Calc III - Ch 15 Part 2 - Extra Practice

$\boxed{\text{SOLUTIONS}}$

15.6

#1b. Evaluate the given integral over the specified region using the three specified orders of integration.

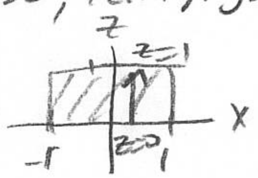
$$\iiint_E \left(xz - y^3\right)\, dV$$

$$E = \{(x,y,z) \mid -1 \le x \le 1,\ 0 \le y \le 2,\ 0 \le z \le 1\}$$

(i) …integrating first with respect to y, then z, and then x.

inner projection

floor: $y=0$, ceiling: $y=2$



$$\int_{-1}^{1}\int_{0}^{1}\int_{0}^{2}(xz-y^3)\,dy\,dz\,dx$$

$$\int_0^2 (xz-y^3)\,dy = \left[xzy - \tfrac{1}{4}y^4\right]_0^2$$

$$\left(xz(2) - \tfrac14(2)^4\right) - \left(xz(0) - \tfrac14(0)^4\right)$$

$$= 2xz - 4$$

$$\int_0^1 (2xz-4)\,dz = \left[xz^2 - 4z\right]_0^1$$

$$= \left(x(1)^2 - 4(1)\right) - \left(x(0)^2 - 4(0)\right)$$

$$= x-4$$

$$\int_{-1}^1 (x-4)\,dx = \left[\tfrac12 x^2 - 4x\right]_{-1}^1$$

$$\left(\tfrac12(1)^2 - 4(1)\right) - \left(\tfrac12(-1)^2 - 4(-1)\right)$$

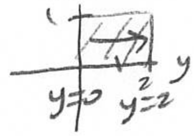
$$\tfrac12 - 4 - \tfrac12 - 4$$

$$= \boxed{-8}$$

(ii) …integrating first with respect to x, then y, and then z.

inner projection

floor: $x=-1$, ceiling: $x=1$



$$\int_0^1\int_0^2\int_{-1}^1 (xz-y^3)\,dx\,dy\,dz$$

$$\int_{-1}^1 (xz-y^3)\,dx = \left[\tfrac12 zx^2 - y^3x\right]_{-1}^1$$

$$= \left[\tfrac12 z(1)^2 - y^3(1)\right) - \left(\tfrac12 z(-1)^2 - y^3(-1)\right]$$

$$= \tfrac12 z - y^3 - \tfrac12 z - y^3 = -2y^3$$

$$\int_0^2 (-2y^3)\,dy = \left[-\tfrac12 y^4\right]_0^2 = \left(-\tfrac12(2)^4\right) - \left(-\tfrac12(0)^4\right)$$

$$= -8$$

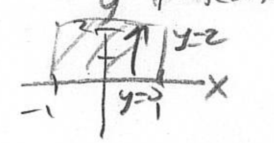
$$\int_0^1 -8\,dz = -8\left[z\right]_0^1 = -8(1-0)$$

$$= \boxed{-8}$$

(iii) …integrating first with respect to z, then y, and then x.

inner projection

floor: $z=0$, ceiling: $z=1$



$$\int_{-1}^1\int_0^2\int_0^1 (xz-y^3)\,dz\,dy\,dx$$

$$\int_0^1 (xz-y^3)\,dz = \left[\tfrac12 xz^2 - y^3z\right]_0^1$$

$$= \left(\tfrac12 x(1)^2 - y^3(1)\right) - \left(\tfrac12 x(0)^2 - y^3(0)\right)$$

$$= \tfrac12 x^2 - y^3$$

$$\int_0^2 \left(\tfrac12 x^2 - y^3\right)dy = \left[\tfrac12 x^2 y - \tfrac14 y^4\right]_0^2$$

$$= \left(\tfrac12 x^2(2) - \tfrac14(2)^4\right) - \left(\tfrac12 x^2(0) - \tfrac14(0)^4\right)$$

$$= x^2 - 4$$

$$\int_{-1}^1 (x^2-4)\,dx = \left[\tfrac13 x^3 - 4x\right]_{-1}^1$$

$$\left(\tfrac13(1)^3 - 4(1)\right) - \left(\tfrac13(-1)^3 - 4(-1)\right)$$

$$\tfrac13 - 4 + \tfrac13 - 4 = \boxed{-8}$$

#2b. Evaluate the iterated integral

$$\int_0^{\sqrt{\pi}}\int_0^{x}\int_0^{xz} x^2 \sin y \, dy\, dz\, dx$$

$$\int_0^{xz} x^2 \sin y\, dy = \left[-x^2\cos y\right]_0^{xz}$$

$$= (-x^2\cos(xz)) - (-x^2\cos(0)))$$

$$= -x^2\cos(xz) + x^2 = x^2 - x^2\cos(xz)$$

$$\int_0^x (x^2 - x^2\cos(xz))dz = x^2\int_0^x 1\, dz - x^2\int_0^x \cos(xz)dz$$

$$= x^2\left[z\right]_0^x - x^2\left[\frac{1}{x}\sin(xz)\right]_0^x$$

$$= x^2(x-0) - x(\sin(x(x)) - \sin(0))$$

$$= x^3 - x\sin(x^2)$$

$$\int_0^{\sqrt{\pi}} (x^3 - x\sin(x^2))dx$$

$$= \int_0^{\sqrt{\pi}} x^3 dx - \int_0^{\sqrt{\pi}} x\sin(x^2)dx$$

u-sub: $u = x^2$

$\frac{du}{dx} = 2x$

$x\,dx = \frac{1}{2}du$

$x = 0 \rightarrow u = (0)^2 = 0$

$x = \sqrt{\pi} \rightarrow u = (\sqrt{\pi})^2 = \pi$

$$= \int_0^{\sqrt{\pi}} x^3 dx - \frac{1}{2}\int_0^{\pi} \sin(u)du$$

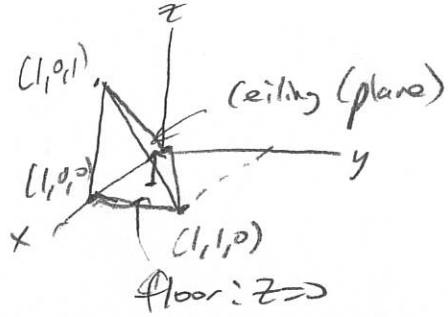
$$= \left[\frac{1}{4}x^4\right]_0^{\sqrt{\pi}} - \frac{1}{2}\left[-\cos(u)\right]_0^{\pi}$$

$$= \frac{1}{4}((\sqrt{\pi})^4 - (0)^4) + \frac{1}{2}(\underbrace{\cos\pi}_{(-1)} - \underbrace{\cos 0}_{(1)})$$

$$= \boxed{\frac{\pi^2}{4} - 1}$$

#3b. Evaluate the integral $\iiint_T xyz\, dV$

where T is the solid tetrahedrom with vertices (0,0,0), (1,0,0), (1,1,0), and (1,0,1).



floor: $z=0$

for ceiling plane, find equation $ax+by+cz = \vec{n}\cdot\vec{r_0}$ w/ 2 vectors in plane: $\vec{v_1} = \langle 1,0,1\rangle$, $\vec{v_2} = \langle 1,1,0\rangle$

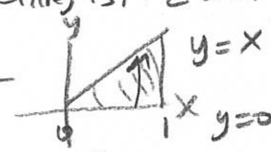
$$\vec{n} = \begin{vmatrix} + & - & + \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = \langle 0-1, -(0-1), 1-0\rangle = \langle -1, 1, 1\rangle$$

$\vec{r_0} = \langle 0,0,0\rangle$ $\quad -x+y+z = \langle -1,1,1\rangle\cdot\langle 0,0,0\rangle$

$-x+y+z = 0$

so ceiling is: $z = x - y$

projection



$$\int_0^1\int_0^x\int_0^{x-y} xyz\, dz\, dy\, dx$$

$$\int_0^{x-y} xyz\, dz = \left[\frac{1}{2}xyz^2\right]_0^{x-y} = \frac{1}{2}xy\left[(x-y)^2 - 0^2\right]$$

$$= \frac{1}{2}xy(x^2 - 2xy + y^2) = \frac{1}{2}x^3y - x^2y^2 + \frac{1}{2}xy^3$$

$$\int_0^x \left(\frac{1}{2}x^3y - x^2y^2 + \frac{1}{2}xy^3\right)dy$$

$$= \left[\frac{1}{4}x^3y^2 - \frac{1}{3}x^2y^3 + \frac{1}{8}xy^4\right]_0^x$$

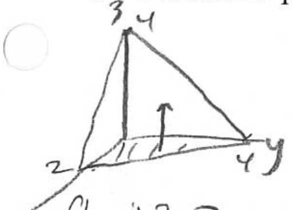
$$= \left(\frac{1}{4}x^3(x)^2 - \frac{1}{3}x^2(x)^3 + \frac{1}{8}x(x)^4\right) - [0]$$

$$= \frac{1}{4}x^5 - \frac{1}{3}x^5 + \frac{1}{8}x^5 = \frac{1}{24}x^5$$

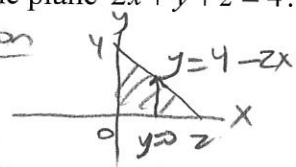
$$\int_0^1 \frac{1}{24}x^5 dx = \frac{1}{144}\left[x^6\right]_0^1 = \frac{1}{144}(1^6 - 0)$$

$$= \boxed{\frac{1}{144}}$$

#4b. Set up (but do not evaluate) a triple integral to find the volume of the tetrahedron enclosed by the coordinate planes and the plane $2x + y + z = 4$.



