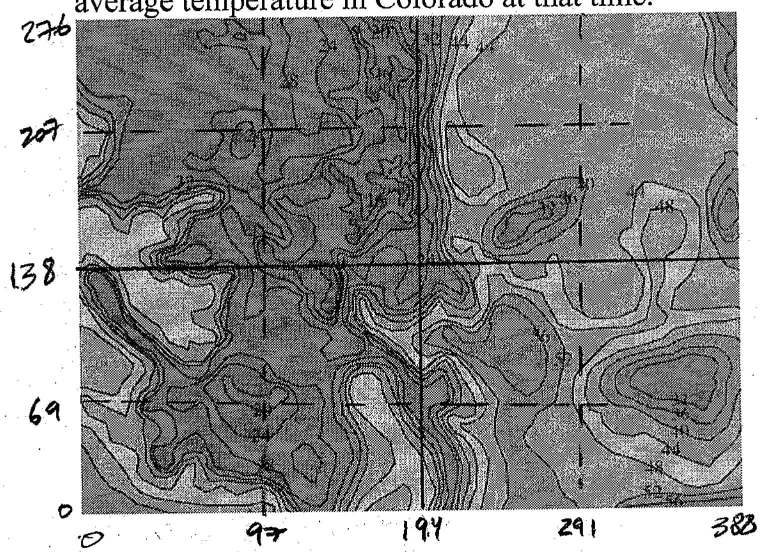
Calc III - Ch 15 Part 1 - Extra Practice

15.1 and 15.2

#1b. The contour map shows the temperature, in degrees Fahrenheit, at 4:00 PM on February, 26, 2007, in Colorado. (The state measures 388 mi east to west and 276 mi north to south.) Use the Midpoint Rule with $m = n = 2$ to estimate the average temperature in Colorado at that time.

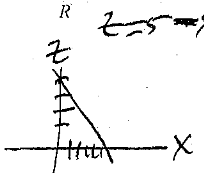


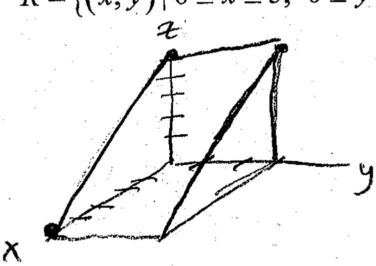
(x,y)	$f(x,y)$	$\cdot\ \Delta A$	$= V$
$(97,69)$	20	$\cdot\ 26772$	$= 535440$
$(97,207)$	24	$\cdot\ 26772$	$= 642528$
$(291,69)$	48	$\cdot\ 26772$	$= 1285056$
$(291,207)$	42	$\cdot\ 26772$	$= 1124424$
			3587440

$$T_{avg} = \frac{3587440}{\text{area}} = \frac{3587440}{(388)(276)} = \boxed{33.5°F}$$

#2b. Evaluate the double integral by first identifying it as the volume of a solid.
$\iint_R (5-x)\, dA, \quad R = \{(x,y) \mid 0 \le x \le 5,\ 0 \le y \le 3\}$.

$z = 5 - x$





$$V = \tfrac{1}{2}(\text{box}) = \tfrac{1}{2}(5)(5)(3) = \boxed{\tfrac{75}{2}}$$

#3b. Find $\int_0^5 f(x,y)\, dx$ and $\int_0^1 f(x,y)\, dy$

$f(x,y) = y + xe^y$.

$$\int_0^5 (y + xe^y)\,dx = y\int_0^5 1\,dx + e^y\int_0^5 x\,dx$$
$$= y\left[x\right]_0^5 + e^y\left[\tfrac{1}{2}x^2\right]_0^5$$
$$= y(5-0) + e^y\left(\tfrac{1}{2}\right)(5^2 - 0^2)$$
$$= \boxed{5y + \tfrac{25}{2}e^y}$$

$$\int_0^1 (y + xe^y)\,dy = \int_0^1 y\,dy + x\int_0^1 e^y\,dy$$
$$= \left[\tfrac{1}{2}y^2\right]_0^1 + x\left[e^y\right]_0^1$$
$$= \tfrac{1}{2}(1^2 - 0^2) + x(e^1 - e^0)$$
$$= \boxed{\tfrac{1}{2} + ex - x}$$

#4b. Evaluate $\int_0^1\int_1^2 (4x^3 - 9x^2y^2)\, dy\, dx$.

$$\int_1^2 (4x^3 - 9x^2y^2)\,dy = x^3\int_1^2 4\,dy - 9x^2\int_1^2 y^2\,dy$$
$$= x^3\left[4x\right]_1^2 - 9x^2\left(\tfrac{1}{3}y^3\right]_1^2$$
$$= x^3(4)(2-1) - 9x^2\left(\tfrac{1}{3}\right)(2^3 - 1^3)$$
$$= 4x^3 - 21x^2 \quad -17x$$

$$\int_0^1 (4x^3 - 21x^2)\,dx$$
$$\left[4\left(\tfrac{1}{4}x^4\right) - 21\tfrac{1}{3}x^3\right]_0^1$$
$$\left[x^4 - 7x^3\right]_0^1$$
$$(1^4 - 7(1)^3) - (0^4 - 7(0)^3)$$
$$1 - 7$$
$$\boxed{-6}$$

#5b. Evaluate $\int_{\pi/6}^{\pi/2}\int_{-1}^{5} x\cos y \, dx\, dy$.

