \boxed{\frac{9\pi}{2}}$

#3b. Using the Divergence Theorem, write out and evaluate the triple-integral which is equivalent to the surface integral $\iint_S \vec{F} \cdot dS$ which calculates the flux of $\vec{F}$ across S if

$\vec{F}(x,y,z) = \langle 4x^3z,\ 4y^3z,\ 3z^4 \rangle$

S is the sphere with radius R and center at the origin.



$$\iiint \operatorname{div} \vec{F}\, dv$$

Spherical coordinates:

$x = \rho \sin\phi \cos\theta$ $\quad \rho = 0$ to $\rho = R$

$y = \rho \sin\phi \sin\theta$ $\quad \phi = 0$ to $\phi = \pi$

$z = \rho \cos\phi$ $\quad \theta = 0$ to $\theta = 2\pi$

$$\operatorname{div}\vec{F} = \frac{\partial}{\partial x}[4x^3z] + \frac{\partial}{\partial y}[4y^3z] + \frac{\partial}{\partial z}[3z^4]$$

$$= 12x^2z + 12y^2z + 12z^3$$

to spherical.. $= 12(\rho\sin\phi\cos\theta)^2(\rho\cos\phi) + 12(\rho\sin\phi\sin\theta)^2(\rho\cos\phi) + 12(\rho\cos\phi)^3$

$$= 12\rho^3\sin^2\phi\cos\phi\cos^2\theta + 12\rho^3\sin^2\phi\cos\phi\sin^2\theta + 12\rho^3\cos^3\phi$$

$$= 12\rho^3\sin^2\phi\cos\phi + 12\rho^3\cos^3\phi = 12\rho^3\cos\phi\,\underline{\sin^2\phi} + 12\rho^3\cos\phi\,\underline{\cos^2\phi}$$

$$= 12\rho^3\cos\phi$$

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{R} (12\rho^3\cos\phi)\,\rho^2\sin\phi\, d\rho\, d\phi\, d\theta$$

$$12\cos\phi\sin\phi \int_0^R \rho^5\, d\rho = 12\cos\phi\sin\phi\left[\frac{1}{6}\rho^6\right]_0^R = 2R^6\cos\phi\sin\phi$$

$$2R^6\int_0^{\pi}\cos\phi\sin\phi\, d\phi$$

$u = \sin\phi$ $\quad \phi = 0 \Rightarrow u = 0$

$\frac{du}{d\phi} = \cos\phi$ $\quad \phi = \pi \Rightarrow u = 0$

$\cos\phi\, d\phi = du$

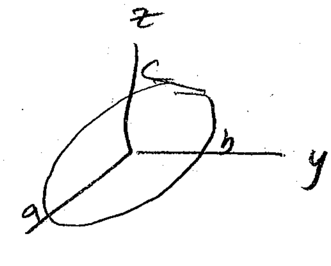
$$2R^6\int_0^0 u\, du = 0$$

so $\int_0^{2\pi} 0\, d\theta = \boxed{0}$

#3c. Using the Divergence Theorem, write out and evaluate the triple-integral which is equivalent to the surface integral $\iint_S \vec{F} \cdot dS$ which calculates the flux of $\vec{F}$ across S if

$\vec{F}(x,y,z) = \langle xy\sin z,\ \cos(xz),\ y\cos z \rangle$

S is the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.



hmm... not spherical,
not rectangular
not great for cylindrical
before we proceed, let's find div $\vec{F}$

$$\operatorname{div}\vec{F} = \frac{\partial}{\partial x}[xy\sin z] + \frac{\partial}{\partial y}[\cos(xz)] + \frac{\partial}{\partial z}[y\cos z]$$

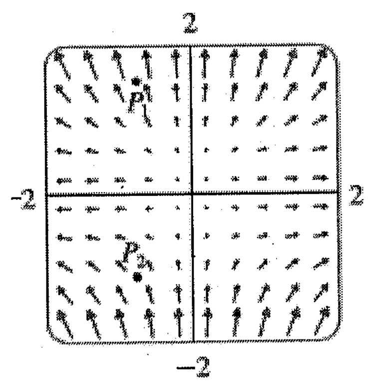
$$= y\sin z + 0 - y\sin z$$

$$= 0$$

So $\iiint \operatorname{div}\vec{F}\,dV \boxed{=0}$ no matter how we carve the volume :)

(maybe we should make it a habit of checking div $\vec{F}$ first, kind of like we should always check for conservative fields w/ closed paths :))

#4b. A vector field $\vec{F}$ is shown.



(i) Determine whether is positive or negative at P_1 and P_2 just from looking at the field picture.

(ii) Given the $\vec{F}(x,y) = \langle x, y^2 \rangle$ for this field, use the definition of divergence to verify your answers in part (i).

(i) at P_1, appears more leaving than entering

so $\boxed{\text{div } \vec{F} \text{ at } P_1 > 0}$ (source)

at P_2, appears more entering than leaving

so $\boxed{\text{div } \vec{F} \text{ at } P_2 < 0}$ (sink)

(ii) $\text{div } \vec{F} = \frac{\partial}{\partial x}[x] + \frac{\partial}{\partial y}[y^2]$

$= 1 + 2y$

at $P_1 \approx (-0.8, 1.5)$ $\text{div } \vec{F} = 1 + 2(1.5) = 4$

is > 0 ✓

at $P_2 \approx (-0.8, -1.1)$ $\text{div } \vec{F} = 1 + 2(-1.1) = -1.2$

is < 0 ✓