projection

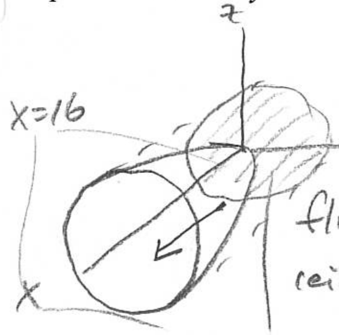


floor: $z=0$

ceiling: $z=4-2x-y$

$$\boxed{\int_0^2 \int_0^{4-2x} \int_0^{4-2x-y} (1)\, dz\, dy\, dx}$$

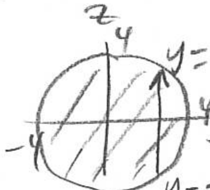
#4c. Set up (but do not evaluate) a triple integral to find the volume of the solid enclosed by the paraboloid $x = y^2 + z^2$ and the plane $x = 16$.



floor: $x=y^2+z^2$

ceiling: $x=16$

projection (intersection): $\begin{cases} x=16 \\ x=y^2+z^2 \end{cases}$ $y^2+z^2=16$

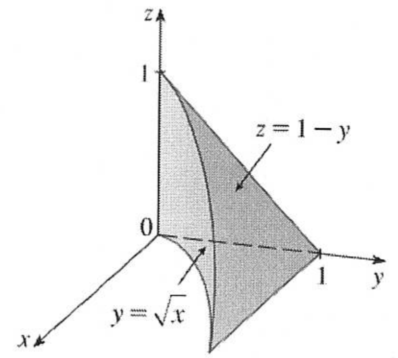


$$\boxed{\int_{-4}^{4} \int_{-\sqrt{16-z^2}}^{+\sqrt{16-z^2}} \int_{16}^{y^2+z^2} (1)\, dx\, dz\, dy}$$

"Can we use polar on projection?"

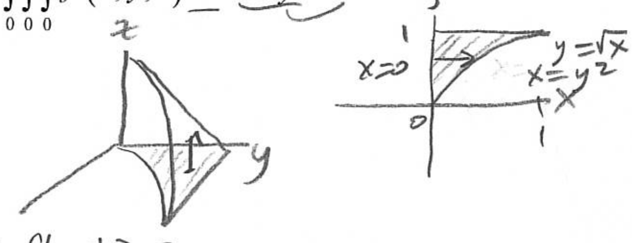
(next section :))

#5b. The figure shows the region of integration for the integral $\int_0^1 \int_{\sqrt{x}}^1 \int_0^{1-y} f(x,y,z)\, dz\, dy\, dx$.



Rewrite this integral (but do not evaluate) as an equivalent iterated integral in the specified integration orders….

(i) $\int_0^? \int_0^? \int_0^? f(x,y,z)\, dz\, dx\, dy$. projection:

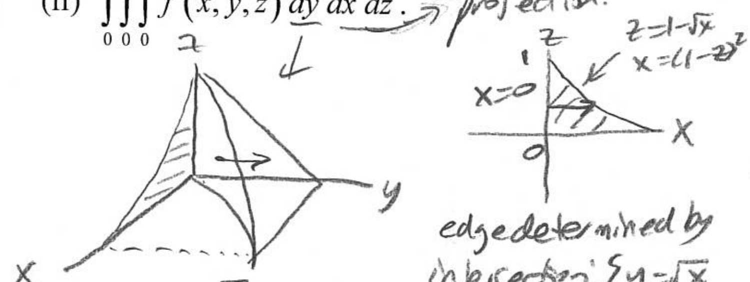


floor: $z=0$

ceiling: $z=1-y$

$$\boxed{\int_0^1 \int_0^{y^2} \int_0^{1-y} f(x,y,z)\, dz\, dx\, dy}$$

(ii) $\int_0^? \int_0^? \int_0^? f(x,y,z)\, dy\, dx\, dz$. projection:



floor: $y=\sqrt{x}$

ceiling: $z=1-y$ $(y=1-z)$

edge determined by intersection: $\begin{cases} y=\sqrt{x} \\ z=1-y \end{cases}$

$z=1-\sqrt{x}$

or $\sqrt{x}=1-z$

$x=(1-z)^2$

$$\boxed{\int_0^1 \int_0^{(1-z)^2} \int_{\sqrt{x}}^{1-z} f(x,y,z)\, dy\, dx\, dz}$$

Extra #6. Write five other iterated integrals that are equal to the given iterated integral:

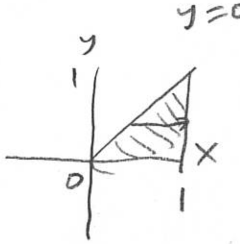
$$\int_0^1\int_y^1\int_0^y f(x,y,z)\,dz\,dx\,dy.$$

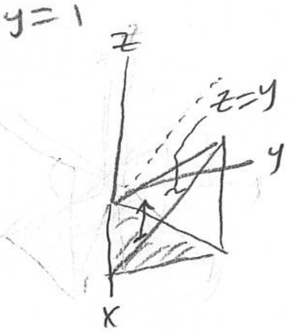
original: floor: $z=0$

ceiling: $z=y$

projection: $x=y$ to $x=1$

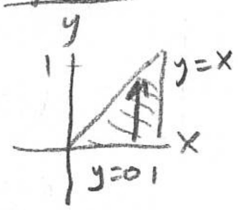
$y=0$ to $y=1$





$\boxed{dz\,dy\,dx}$

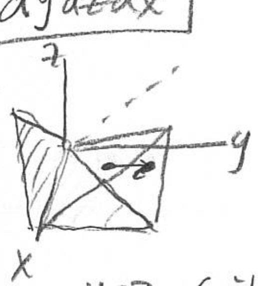
projection:



floor: $z=0$, ceiling: $z=y$

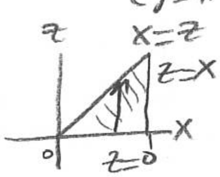
$$\boxed{\int_0^1\int_0^x\int_0^y f(x,y,z)\,dz\,dy\,dx}$$

$\boxed{dy\,dz\,dx}$



projection:

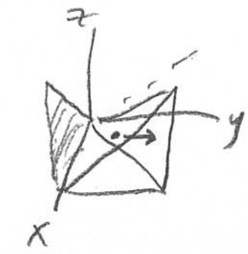
intersection: $\begin{cases} y=z \\ y=x \end{cases}$ $x=z$



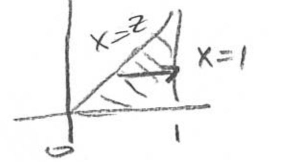
floor: $y=z$, Ceiling: $y=x$

$$\boxed{\int_0^1\int_0^x\int_z^x f(x,y)\,dy\,dz\,dx}$$

$\boxed{dy\,dx\,dz}$



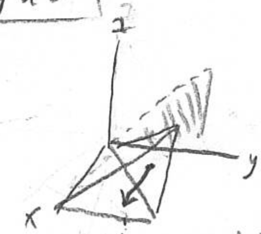
Projection



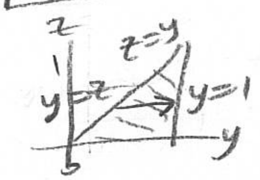
floor: $y=z$, ceiling: $y=x$

$$\boxed{\int_0^1\int_z^1\int_z^x f(x,y)\,dy\,dx\,dz}$$

$\boxed{dx\,dy\,dz}$



projection:



floor: plane containing $(0,0,0)(1,1,0)(1,1,1)$

$\vec{v}_1=\langle 1,1,0\rangle$ $\vec{v}_2=\langle 1,1,1\rangle$

$$\vec{n}=\begin{vmatrix} + & - & + \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix}=\langle 1-0,-(1-0),1-1\rangle$$

$=\langle 1,-1,0\rangle$ $\vec{r}_0=\langle 0,0,0\rangle$

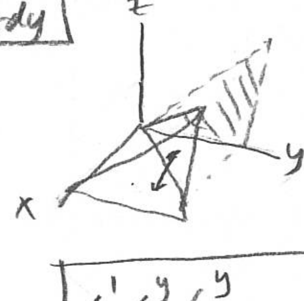
$ax+by+cz=\vec{n}\cdot\vec{r}_0=x-y=0$

floor: $x=y$

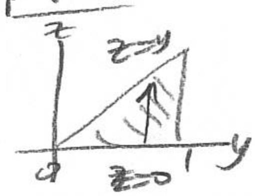
Ceiling: $x=1$

$$\boxed{\int_0^1\int_z^1\int_1^y f(x,y,z)\,dx\,dy\,dz}$$

$\boxed{dx\,dz\,dy}$



projection

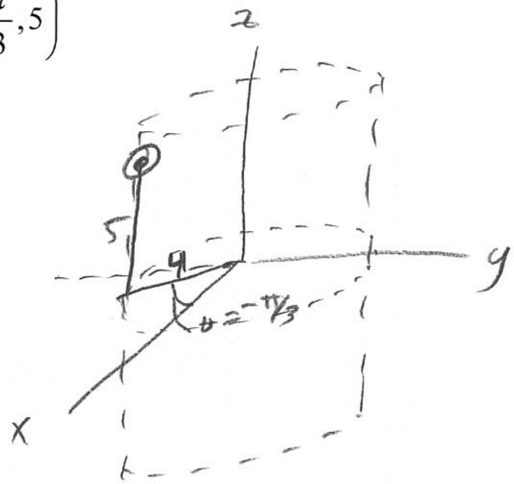


$$\boxed{\int_0^1\int_0^y\int_1^y f(x,y,z)\,dx\,dz\,dy}$$

15.7

#1b. Plot the point whose cylindrical coordinates are given:

$\left(4, -\frac{\pi}{3}, 5\right)$



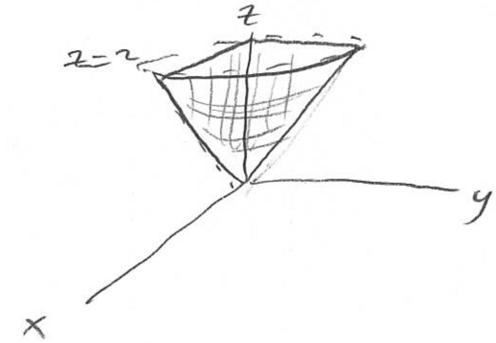
#3b. Sketch the solid described by the inequalities:

$0 \le \theta \le \frac{\pi}{2}, \quad r \le z \le 2$

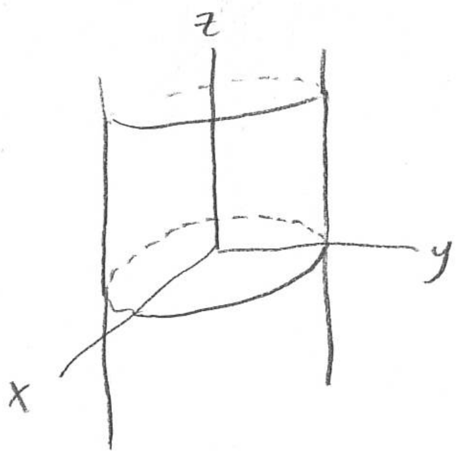
$\theta = 0$ to $\theta = \frac{\pi}{2}$ $\quad z = r$ to $z = 2$

$z = \sqrt{x^2 + y^2}$

$z^2 = x^2 + y^2$ a cone



#2b. Sketch and describe in words the surface whose equation is given: $r = 5$



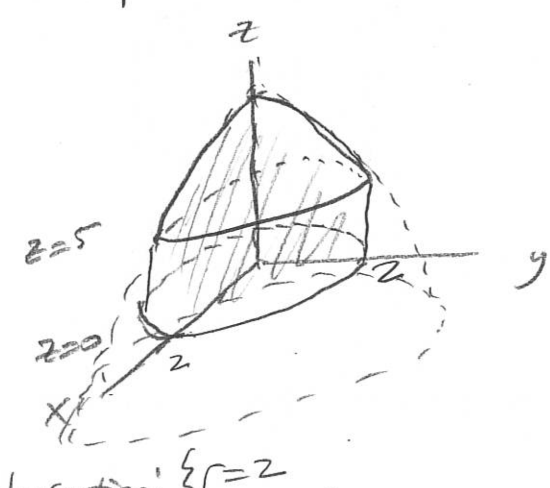
a circular cylinder with axis of symmetry in z direction.

#4b. Sketch the solid whose volume is given by the integral: $\int_0^{\pi/2}\int_0^{2}\int_0^{9-r^2} r\, dz\, dr\, d\theta$

$z = 0$ to $z = 9 - r^2$ $\quad r = 0$ to $r = 2$ $\quad \theta = 0$ to $\theta = \frac{\pi}{2}$

$z = 9 - (x^2 + y^2)$

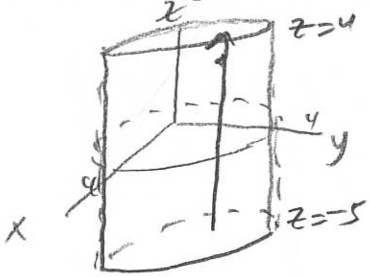
(a paraboloid)



intersection: $\begin{cases} r = 2 \\ z = 9 - r^2 \end{cases}$

$z = 9 - (2)^2 = 5$

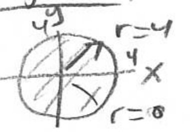
#5b. Evaluate $\iiint\limits_E \sqrt{x^2+y^2}\, dV$, where E is the region the lies inside the cylinder $x^2+y^2=16$, and between the planes $z=-5$ and $z=4$.



floor: $z=-5$

ceiling: $z=4$

projection:



Integrand: $\sqrt{x^2+y^2}=\sqrt{r^2}=r$

$$\int_0^{2\pi}\int_0^4\int_{-5}^{4} (r)\, r\, dz\, dr\, d\theta$$

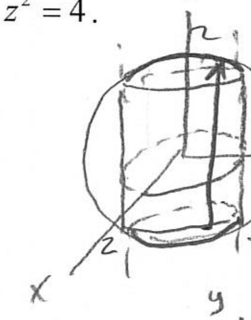
$$\int_{-5}^{4} r^2\, dz = r^2\left[z\right]_{-5}^{4} = r^2(4-(-5)) = 9r^2$$

$$\int_0^4 9r^2\, dr = 3\left[r^3\right]_0^4 = 3((4)^3-(0)^3) = 192$$

$$\int_0^{2\pi} 192\, d\theta = 192\left[\theta\right]_0^{2\pi} = \boxed{384\pi}$$

(integrand $=1$)

#6b. Find the volume of the solid that lies within both the cylinder $x^2+y^2=1$ and the sphere $x^2+y^2+z^2=4$.



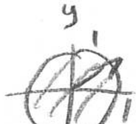
ceiling: $z=+\sqrt{4-x^2-y^2}$

$z=+\sqrt{4-r^2}$

floor: $z=-\sqrt{4-x^2-y^2}$

$z=-\sqrt{4-r^2}$

Projection:



$r=0$ to $r=1$

$\theta=0$ to $\theta=2\pi$

$$\int_0^{2\pi}\int_0^1\int_{-\sqrt{4-r^2}}^{+\sqrt{4-r^2}} (1)\, r\, dz\, dr\, d\theta$$

$$\int_{-\sqrt{4-r^2}}^{+\sqrt{4-r^2}} r\, dz = r\left[z\right]_{-\sqrt{4-r^2}}^{+\sqrt{4-r^2}}$$

$$= r\left[\sqrt{4-r^2}-\left(-\sqrt{4-r^2}\right)\right] = 2r\sqrt{4-r^2}$$

$$\int_0^1 2r\sqrt{4-r^2}\, dr$$

$u=4-r^2$ $\quad$ $r=0 \to u=4$

$\frac{du}{dr}=-2r$ $\quad$ $r=1 \to u=3$

$2r\,dr=-du$

$$= -\int_4^3 u^{1/2}\, du = \int_3^4 u^{1/2}\, du = \frac{2}{3}\left[u^{3/2}\right]_3^4$$

$$= 2/3\left[(4)^{3/2}-(3)^{3/2}\right] = \frac{2}{3}\left[8-\sqrt{27}\right]$$

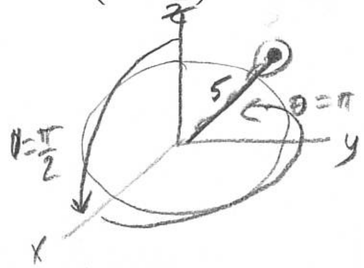
$$\int_0^{2\pi} \frac{2}{3}\left[8-\sqrt{27}\right] d\theta = \frac{2}{3}\left(8-\sqrt{27}\right)\left[\theta\right]_0^{2\pi}$$

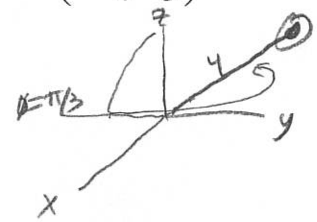
$$= \frac{2}{3}\left(8-\sqrt{27}\right)2\pi = \boxed{\frac{4\pi}{3}\left(8-\sqrt{27}\right)}$$

15.8

#1b. Plot the point whose ~~cylindrical~~ spherical coordinates are given and find the rectangular coordinates of the point:

(i) $\left(5, \pi, \frac{\pi}{2}\right)$ $\quad (\rho, \theta, \phi)$ (ii) $\left(4, \frac{3\pi}{4}, \frac{\pi}{3}\right)$





(i)

$x = \rho \sin\phi \cos\theta$
$= (5)\sin\frac{\pi}{2}\cos\pi = (5)(1)(-1)$
$= -5$

$y = \rho \sin\phi \sin\theta$
$= (5)\sin\frac{\pi}{2}\sin\pi = (5)(1)(0)$
$= 0$

$z = \rho\cos\phi = (5)\cos\frac{\pi}{2} = (5)(0) = 0$

$\boxed{(-5, 0, 0)}$

(ii)

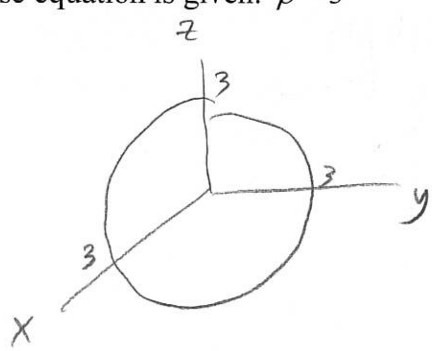
$x = \rho\sin\phi\cos\theta$
$= (4)\sin\frac{\pi}{3}\cos\frac{3\pi}{4}$
$= (4)\left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) = \sqrt{6}$

$x = \rho\sin\phi\sin\theta$
$= (4)\sin\frac{\pi}{3}\sin\frac{3\pi}{4}$
$= (4)\left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) = \sqrt{6}$

$z = \rho\cos\phi = (4)\cos\frac{\pi}{3}$
$= (4)\left(\frac{1}{2}\right) = 2$

$\boxed{(\sqrt{6}, \sqrt{6}, 2)}$

#2b. Sketch and describe in words the surface whose equation is given: $\rho = 3$



a sphere, centered at the origin with radius = 3

#3b. Identify the surface whose equation is given: $\rho^2\left(\sin^2\phi\sin^2\theta + \cos^2\phi\right) = 9$

$\rho^2\sin^2\phi\sin^2\theta + \rho^2\cos^2\phi = 9$

$(\rho\sin\phi\sin\theta)^2 + (\rho\cos\phi)^2 = 9$

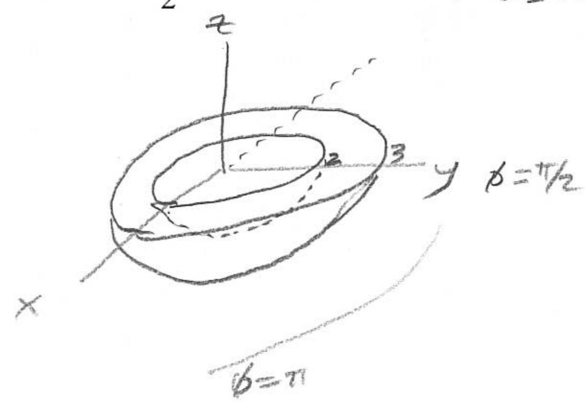
$y^2 + z^2 = 9$

Circular cylinder w/ radius = 3 and axis of symmetry in x-direction.