(split if possible) :)

$\int_{-1}^{5} x\,dx \int_{\pi/6}^{\pi/2} \cos y\,dy$

$\left[\frac{1}{2}x^2\right]_{-1}^{5} \left[\sin y\right]_{\pi/6}^{\pi/2}$

$\left(\frac{1}{2}5^2 - \frac{1}{2}(-1)^2\right)\left(\sin \pi/2 - \sin \pi/6\right)$

$\left(\frac{25}{2} - \frac{1}{2}\right)\left(1 - \frac{1}{2}\right)$

$(12)\left(\frac{1}{2}\right) = \boxed{6}$

#6b. Evaluate $\int_0^1\int_0^1 xy\sqrt{x^2+y^2}\, dy\, dx$.

$x\int_0^1 y(x^2+y^2)^{1/2}dy$

$u = x^2+y^2$ $\quad y=0 \to u = x^2$

$\frac{du}{dy} = 2y$ $\quad y=1 \to u = x^2+1$

$y\,dy = \frac{1}{2}du$

$x\int_{x^2}^{x^2+1} u^{1/2}\left(\frac{1}{2}du\right) = \frac{1}{2}x\left[\frac{2}{3}u^{3/2}\right]_{x^2}^{x^2+1}$

$\frac{1}{2}x\left(\frac{2}{3}\right)\left[(x^2+1)^{3/2} - (x^2)^{3/2}\right] = \frac{1}{3}x(x^2+1)^{3/2} - \frac{1}{3}x^4$

$\int_0^1 \frac{1}{3}x(x^2+1)^{3/2}dx - \frac{1}{3}\int_0^1 x^4\,dx$

$u = x^2+1$ $\quad x=0 \to u=1$

$\frac{du}{dx} = 2x$ $\quad x=1 \to u=2$

$x\,dx = \frac{1}{2}du$

$\frac{1}{3}\frac{1}{2}\int_1^2 u^{3/2}du - \frac{1}{3}\int_0^1 x^4 dx$

$\frac{1}{6}\left[\frac{2}{5}u^{5/2}\right]_1^2 - \frac{1}{3}\left(\frac{1}{5}x^5\right]_0^1$

$\frac{1}{15}\left(2^{5/2} - 1^{5/2}\right) - \frac{1}{15}(1^5 - 0^5) = \boxed{0.2438}$

#6c. Evaluate $\int_0^2\int_0^{\pi} r\sin^2\theta\, d\theta\, dr$.

$\int_0^2 r\,dr \int_0^{\pi} \sin^2\theta\, d\theta$ $\quad \sin^2\theta = \frac{1}{2}(1-\cos(2\theta))$

$\int_0^2 r\,dr \int_0^{\pi} \frac{1}{2}(1-\cos(2\theta))d\theta$

$\left[\frac{1}{2}r^2\right]_0^2 \left[\frac{1}{2}\theta - \frac{1}{4}\sin(2\theta)\right]_0^{\pi}$

$\frac{1}{2}(2^2 - 0^2)\left[\left(\frac{1}{2}(\pi) - \frac{1}{4}\sin 2\pi\right) - \left(\frac{1}{2}(0) - \frac{1}{4}\sin 0\right)\right]$

$2\left(\frac{\pi}{2} - 0\right) - (0 - 0))$

$= \boxed{\pi}$

#7c. Evaluate the double integral:

$\iint_R \frac{xy^2}{x^2+1}\, dA, \quad R = \{(x,y) \mid 0 \le x \le 1, -3 \le y \le 3\}$

(split if possible, if not must do iteratively)