#4b. Sketch the solid described by the given inequalities:

$2 \le \rho \le 3, \quad \frac{\pi}{2} \le \phi \le \pi$

no restriction on θ so $0 \le \theta \le 2\pi$



#5b. Write the equation in spherical coordinates:

(i) $x^2 - 2x + y^2 + z^2 = 0$

$x^2+y^2+z^2-2x=0$

$\rho^2-2(\rho\sin\phi\cos\theta)=0$

$\rho(\rho-2\sin\phi\cos\theta)=0$

$\boxed{\rho=0}$ or $\boxed{\rho=2\sin\phi\cos\theta}$

(ii) $x + 2y + 3z = 1$

$\boxed{\rho\sin\phi\cos\theta + 2\rho\sin\phi\sin\theta + 3\rho\cos\phi = 1}$

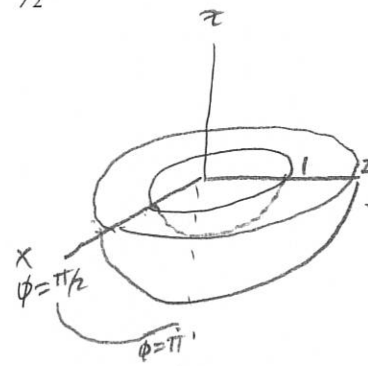
or

$\rho(\sin\phi\cos\theta + 2\sin\phi\sin\theta + 3\cos\phi) = 1$

$\boxed{\rho = \dfrac{1}{\sin\phi\cos\theta + 2\sin\phi\sin\theta + 3\cos\phi}}$

#6b. Sketch the solid whose volume is given by the integral and evaluate the integral.

$$\int_0^{2\pi}\int_{\pi/2}^{\pi}\int_1^2 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$



$\int_1^2 \rho^2\sin\phi \, d\rho = \frac{1}{3}\sin\phi\left[\rho^3\right]_1^2$

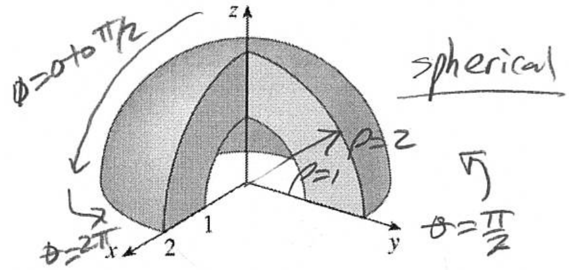
$= \frac{1}{3}\sin\phi\left[(2)^3-(1)^3\right] = \frac{7}{3}\sin\phi$

$\int_{\pi/2}^{\pi} \frac{7}{3}\sin\phi \, d\phi = -\frac{7}{3}\left[\cos\phi\right]_{\pi/2}^{\pi}$

$= -\frac{7}{3}\left[\cos\pi - \cos\frac{\pi}{2}\right] = -\frac{7}{3}(-1-0) = \frac{7}{3}$

$\int_0^{2\pi} \frac{7}{3} d\theta = \frac{7}{3}\left[\theta\right]_0^{2\pi} = \boxed{\frac{14\pi}{3}}$

#7b. Set up the triple integral of an arbitrary continuous function $f(x,y,z)$ in cylindrical or spherical coordinates over the solid shown.



$$\int_{\frac{\pi}{2}}^{2\pi}\int_{0}^{\pi/2}\int_{1}^{2} f(x,y,z)\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

convert to spherical

$x = \rho\sin\phi\cos\theta$

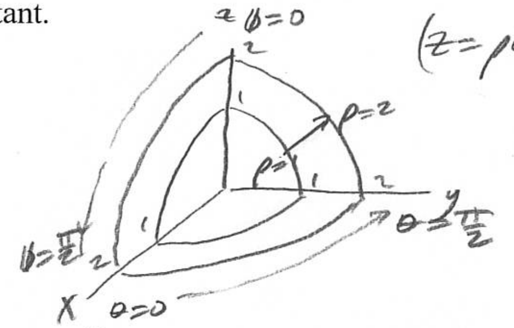
$y = \rho\sin\phi\sin\theta$

$z = \rho\cos\phi$

$$\boxed{\int_{\frac{\pi}{2}}^{2\pi}\int_{0}^{\pi/2}\int_{1}^{2} f(\rho\sin\phi\cos\theta,\ \rho\sin\phi\sin\theta,\ \rho\cos\phi)\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta}$$

#8b. Evaluate using spherical coordinates $\iiint\limits_E z\,dV$, where E lies between the spheres $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 4$ in the first octant.

$(z = \rho\cos\phi)$



$$\int_{0}^{\pi/2}\int_{0}^{\pi/2}\int_{1}^{2} (\rho\cos\phi)\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$\int_1^2 \rho^3\cos\phi\sin\phi\,d\rho = \tfrac{1}{4}\cos\phi\sin\phi\,[\rho^4]_1^2$$

$$= \tfrac{1}{4}\cos\phi\sin\phi\,[(2)^4 - (1)^4] = \tfrac{15}{4}\cos\phi\sin\phi$$

$$\int_0^{\pi/2} \tfrac{15}{4}\cos\phi\sin\phi\,d\phi \qquad u = \sin\phi \qquad \phi = 0 \to u = 0$$

$$\frac{du}{d\phi} = \cos\phi \qquad \phi = \pi/2 \to u = 1$$

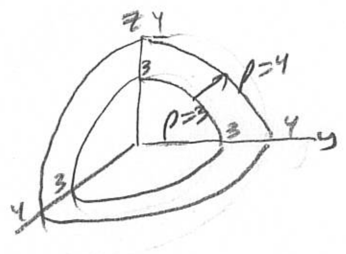
$$\cos\phi\,d\phi = du$$

$$\tfrac{15}{4}\int_0^1 u\,du = \tfrac{15}{4}\cdot\tfrac{1}{2}[u^2]_0^1 = \tfrac{15}{8}[1^2 - 0^2] = \tfrac{15}{8}$$

$$\int_0^{\pi/2}\tfrac{15}{8}\,d\theta = \tfrac{15}{8}[\theta]_0^{\pi/2} = \tfrac{15}{8}(\pi/2 - 0)$$

$$= \boxed{\frac{15\pi}{16}}$$

#9b. Evaluate using spherical coordinates $\iiint_E x^2\,dV$, where E is bounded by the xz-plane and the hemispheres $y=\sqrt{9-x^2-z^2}$ and $y=\sqrt{16-x^2-z^2}$ in the first octant.



$x^2 \Rightarrow (\rho\sin\phi\cos\theta)^2$

$$\int_0^{\pi/2}\int_0^{\pi/2}\int_3^4 (\rho\sin\phi\cos\theta)^2\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$\int_3^4 \rho^4\sin^3\phi\cos^2\theta\,d\rho = \frac{1}{5}\sin^3\phi\cos^2\theta\left[\rho^5\right]_3^4$$

$$= \frac{1}{5}\sin^3\phi\cos^2\theta\left[(4)^5-(3)^5\right] = \frac{781}{5}\sin^3\phi\cos^2\theta$$

$$\int_0^{\pi/2}\frac{781}{5}\cos^2\theta\sin^3\phi\,d\phi = \frac{781}{5}\cos^2\theta\int_0^{\pi/2}\sin^3\phi\,d\phi$$

$$\frac{781}{5}\cos^2\theta\int_0^{\pi/2}\sin^2\phi\sin\phi\,d\phi = \frac{781}{5}\cos^2\theta\int_0^{\pi/2}(1-\cos^2\phi)\sin\phi\,d\phi$$

$$\frac{781}{5}\cos^2\theta\left[\int_0^{\pi/2}\sin\phi\,d\phi - \int_0^{\pi/2}\cos^2\phi\sin\phi\,d\phi\right]$$

$u=\cos\phi$

$\frac{du}{d\phi}=-\sin\phi$

$\sin\phi\,d\phi=-du$

$\phi=0\to u=1$

$\phi=\pi/2\to u=0$

$$\frac{781}{5}\cos^2\theta\left[\left[-\cos\phi\right]_0^{\pi/2}+\int_1^0 u^2\,du\right]$$

$$\frac{781}{5}\cos^2\theta\left[-(\cos\pi/2-\cos 0)+\frac{1}{3}\left[(0)^3-(1)^3\right]\right]$$

$$\frac{781}{5}\cos^2\theta\left(-(0-1)+\frac{1}{3}\right] = \frac{1562}{15}\cos^2\theta$$

$$\frac{1562}{15}\int_0^{\pi/2}\cos^2\theta\,d\theta = \frac{1562}{15}\int_0^{\pi/2}\left(\frac{1}{2}+\frac{1}{2}\cos(2\theta)\right)d\theta$$

$$\frac{1562}{15}\left[\frac{1}{2}\theta+\frac{1}{4}\sin(2\theta)\right]_0^{\pi/2}$$

$$\frac{1562}{15}\left[\frac{1}{2}(\pi/2-0)+\frac{1}{4}(\underbrace{\sin\pi}_{0}-\underbrace{\sin 0}_{0})\right]$$

$$\boxed{\frac{781\pi}{30}}$$

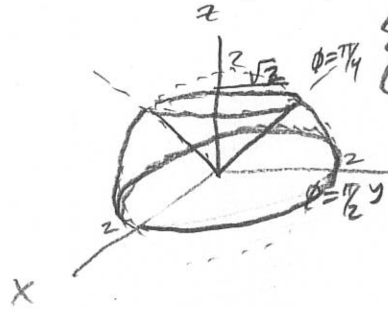
#10b. Find the volume of the solid that lies ~~above~~ within the sphere $x^2+y^2+z^2=4$, above the xy-plane, and below the cone $z=\sqrt{x^2+y^2}$. intersection:

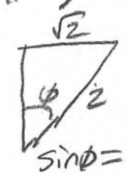
$\begin{cases} x^2+y^2+z^2=4 \\ z=\sqrt{x^2+y^2} \end{cases}$

$x^2+y^2+(x^2+y^2)=4$

$2x^2+2y^2=4$

$x^2+y^2=2$





$\sin\phi=\frac{\sqrt{2}}{2}$

$\phi=\pi/4$

$$\int_0^{2\pi}\int_{\pi/4}^{\pi/2}\int_0^2 (1)\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$\int_0^2\rho^2\sin\phi\,d\rho = \frac{1}{3}\sin\phi\left[\rho^3\right]_0^2$$

$$=\frac{1}{3}\sin\phi\left[(2)^3-(0)^3\right] = \frac{8}{3}\sin\phi$$

$$\int_{\pi/4}^{\pi/2}\frac{8}{3}\sin\phi\,d\phi = -\frac{8}{3}\left[\cos\phi\right]_{\pi/4}^{\pi/2}$$

$$=-\frac{8}{3}\left[\cos\pi/2-\cos\pi/4\right]=-\frac{8}{3}\left(0-\frac{\sqrt{2}}{2}\right)$$

$$=\frac{4\sqrt{2}}{3}$$

$$\int_0^{2\pi}\frac{4\sqrt{2}}{3}\,d\theta = \frac{4\sqrt{2}}{3}\left[\theta\right]_0^{2\pi}$$

$$=\boxed{\frac{8\sqrt{2}\pi}{3}}$$

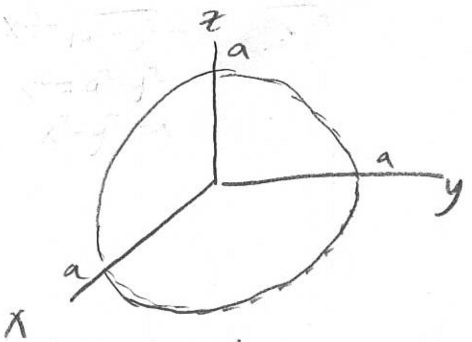
#11b. Evaluate the integral by changing to spherical coordinates:

$$\int_{-a}^{a}\int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}}\int_{-\sqrt{a^2-x^2-y^2}}^{\sqrt{a^2-x^2-y^2}}\left(x^2z+y^2z+z^3\right)dz\,dx\,dy$$

$z = -\sqrt{a^2-x^2-y^2}$ to $+\sqrt{a^2-x^2-y^2}$

$z^2 = a^2-x^2-y^2$

$x^2+y^2+z^2=a^2$ (sphere)



the entire sphere

Integrand:

$x^2z+y^2z+z^3$

$(x^2+y^2)z+z^3$

r^2z+z^3

$(r^2+z^2)z$

$[(\rho\sin\phi)^2+(\rho\cos\phi)^2]\rho\cos\phi$

$[\rho^2\sin^2\phi+\rho^2\cos^2\phi]\rho\cos\phi$

$\rho^2(\sin^2\phi+\cos^2\phi)\,\rho\cos\phi$

$\rho^2(1)\rho\cos\phi$

$\rho^3\cos\phi$

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{a}(\rho^3\cos\phi)\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$\int_0^a \rho^5\cos\phi\sin\phi\,d\rho = \frac{1}{6}\cos\phi\sin\phi[\rho^6]_0^a$

$= \frac{1}{6}\cos\phi\sin\phi[a^6-0^6] = \frac{a^6}{6}\cos\phi\sin\phi$

$\frac{a^6}{6}\int_0^{\pi}\cos\phi\sin\phi\,d\phi$ $\quad u=\sin\phi \quad \frac{du}{d\phi}=\cos\phi \quad \cos\phi\,d\phi=du$ $\quad \phi=0\Rightarrow u=0 \quad \phi=\pi\Rightarrow u=0$

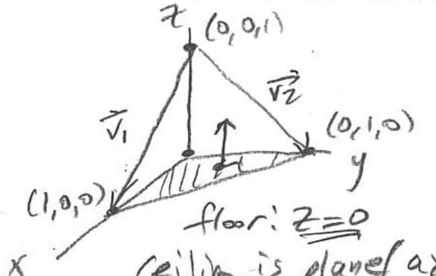
$\frac{a^6}{6}\int_0^0 u\,du = 0$

$\int_0^a 0\,d\theta = \boxed{0}$

Ch 15 Part 2 Test Review

***NOTE:* On all test review problems, set up the integral including limits of integration but <u>do not evaluate</u>.**

#1. $\iiint_T x^2\, dV$, where T is the solid tetrahedron with vertices (0,0,0), (1,0,0), (0,1,0), and (0,0,1).



floor: $\underline{\underline{z=0}}$

ceiling is plane ($ax+by+cz=\vec{n}\cdot\vec{r_0}$)

$\vec{v_1}=\langle 1-0,0-0,0-1\rangle=\langle 1,0,-1\rangle$

$\vec{v_2}=\langle 0-0,1-0,0-1\rangle=\langle 0,1,-1\rangle$

$\vec{n}=\vec{v_1}\times\vec{v_2}=\begin{vmatrix} + & - & + \\ 1 & 0 & -1 \\ 0 & 1 & -1\end{vmatrix}$

$=\langle 0+1,-(-1-0),1-0\rangle$

$=\langle 1,1,1\rangle \qquad \vec{r_0}=\langle 1,0,0\rangle$

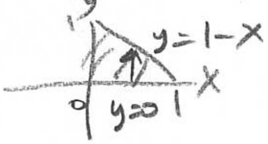
$ax+by+cz=\vec{n}\cdot\vec{r_0}$

$x+y+z=\langle 1,1,1\rangle\cdot\langle 1,0,0\rangle$

$x+y+z=(1)(1)+(1)(0)+(1)(0)$

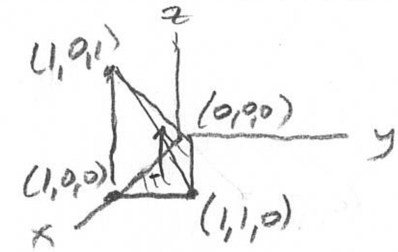
ceiling: $x+y+z=1$, $\underline{\underline{z=1-x-y}}$

projection:



$$\boxed{\int_0^1\int_0^{1-x}\int_0^{1-x-y} x^2\, dz\, dy\, dx}$$

#2. $\iiint_T xyz\, dV$, where T is the solid tetrahedron with vertices (0,0,0), (1,0,0), (1,1,0), and (1,0,1).



floor: $\underline{\underline{z=0}}$

ceiling is plane: $ax+by+cz=\vec{n}\cdot\vec{r_0}$

$\vec{v_1}=\langle 1-0,0-0,1-0\rangle=\langle 1,0,1\rangle$

$\vec{v_2}=\langle 1-0,1-0,0-0\rangle=\langle 1,1,0\rangle$

$\vec{n}=\vec{v_1}\times\vec{v_2}=\begin{vmatrix} + & - & + \\ 1 & 0 & 1 \\ 1 & 1 & 0\end{vmatrix}$

$=\langle 0-1,-(0-1),1-0\rangle=\langle -1,1,1\rangle$

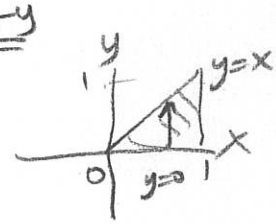
$\vec{r_0}=\langle 0,0,0\rangle \quad ax+by+cz=\vec{n}\cdot\vec{r_0}$

$-x+y+z=\langle -1,1,1\rangle\langle 0,0,0\rangle$

floor: $-x+y+z=(-1)(0)+(1)(0)+(1)(0)=0$

$\underline{\underline{z=x-y}}$

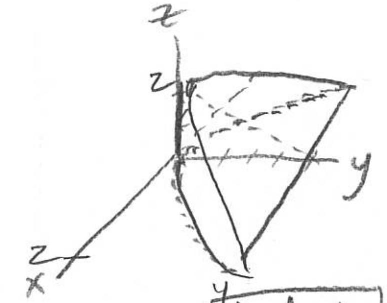
projection:



$$\boxed{\int_0^1\int_0^{x}\int_0^{x-y} xyz\, dz\, dy\, dx}$$

#3. Express the integral $\iiint_E f(x,y,z)\,dV$, as an iterated integral in all six integration orderings, where E is the solid bounded by the given surface:

$y = x^2, \quad z = 0, \quad y + 2z = 4$



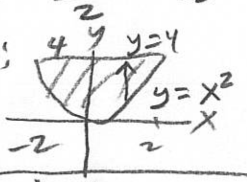
$\boxed{dz\,dy\,dx}$

floor $z=0$

ceiling: $y+2z=4$

$z = \frac{4-y}{2} = -\frac{1}{2}y+2$

projection:



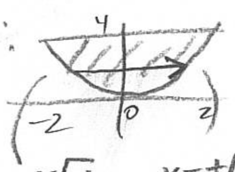
$$\boxed{\int_{-2}^{2}\int_{x^2}^{4}\int_{0}^{-\frac{1}{2}y+2} f(x,y,z)\,dz\,dy\,dx}$$

$\boxed{dz\,dx\,dy}$ same floor/ceiling:

floor: $z=0$

ceiling: $z=-\frac{1}{2}y+2$

projection:



$x=-\sqrt{y}$ $\quad x=+\sqrt{y}$

$$\boxed{\int_{0}^{4}\int_{-\sqrt{y}}^{+\sqrt{y}}\int_{0}^{-\frac{1}{2}y+2} f(x,y,z)\,dz\,dx\,dy}$$

$\boxed{dy\,dz\,dx}$

intersection: $\begin{cases} y=x^2 \\ y+2z=4 \end{cases}$

$x^2+2z=4$

$2z=4-x^2$

$z=2-\frac{1}{2}x^2$

projection:

floor: $y=x^2$,

ceiling: $y+2z=4$, $y=4-2z$

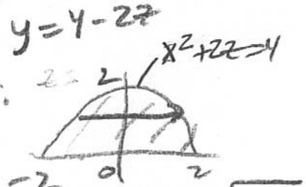
$$\boxed{\int_{-2}^{2}\int_{0}^{2-\frac{1}{2}x^2}\int_{x^2}^{4-2z} f(x,y,z)\,dy\,dz\,dx}$$

$\boxed{dy\,dx\,dz}$ same floor/ceiling:

floor: $y=x^2$

ceiling: $y=4-2z$

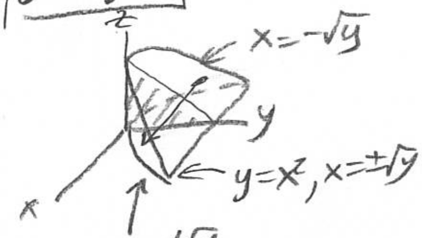
projection:



$x=-\sqrt{4-2z}$ $\quad x=+\sqrt{4-2z}$

$$\boxed{\int_{0}^{2}\int_{-\sqrt{4-2z}}^{+\sqrt{4-2z}}\int_{x^2}^{4-2z} f(x,y,z)\,dy\,dx\,dz}$$

$\boxed{dx\,dy\,dz}$

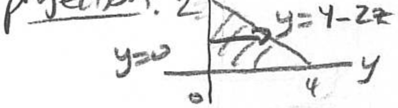


$x=+\sqrt{y}$

floor: $x=-\sqrt{y}$

ceiling: $x=+\sqrt{y}$

projection:



$$\boxed{\int_{0}^{2}\int_{0}^{4-2z}\int_{-\sqrt{y}}^{\sqrt{y}} f(x,y,z)\,dx\,dy\,dz}$$

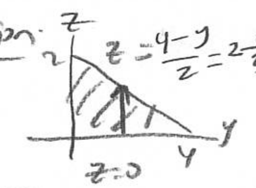
$\boxed{dx\,dz\,dy}$

same floor/ceiling:

floor: $-\sqrt{y}$

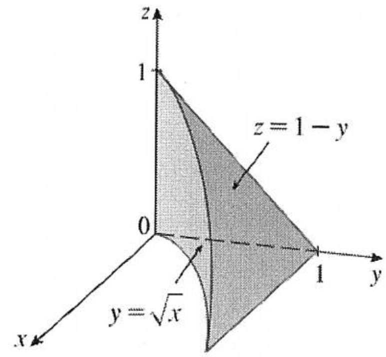
ceiling: $+\sqrt{y}$

projection:



$$\boxed{\int_{0}^{4}\int_{0}^{2-\frac{1}{2}y}\int_{-\sqrt{y}}^{\sqrt{y}} f(x,y,z)\,dx\,dz\,dy}$$

#4. The figure shows the region of integration for the integral $\int_0^1 \int_{\sqrt{x}}^1 \int_0^{1-y} f(x, y, z)\, dz\, dy\, dx$.



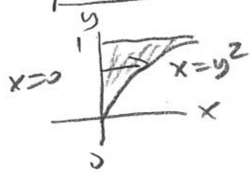
Rewrite this integral in the other five integration orders.

$\boxed{dz\,dx\,dy}$

← same floor/ceiling; floor: $z=0$

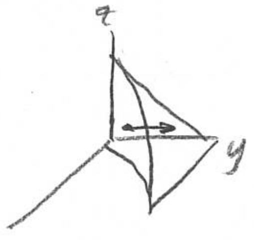
ceiling: $z = 1-y$

projection:



$$\boxed{\int_0^1 \int_0^{y^2} \int_0^{1-y} f(x,y,z)\, dz\, dx\, dy}$$

$\boxed{dy\,dx\,dz}$

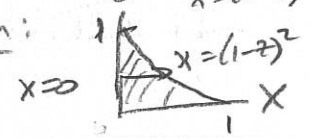


floor: $y = \sqrt{x}$

ceiling: $z = 1-y$, $y = 1-z$

intersection: $\begin{cases} y = \sqrt{x} \\ z = 1-y \end{cases}$ $z = 1-\sqrt{x}$, $\sqrt{x} = 1-z$, $x = (1-z)^2$

projection:



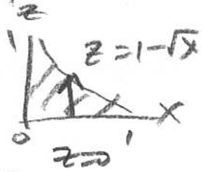
$$\boxed{\int_0^1 \int_0^{(1-z)^2} \int_{\sqrt{x}}^{1-z} f(x,y,z)\, dy\, dx\, dz}$$

$\boxed{dy\,dz\,dx}$ same floor/ceiling:

floor: $y = \sqrt{x}$

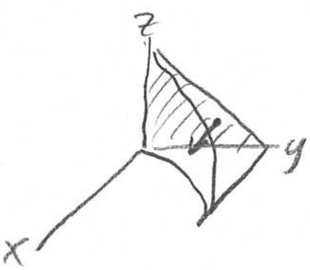
ceiling: $y = 1-z$

projection:



$$\boxed{\int_0^1 \int_0^{1-\sqrt{x}} \int_{\sqrt{x}}^{1-z} f(x,y,z)\, dy\, dz\, dx}$$

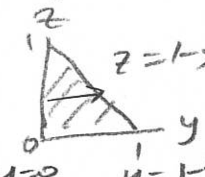
$\boxed{dx\,dy\,dz}$



floor: $x = 0$

ceiling: $y = \sqrt{x}$, $x = y^2$

projection:



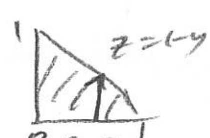
$$\boxed{\int_0^1 \int_0^{1-z} \int_0^{y^2} f(x,y,z)\, dx\, dy\, dz}$$

$\boxed{dx\,dz\,dy}$ same floor/ceiling:

floor: $x = 0$

ceiling: $x = y^2$

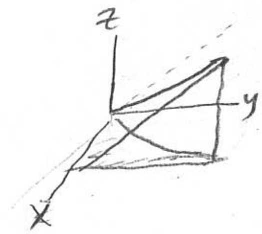
projection:



$$\boxed{\int_0^1 \int_0^{1-y} \int_0^{y^2} f(x,y,z)\, dx\, dz\, dy}$$

#5. Write five other iterated integrals that are equivalent to $\int_0^1\int_0^{x^2}\int_0^y f(x,y,z)\,dz\,dy\,dx$.

$z=0$ to $z=y$, $y=0$ to $y=x^2$, $x=0$ to $x=1$

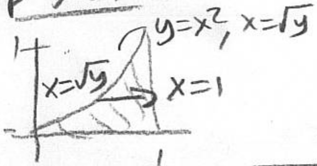


$\boxed{dz\,dx\,dy}$ same floor/ceiling:

floor: $z=0$

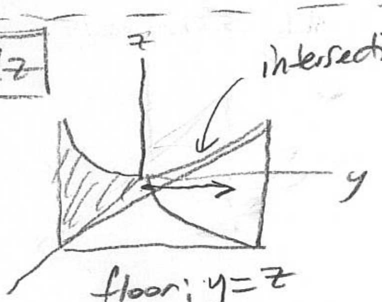
ceiling: $z=y$

projection:



$$\boxed{\int_0^1\int_{\sqrt{y}}^1\int_0^y f(x,y,z)\,dz\,dx\,dy}$$

$\boxed{dy\,dx\,dz}$



floor: $y=z$

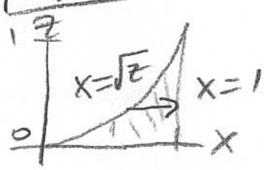
ceiling: $y=x^2$

intersection

$\begin{cases} z=y \\ y=x^2 \end{cases}$

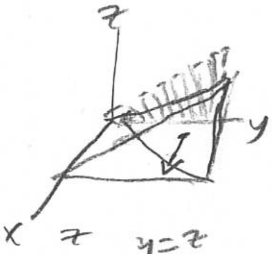
$z=x^2$

projection:



$$\boxed{\int_0^1\int_{\sqrt{z}}^1\int_z^{x^2} f(x,y,z)\,dy\,dx\,dz}$$

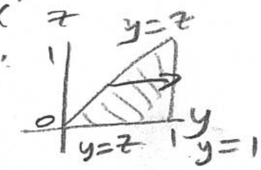
$\boxed{dx\,dy\,dz}$



floor: $y=x^2$, $x=\sqrt{y}$

ceiling: $x=1$

projection:

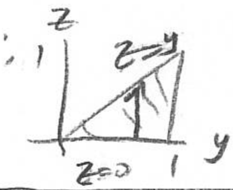


$$\boxed{\int_0^1\int_z^1\int_{\sqrt{y}}^1 f(x,y,z)\,dx\,dy\,dz}$$

$\boxed{dx\,dz\,dy}$ same floor/ceiling: floor: $x=\sqrt{y}$

ceiling: $x=1$

projection:



$$\boxed{\int_0^1\int_0^y\int_{\sqrt{y}}^1 f(x,y,z)\,dx\,dz\,dy}$$

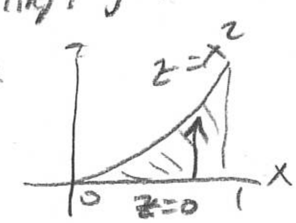
$\boxed{dy\,dz\,dx}$

same floor/ceiling:

floor: $y=z$

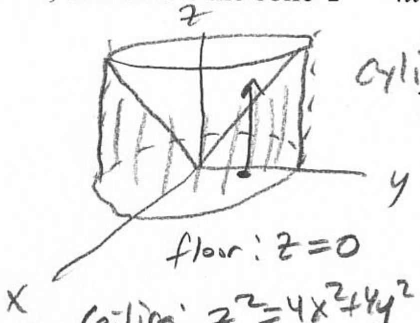
ceiling: $y=x^2$

projection:



$$\boxed{\int_0^1\int_0^{x^2}\int_z^{x^2} f(x,y,z)\,dy\,dz\,dx}$$

#6. Evaluate $\iiint_E x^2\, dV$, where E is the solid that lies within the cylinder $x^2 + y^2 = 1$, above the plane $z = 0$, and below the cone $z^2 = 4x^2 + 4y^2$.