$\int_{-3}^{3} y^2 dy \cdot \int_0^1 \frac{x}{x^2+1}dx$

$u = x^2+1$ $\quad x=0 \to u=1$

$\frac{du}{dx} = 2x$ $\quad x=1 \to u=2$

$x\,dx = \frac{1}{2}du$

$\int_{-3}^{3} y^2 dy \left(\frac{1}{2}\int_1^2 \frac{1}{u}du\right)$

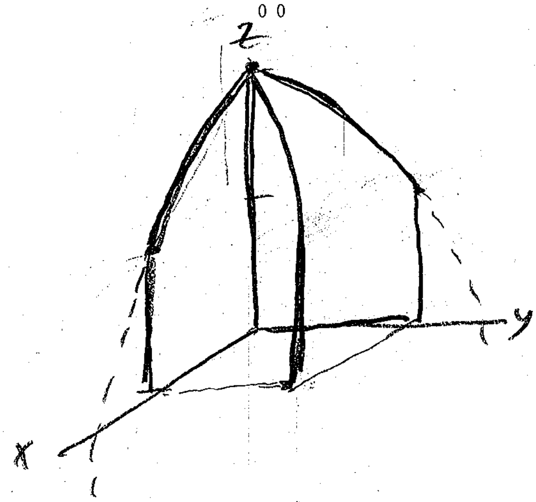
$\left[\frac{1}{3}y^3\right]_{-3}^{3}\left(\frac{1}{2}\right)\left[\ln|u|\right]_1^2$

$\frac{1}{3}(3^3 - (-3)^3)\frac{1}{2}(\ln 2 - \cancel{\ln 1}^{0})$

$\frac{1}{3}(27+27)\frac{1}{2}\ln 2$

$\boxed{9\ln 2}$

#8c. Sketch the solid whose volume is given by the iterated integral $\int_0^1\int_0^1 (2 - x^2 - y^2)\, dy\, dx$



($z = 2 - x^2 - y^2$ is an elliptical paraboloid)

#9c. Find the volume of the solid that lies under the elliptic paraboloid $\frac{x^2}{4}+\frac{y^2}{9}+z=1$ and above the rectangle $R=\{(x,y)\,|\,-1\le x\le 1,\ -2\le y\le 2\}$.

$$V=\int_{-2}^{2}\int_{-1}^{1}\left(1-\frac{x^2}{4}-\frac{y^2}{9}\right)dx\,dy$$

$$\int_{-1}^{1}1\,dx-\frac{1}{4}\int_{-1}^{1}x^2\,dx-y^2\int_{-1}^{1}\frac{1}{9}\,dx$$

$$\left[y\right]_{-1}^{1}-\frac{1}{4}\left[\frac{1}{3}x^3\right]_{-1}^{1}-y^2\left[\frac{1}{9}x\right]_{-1}^{1}$$

$$\left[1-(-1)\right]-\frac{1}{12}\left(1^3-(-1)^3\right)-y^2\left(\frac{1}{9}\right)\left(1-(-1)\right)$$

$$2-\frac{1}{12}(2)-\frac{1}{9}y^2(2)$$

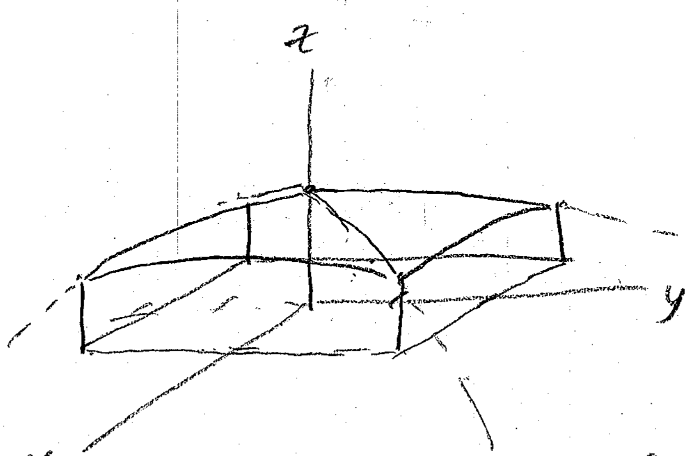
$$\frac{11}{6}-\frac{2}{9}y^2$$

$$\int_{-2}^{2}\left(\frac{11}{6}-\frac{2}{9}y^2\right)dy$$

$$\left[\frac{11}{6}y-\frac{2}{9}\cdot\frac{1}{3}y^3\right]_{-2}^{2}$$

$$\left(\frac{11}{6}(2)-\frac{2}{27}(2)^3\right)-\left(\frac{11}{6}(-2)-\frac{2}{27}(-2)^3\right)$$

$$\frac{11}{3}-\frac{16}{27}+\frac{11}{3}-\frac{16}{27}=\boxed{\frac{166}{27}}$$



(sketch not required)

#10. (hints)
Use 3D graphing software to picture…the surface is an elliptical paraboloid which starts at z = 2 and extends towards more positive z.

That means the volume of the solid between this surface and z =1 has a height which is just the surface minus 1: height $= 2+x^2+(y-2)^2-1$ so this is the integrand for the volume.