cylindrical

floor: $z=0$

ceiling: $z^2=4x^2+4y^2$

$z=\sqrt{4x^2+4y^2}=\sqrt{4(x^2+y^2)}$

$=\sqrt{4r^2}=\sqrt{4}\sqrt{r^2}=2r$

projection:

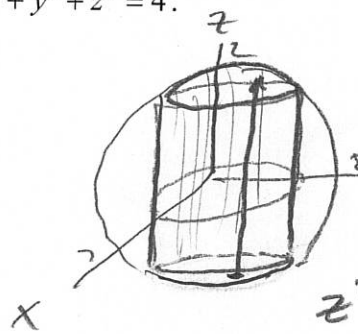
$r=1$

$\theta=0\to2\pi$

integrand: $x^2=(r\cos\theta)^2$

$$\boxed{\int_0^{2\pi}\int_0^{1}\int_0^{2r}(r\cos\theta)^2 r\,dz\,dr\,d\theta}$$

#7. Find the volume of the solid that lies within both the cylinder $x^2 + y^2 = 1$ and the sphere $x^2 + y^2 + z^2 = 4$.



cylindrical is the best

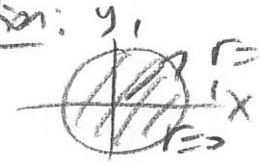
floor/ceiling both set by $x^2+y^2+z^2=4$

$z^2=4-(x^2+y^2)$

$z=\pm\sqrt{4-(x^2+y^2)}$

$z=\pm\sqrt{4-r^2}$

projection:

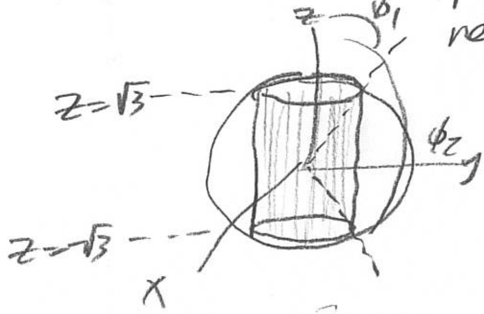


$r=1$ $\theta=0\to2\pi$

$$\boxed{\int_0^{2\pi}\int_0^{1}\int_{-\sqrt{4-r^2}}^{+\sqrt{4-r^2}}(1)\,r\,dz\,dr\,d\theta}$$

See why you shouldn't use spherical →

here is why you shouldn't use spherical for #7...



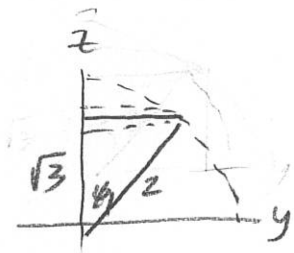
need to determine angles ϕ_1 & ϕ_2:

intersection: $\begin{cases} x^2+y^2=1 \\ x^2+y^2+z^2=4 \end{cases}$

$1+z^2=4$

$z^2=3$

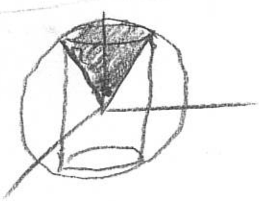
$z=\pm\sqrt{3}$



$\cos\phi=\frac{\sqrt{3}}{2}$, means $\phi_1=\frac{\pi}{6}$

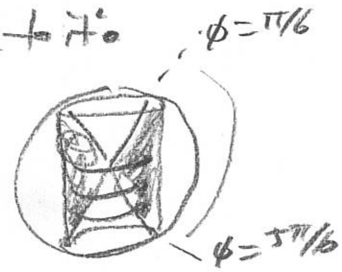
If you find $\int_0^{2\pi}\int_0^{\pi/6}\int_0^{2}(1)\rho^2\sin\phi\, d\rho\, d\phi\, d\theta$

you've found this volume:



(so you need to double this) but...

∴ you need to add the middle volume to it:



requires expressing the cylinder boundary in spherical:

$x^2+y^2=1$

$r^2=1$

$(\rho\sin\phi)^2=1$

$\rho^2\sin^2\phi=1$

$\rho^2=\frac{1}{\sin^2\phi}$

$\rho=\frac{1}{\sin\phi}$

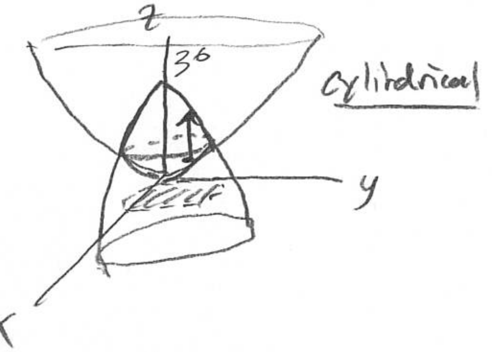
and adding this integral...

$$\int_0^{2\pi}\int_{\pi/6}^{5\pi/6}\int_0^{\frac{1}{\sin\phi}}(1)\rho^2\sin\phi\, d\rho\, d\phi\, d\theta$$

so spherical solution would be:

$$\boxed{2\int_0^{2\pi}\int_0^{\pi/6}\int_0^{2}(1)\rho^2\sin\phi\, d\rho\, d\phi\, d\theta+\int_0^{2\pi}\int_{\pi/6}^{\frac{5\pi}{6}}\int_0^{\frac{1}{\sin\phi}}(1)\rho^2\sin\phi\, d\rho\, d\phi\, d\theta}$$

#8. Find the volume of the region bounded by the paraboloids $z = x^2 + y^2$ and $z = 36 - 3x^2 - 3y^2$.



cylindrical

floor: $z = x^2+y^2$

ceiling: $z = 36-3x^2-3y^2$

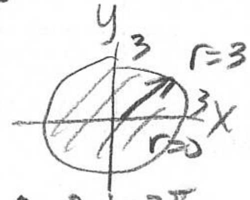
projection (determined by intersection)

$$\begin{cases} z = x^2+y^2 \\ z = 36-3x^2-3y^2 \end{cases}$$

$x^2+y^2 = 36-3x^2-3y^2$

$4x^2+4y^2 = 36$

$x^2+y^2 = 9$



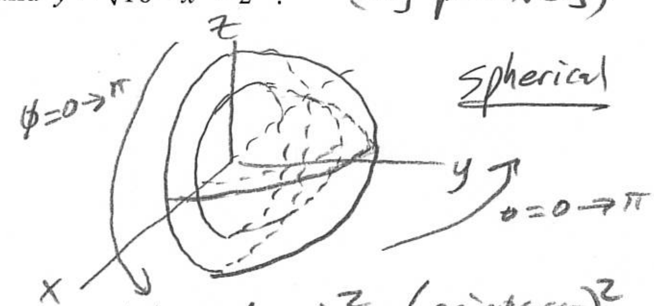
$\theta = 0$ to 2π

$$\int_0^{2\pi}\int_0^{3}\int_{x^2+y^2}^{36-3x^2-3y^2} (1)\, r\, dz\, dr\, d\theta$$

convert to cylindrical

$$\boxed{\int_0^{2\pi}\int_0^{3}\int_{r^2}^{36-3r^2} (1)\, r\, dz\, dr\, d\theta}$$

#9. Evaluate $\iiint_E x^2\, dV$, where E is bounded by the xz-plane and the hemispheres $y = \sqrt{9 - x^2 - z^2}$, and $y = \sqrt{16 - x^2 - z^2}$. (only positive y)

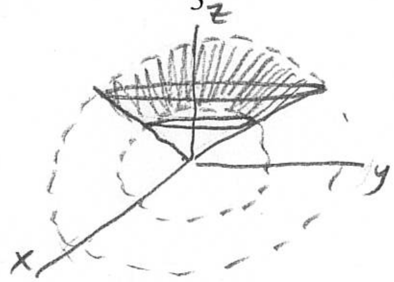


spherical

integrand: $x^2 = (\rho \sin\phi \cos\theta)^2$

$$\boxed{\int_0^{\pi}\int_0^{\pi}\int_3^{4} (\rho \sin\phi \cos\theta)^2 \rho^2 \sin\phi\, d\rho\, d\theta\, d\phi}$$

#10. Evaluate $\iiint\limits_E xyz\, dV$, where E lies between the spheres $\rho = 2$ and $\rho = 4$ and above the cone $\phi = \frac{\pi}{3}$.



Spherical

integrand: xyz

$(\rho\sin\phi\cos\theta)(\rho\sin\phi\sin\theta)(\rho\cos\phi)$

$\rho^3\sin^2\phi\cos\phi\sin\theta\cos\theta$

$$\boxed{\int_0^{2\pi}\int_0^{\pi/3}\int_2^{4}(\rho^3\sin^2\phi\cos\phi\sin\theta\cos\theta)\rho^2\sin\phi\, d\rho\, d\phi\, d\theta}$$

#11. Evaluate the integral by changing to spherical coordinates:

$$\int_0^{9\ (3)}\int_0^{\sqrt{9-x^2}}\int_0^{\sqrt{9-x^2-y^2}}\left(x^2z+y^2z+z^3\right)dz\, dx\, dy \quad dy\,dx$$

sphere w/ radius = 3

only positive x, y, z (1st octant)

Integrand: $x^2z+y^2z+z^3$

$(\rho\sin\phi\cos\theta)^2(\rho\cos\phi)+(\rho\sin\phi\sin\theta)^2(\rho\cos\phi)+(\rho\cos\phi)^3$

$\rho^3\sin^2\phi\cos^2\theta\cos\phi+\rho^3\sin^2\phi\sin^2\theta\cos\phi+\rho^3\cos^3\phi$

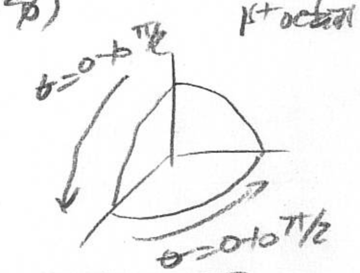
$\rho^3\sin^2\phi\cos\phi(\cos^2\theta+\sin^2\theta)+\rho^3\cos^3\phi$

$\rho^3\sin^2\phi\cos\phi\,(1)+\rho^3\cos^3\phi$

$\rho^3\sin^2\phi\cos\phi+\rho^3\cos^2\phi\cos\phi$

$\rho^3\cos\phi(\sin^2\phi+\cos^2\phi)$

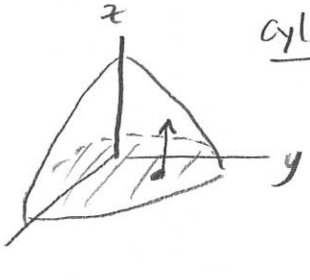
$\rho^3\cos\phi$



$$\boxed{\int_0^{\pi/2}\int_0^{\pi/2}\int_0^{3}(\rho^3\cos\phi)\rho^2\sin\phi\, d\rho\, d\phi\, d\theta}$$

#12. Evaluate $\iiint_E x^2y^2\,dV$, where E is bounded by the paraboloid $z = 1 - x^2 - y^2$ and the plane $z = 0$.

cylindrical



floor: $z=0$

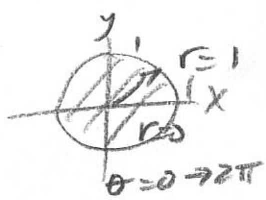
ceiling: $z = 1-x^2-y^2 = 1-r^2$

integrand: $x^2y^2 = (r\cos\theta)^2(r\sin\theta)^2$

$r^2\cos^2\theta\sin^2\theta$

projection:

(intersection):



$\begin{cases} z=0 \\ z=1-x^2-y^2 \end{cases}$

$0 = 1-x^2-y^2$

$x^2+y^2=1$

$$\boxed{\int_0^{2\pi}\int_0^{1}\int_0^{1-r^2} (r^2\cos^2\theta\sin^2\theta)\,r\,dz\,dr\,d\theta}$$

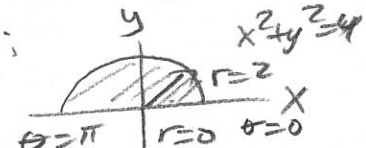
#13. Evaluate $\iiint_E yz\,dV$, where E lies above the plane $z = 0$, below the plane $z = y$, and inside the cylinder $x^2 + y^2 = 4$.

cylindrical

floor: $z=0$

ceiling: $z = y = r\sin\theta$

projection:

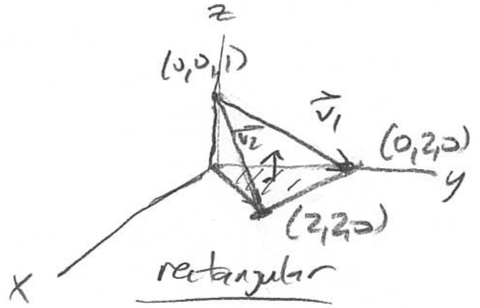


integrand: yz

$(r\sin\theta)z$

$$\boxed{\int_0^{\pi}\int_0^{2}\int_0^{r\sin\theta} (r\sin\theta\, z)\,r\,dz\,dr\,d\theta}$$

#14. Find the volume of the solid tetrahedron with vertices (0,0,0), (0,0,1), (0,2,0), and (2,2,0).



rectangular

floor: $z=0$

ceiling: plane $ax+by+cz=\vec{n}\cdot\vec{r_0}$

$\vec{v_1}=\langle 0-0,2-0,0-1\rangle=\langle 0,2,-1\rangle$

$\vec{v_2}=\langle 2-0,2-0,0-1\rangle=\langle 2,2,-1\rangle$

$$\vec{n}=\vec{v_1}\times\vec{v_2}=\begin{vmatrix} + & - & + \\ 0 & 2 & -1 \\ 2 & 2 & -1 \end{vmatrix}$$

$=\langle -2+2,-(0+2),0-4\rangle=\langle 0,-2,-4\rangle$

$\vec{r_0}=\langle 0,0,1\rangle$

$ax+by+cz=\vec{n}\cdot\vec{r_0}$

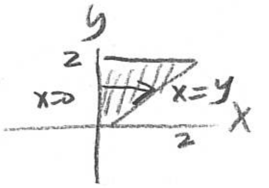
$-2y-4z=\langle 0,-2,-4\rangle\cdot\langle 0,0,1\rangle$

$-2y-4z=(0)(0)+(-2)(0)+(-4)(1)=-4$

$y+2z=2$, ceiling; $2z=2-y$

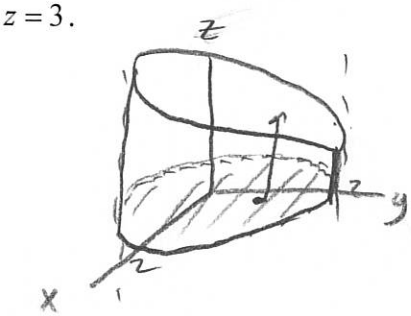
$z=1-\frac{1}{2}y$

projection:



$$\boxed{\int_0^2\int_0^y\int_0^{1-\frac{1}{2}y}(1)\,dz\,dx\,dy}$$

#15. Find the volume of the solid bounded by the cylinder $x^2+y^2=4$ and the planes $z=0$ and $y+z=3$.

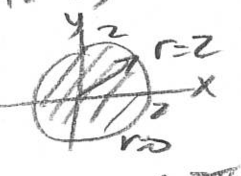


Cylindrical

floor: $z=0$

ceiling: $y+z=3$, $z=3-y=3-r\sin\theta$

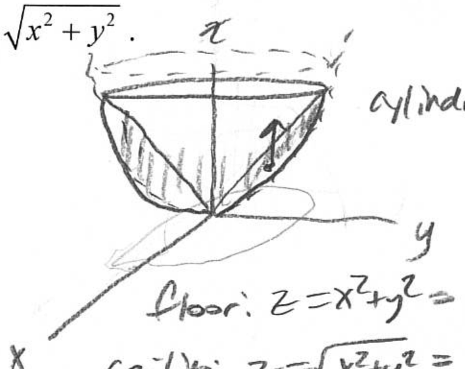
projection:



$\theta=0$ to 2π

$$\boxed{\int_0^{2\pi}\int_0^2\int_0^{3-r\sin\theta}(1)\,r\,dz\,dr\,d\theta}$$

#16. Find the volume of the solid above the paraboloid $z = x^2 + y^2$ and below the half-cone $z = \sqrt{x^2 + y^2}$.



cylindrical

floor: $z = x^2+y^2 = r^2$

ceiling: $z = \sqrt{x^2+y^2} = r$

projection (established by intersection)

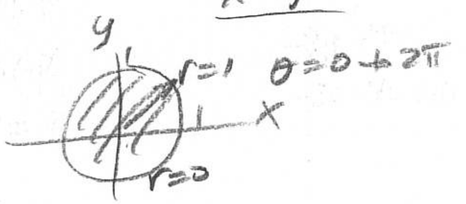
$$\begin{cases} z = x^2+y^2 \\ z = \sqrt{x^2+y^2} \end{cases} \quad x^2+y^2 = \sqrt{x^2+y^2} \quad (x^2+y^2)^2 = x^2+y^2$$

$(x^2+y^2)^2 - (x^2+y^2) = 0$

$(x^2+y^2)[(x^2+y^2)-1] = 0$

$x^2+y^2 \neq 0$ so $x^2+y^2-1=0$

$x^2+y^2=1$

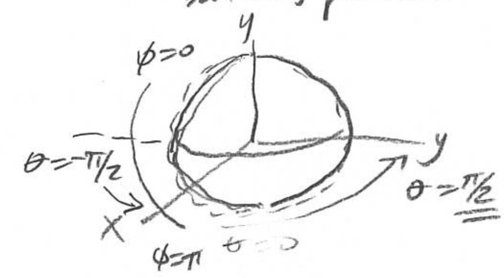


$$\boxed{\int_0^{2\pi}\int_0^1\int_{r^2}^{r} (1)\, r\, dz\, dr\, d\theta}$$

#17. Convert the integral to spherical coordinates:

$$\int_{-2}^{2}\int_{0}^{\sqrt{4-y^2}}\int_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}} y^2\sqrt{x^2+y^2+z^2}\; dz\, dx\, dy.$$

↑ sphere of radius 2 but only positive x



$\rho = 0$ to 2

Integrand: $y^2\sqrt{x^2+y^2+z^2}$

$(\rho\sin\phi\sin\theta)^2\sqrt{\rho^2}$

$\rho^2\sin^2\phi\sin^2\theta\,\rho$

$\rho^3\sin^2\phi\sin^2\theta$

$$\boxed{\int_{-\pi/2}^{\pi/2}\int_0^{\pi}\int_0^{2}\left(\rho^3\sin^2\phi\sin^2\theta\right)\rho^2\sin\phi\, d\rho\, d\phi\, d\theta}$$