15.3

#1b. Evaluate $\int_0^{\pi/2}\int_0^{\cos\theta} e^{\sin\theta}\, dr\, d\theta$.

$$\int_0^{\cos\theta} e^{\sin\theta} dr = e^{\sin\theta}[r]_0^{\cos\theta}$$

no r
so this is a 'constant'

$$e^{\sin\theta}(\cos\theta - 0) = e^{\sin\theta}\cos\theta$$

$$\int_0^{\pi/2} e^{\sin\theta}\cos\theta\, d\theta \qquad u=\sin\theta$$

$\frac{du}{d\theta}=\cos\theta$, $\cos\theta\, d\theta = du$

$\theta=\pi/2 \to u=\sin\pi/2=1$

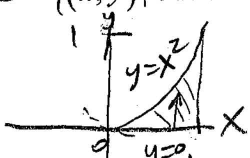
$\theta=0 \to u=\sin 0=0$

$$\int_0^1 e^u du$$

$$[e^u]_0^1 = e^1 - e^0 = \boxed{e-1}$$

#2b. Evaluate $\iint_D \frac{y}{x^5+1}\, dA$

$D=\{(x,y)\,|\,0\le x\le 1, 0\le y\le x^2\}$



$$\int_0^1\int_0^{x^2} \frac{y}{x^5+1} dy\,dx$$

$$\int_0^{x^2} \frac{y}{x^5+1} dy = \frac{1}{x^5+1}\left[\frac{1}{2}y^2\right]_0^{x^2}$$

$$\frac{1}{x^5+1}\left(\frac{1}{2}\right)\left[(x^2)^2-(0)^2\right] = \frac{1}{2}\frac{x^4}{x^5+1}$$

$$\frac{1}{2}\int_0^1 \frac{x^4}{x^5+1}dx \qquad u=x^5+1$$

$\frac{du}{dx}=5x^4$, $x^4=\frac{1}{5}du$

$x=0 \to u=(0)^5+1=1$

$x=1 \to u=(1)^5+1=2$

$$\frac{1}{2}\int_1^2 \frac{1}{u}\left(\frac{1}{5}\right)du$$

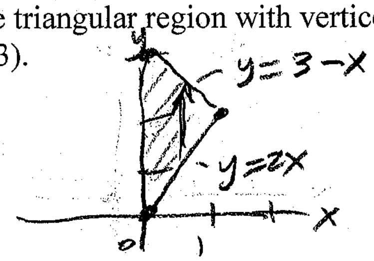
$$\frac{1}{10}[\ln|u|]_1^2$$

$$\frac{1}{10}(\ln 2 - \ln 1)$$

$$=\boxed{\frac{1}{10}\ln 2}$$

#3b. Evaluate $\iint_D 2xy\, dA$

D is the triangular region with vertices (0,0), (1,2), and (0,3).



$$\int_0^1\int_{2x}^{3-x} 2xy\, dy\, dx$$

$$\int_{2x}^{3-x} 2xy\, dy = x[y^2]_{2x}^{3-x}$$

$$= x\left((3-x)^2-(2x)^2\right) = x(9-6x+x^2-4x^2)$$

$$= 9x-6x^2-3x^3$$

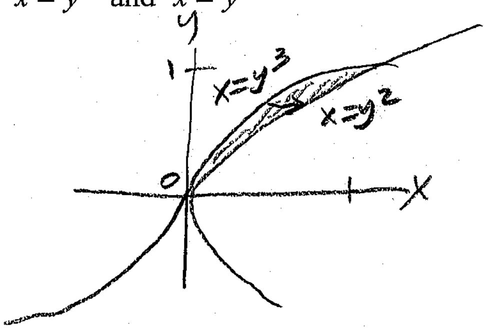
$$\int_0^1 (9x-6x^2-3x^3)dx$$

$$\left[\frac{9}{2}x^2 - 2x^3 - \frac{3}{4}x^4\right]_0^1$$

$$\left(\frac{9}{2}(1)^2-2(1)^3-\frac{3}{4}(1)^4\right)-(0-0-0)$$

$$=\boxed{\frac{7}{4}}$$

#4b. Find the volume of the solid under the surface $z = 2x + y^2$ and above the region bounded by $x = y^2$ and $x = y^3$



$$\int_0^1 \int_{y^3}^{y^2} (2x+y^2)\,dx\,dy$$

$$\int_{y^3}^{y^2} (2x+y^2)\,dx = \left[x^2 + y^2 x\right]_{y^3}^{y^2}$$

$$((y^2)^2 + y^2(y^2)) - ((y^3)^2 + y^2(y^3))$$

$$y^4 + y^4 - y^6 - y^5 = 2y^4 - y^6 - y^5$$

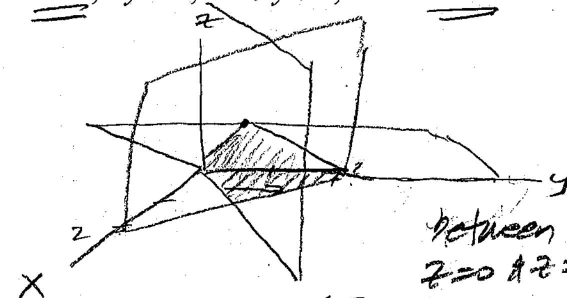
$$\int_0^1 (2y^4 - y^6 - y^5)\,dy$$

$$\left[\tfrac{2}{5}y^5 - \tfrac{1}{7}y^7 - \tfrac{1}{6}y^6\right]_0^1$$

$$\left(\tfrac{2}{5}(1)^5 - \tfrac{1}{7}(1)^7 - \tfrac{1}{6}(1)^6\right) - (0-0-0)$$

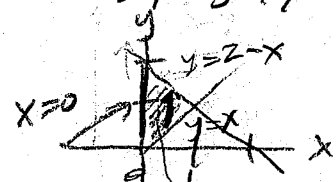
$$= \boxed{\frac{19}{210}}$$

#5b. Find the volume of the solid bounded by the planes $z = x$, $y = x$, $x + y = 2$, and $z = 0$.



between $z=0$ & $z=x$ so $z=x$ is 'ceiling'

project onto x-y plane: and graph $y=x$, $x+y=2$;



so this region

volume must include intersection of $z=0$ & $z=x$ which is $x=0$

$\begin{cases} x+y=2 \\ y=x \end{cases}$

$x+x=2$

$x=1$

$$\int_0^1 \int_x^{2-x} (x)\,dy\,dx$$

$$\int_x^{2-x} x\,dy = x\left[y\right]_x^{2-x}$$

$$x((2-x)-x) = x(2-2x) = 2x - 2x^2$$

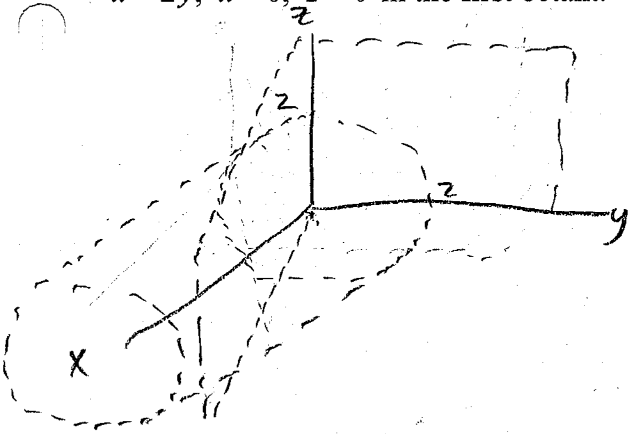
$$\int_0^1 (2x - 2x^2)\,dx$$

$$\left[x^2 - \tfrac{2}{3}x^3\right]_0^1$$

$$\left((1)^2 - \tfrac{2}{3}(1)^3\right) - (0-0)$$

$$= \boxed{\frac{1}{3}}$$

#6b. Find the volume of the solid bounded by the cylinder $y^2+z^2=4$ and the planes $x=2y$, $x=0$, $z=0$ in the first octant.

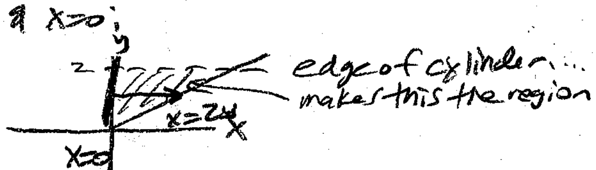


think of +z as "up"

floor: $z=0$

ceiling: cylinder; $z=\sqrt{4-y^2}$

project onto x-y, so graph $x=2y$ & $x=0$;



$$\int_0^2\int_0^{2y}\sqrt{4-y^2}\,dx\,dy$$

$$\int_0^{2y}\sqrt{4-y^2}\,dx=\sqrt{4-y^2}\,[x]_0^{2y}$$

$$\sqrt{4-y^2}(2y-0)=2y\sqrt{4-y^2}$$

$$\int_0^2 2y\sqrt{4-y^2}\,dy \qquad u=4-y^2,\quad \frac{du}{dy}=-2y,\quad 2y\,dy=-du$$

$$-\int_4^0 u^{1/2}\,du \qquad y=0\rightarrow u=4-0^2=4,\quad y=2\rightarrow u=4-2^2=0$$

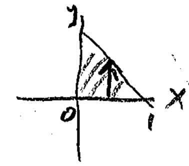
$$=\int_0^4 u^{1/2}\,du$$

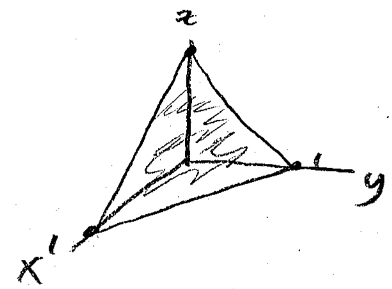
$$\left[\frac{2}{3}u^{3/2}\right]_0^4=\frac{2}{3}\left[4^{3/2}-0^{3/2}\right]$$

$$\frac{2(\sqrt{4})^3}{3}=\boxed{\frac{16}{3}}$$

#7b. Sketch the solid whose volume is given by the iterated integral $\int_0^1\int_0^{1-x}(1-x-y)\,dy\,dx$

$y=0$ to $y=1-x$, $x=0$ to $x=1$



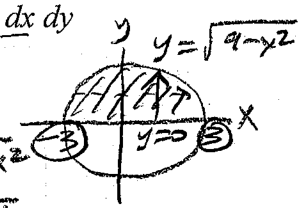


#8b. Sketch the region of integration and change the order of integration (write the new integral but do not evaluate) $\int_0^3\int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} f(x,y)\,dx\,dy$

$x=-\sqrt{9-y^2}$ to $+\sqrt{9-y^2}$, $y=0,3$

$x^2+y^2=9$

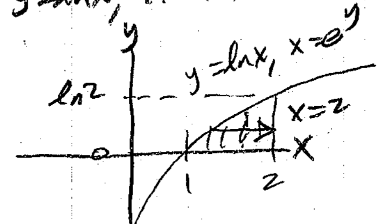
re-solve for y: $y=\pm\sqrt{9-x^2}$



$$\boxed{\int_{-3}^{3}\int_0^{\sqrt{9-x^2}} f(x,y)\,dy\,dx}$$

#8c. Sketch the region of integration and change the order of integration (write the new integral but do not evaluate) $\int_1^2\int_0^{\ln x} f(x,y)\,dy\,dx$

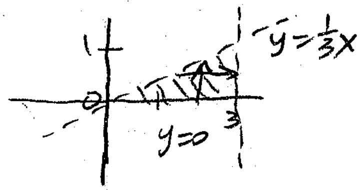
$y=0$ to $y=\ln x$, $x=1$ to $x=2$



$$\boxed{\int_0^{\ln 2}\int_{e^y}^{2} f(x,y)\,dx\,dy}$$

#9c. Evaluate the integral by reversing the order of integration $\int_0^1\int_{3y}^3 e^{x^2}\,dx\,dy$

$x=3y$ to $x=3$, $y=0$ to $y=1$



$$\int_0^3\int_0^{\frac{1}{3}x} \underbrace{e^{x^2}}_{\text{a 'constant' for } y}\,dy\,dx$$

$$\int_0^{1/3x} e^{x^2}\,dy = e^{x^2}\left[y\right]_0^{1/3x}$$

$$= e^{x^2}\left[1/3x - 0\right] = \frac{1}{3}xe^{x^2}$$

$$\frac{1}{3}\int_0^3 xe^{x^2}\,dx$$

$u=x^2$

$\frac{du}{dx}=2x$

$du=2x\,dx$

$x\,dx=\frac{1}{2}du$

$x=0\to u=0$

$x=3\to u=9$

$$\frac{1}{2}\frac{1}{3}\int_0^9 e^u\,du$$

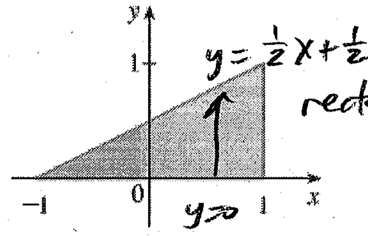
$$\frac{1}{6}\left[e^u\right]_0^9$$

$$\frac{1}{6}(e^9-e^0)$$

$$\boxed{\frac{1}{6}(e^9-1)}$$

15.4

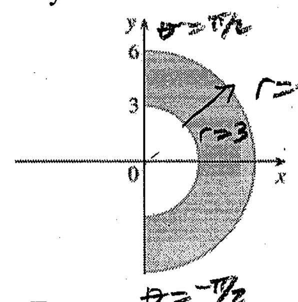
#1b. A region R is shown. Decide whether to use polar coordinates or rectangular coordinates and write $\iint_R f(x,y)\,dA$ as an iterated integral, where f is an arbitrary continuous function on R.



rectangular

$$\boxed{\int_{-1}^{1}\int_0^{\frac{1}{2}x+\frac{1}{2}} f(x,y)\,dy\,dx}$$

#2b. A region R is shown. Decide whether to use polar coordinates or rectangular coordinates and write $\iint_R f(x,y)\,dA$ as an iterated integral, where f is an arbitrary continuous function on R.



polar

$f(x,y)$

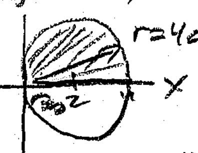
$\to f(r\cos\theta, r\sin\theta)$

$$\boxed{\int_{-\pi/2}^{\pi/2}\int_3^6 f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta}$$

#3b. Sketch the region whose area is given by the integral and evaluate the integral: $\int_0^{\pi/2}\int_0^{4\cos\theta} r\,dr\,d\theta$

$r=0$ to $r=4\cos\theta$, $\theta=0$ to $\theta=\pi/2$

(use calculator polar graph)



$\int_0^{4\cos\theta} r\,dr = \left[\frac12 r^2\right]_0^{4\cos\theta} = \frac12(4\cos\theta)^2 - 0$

$= 8\cos^2\theta$

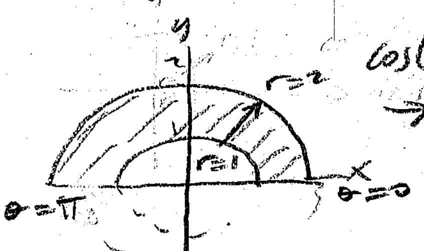
[identity: $\cos^2\theta = \frac12+\frac12\cos(2\theta)$]

$8\int_0^{\pi/2}\cos^2\theta\,d\theta = 8\int_0^{\pi/2}\left(\frac12+\frac12\cos(2\theta)\right)d\theta$

$= 8\left[\frac12\theta + \frac12\left(\frac12\sin(2\theta)\right)\right]_0^{\pi/2} = 8\left(\frac12\frac{\pi}{2}+\frac14\sin\pi\right) - 8\left(\frac12(0)+\frac14\sin 0\right)$

$= 2\pi + 0 - 0 - 0 = \boxed{2\pi}$

#4b. Evaluate the given integral by changing to polar coordinates: $\iint_R \cos\left(x^2+y^2\right) dA$ where R is the region that lies above the x-axis between the circles $x^2+y^2=1$ and $x^2+y^2=4$.



$\cos(x^2+y^2) \to \cos(r^2)$

$\int_0^{\pi}\int_1^2 \cos(r^2)\cdot r\,dr\,d\theta$

$\int_0^{\pi} d\theta \int_1^2 \cos(r^2)\cdot r\,dr$

$u=r^2$, $\frac{du}{dr}=2r$, $r\,dr=\frac12 du$, $r=1 \to u=1$, $r=2 \to u=4$

$\int_0^{\pi} d\theta\ \frac12\int_1^4 \cos u\,du$

$[\theta]_0^{\pi}\ \frac12[\sin u]_1^4$

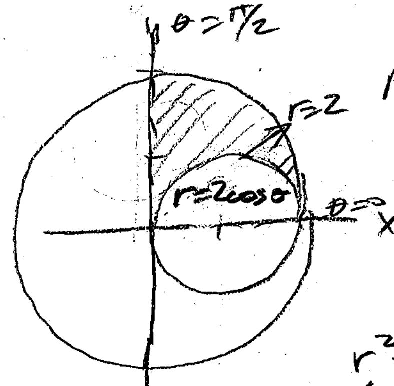
$(\pi-0)\left(\frac12\right)(\sin 4-\sin 1)$

$\boxed{\frac{\pi}{2}(\sin(4)-\sin(1))}$

#5b. Evaluate the given integral by changing to polar coordinates: $\iint_D x\,dA$ where D is the region in the first quadrant the lies between the circles $x^2+y^2=4$ and $x^2+y^2=2x$.

$x^2-2x+1+y^2=0+1$

$(x-1)^2+y^2=1$



polar:

$x^2+y^2=4$

$r^2=4$

$r=2$

$x^2+y^2=2x$

$r^2=2r\cos\theta$

$r^2-2r\cos\theta=0$

$r(r-2\cos\theta)=0$

$r=0$ (origin) or $r-2\cos\theta=0$, $r=2\cos\theta$

Integrand: $x=r\cos\theta$

$\int_0^{\pi/2}\int_{2\cos\theta}^{2} (r\cos\theta)\,r\,dr\,d\theta$

(can't split)

$\int_{2\cos\theta}^{2} r^2\cos\theta\,dr = \cos\theta\left[\frac13 r^3\right]_{2\cos\theta}^{2}$

$\cos\theta\left(\frac13(2)^3-\frac13(2\cos\theta)^3\right)$

$\frac83\cos\theta - \frac83\cos^4\theta$

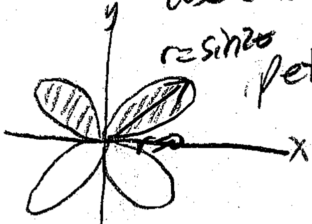
$\frac83\int_0^{\pi/2}(\cos\theta-\cos^4\theta)\,d\theta = 1.09587$ (math 9)

(we won't do manual integrations this hard on tests)

Could be done though... want a challenge? $\cos^4\theta=(\cos^2\theta)^2$

and $\cos^2\theta=\frac12+\frac12\cos2\theta$ (applied 3 times)

#6b. Use a double integral to find the area of the region: the part of the rose $r = \sin 2\theta$ which has positive y. (use calculator polar graph)



petals start, end when $r=0$:

$\sin(2\theta)=0$

Let $\phi = 2\theta$

$\sin\phi = 0$ when $\phi = 0\pi, 2\pi, \ldots$

$\to \phi = 2\theta = 0$ or $2\theta = \pi$

$\theta = 0,\ \theta = \pi/2$

is first petal

by symmetry, area w/ positive y is 2× 1st petal:

$$2\int_0^{\pi/2}\int_0^{\sin 2\theta} (1)\, r\,dr\,d\theta$$

for physical area, use 1 for integrand

$$\int_0^{\sin 2\theta} r\,dr = \left[\tfrac12 r^2\right]_0^{\sin 2\theta}$$

$$\tfrac12\left((\sin 2\theta)^2 - 0^2\right) = \tfrac12 \sin^2(2\theta)$$

[identity: $\sin^2 x = \frac12 - \frac12\cos(2x)$]

$$\tfrac12\left(\tfrac12 - \tfrac12\cos(2\theta)\right)$$

$$2\int_0^{\pi/2} \tfrac12\left(\tfrac12 - \tfrac12\cos(2\theta)\right) d\theta$$

$$\int_0^{\pi/2} \tfrac12\, d\theta - \tfrac12\int_0^{\pi/2}\cos(2\theta)\,d\theta$$

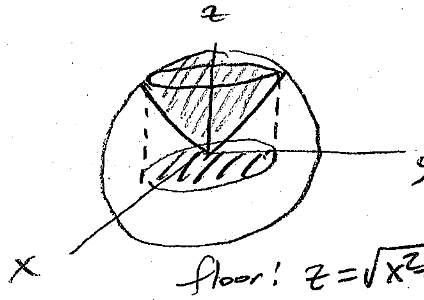
$$\tfrac12[\theta]_0^{\pi/2} - \tfrac12\left[\tfrac12\sin(2\theta)\right]_0^{\pi/2}$$

$$\tfrac12[\pi/2 - 0] - \tfrac14[\sin\pi - \sin 0]$$

$0 - 0$

$$= \boxed{\frac{\pi}{4}}$$

#7b. Use polar coordinates to find the volume of the given solid: above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = 1$.



floor: $z = \sqrt{x^2+y^2}$

ceiling: $x^2+y^2+z^2=1$

$z^2 = 1 - x^2 - y^2$

$z = \sqrt{1-x^2-y^2}$

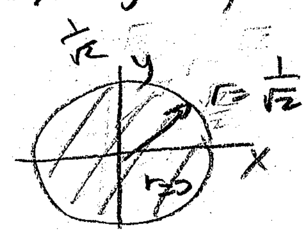
domain defined by intersection of sphere and cone:

$$\begin{cases} z = \sqrt{x^2+y^2} \\ x^2+y^2+z^2 = 1 \end{cases}$$

$x^2+y^2+\left(\sqrt{x^2+y^2}\right)^2 = 1$

$2x^2+2y^2 = 1,\ x^2+y^2 = \frac12$

radius $= \frac{1}{\sqrt2}$



height = ceiling − floor

$= \sqrt{1-x^2-y^2} - \sqrt{x^2+y^2}$

in polar: $\sqrt{1-r^2} - \sqrt{r^2} = \sqrt{1-r^2} - r$

$$\int_0^{2\pi} d\theta \int_0^{\frac{1}{\sqrt2}} \left(\sqrt{1-r^2} - r\right) r\,dr$$

$$\int_0^{\frac{1}{\sqrt2}} r\sqrt{1-r^2}\,dr - \int_0^{\frac{1}{\sqrt2}} r^2\,dr$$

$u = 1 - r^2$

$\frac{du}{dr} = -2r$

$du = -2r\,dr$

$r\,dr = -\frac12 du$

$r=0 \to u=1$

$r = \frac{1}{\sqrt2} \to u = \frac12$

$$-\tfrac12\int_1^{1/2} u^{1/2}\,du - \left[\tfrac13 r^3\right]_0^{\frac{1}{\sqrt2}}$$

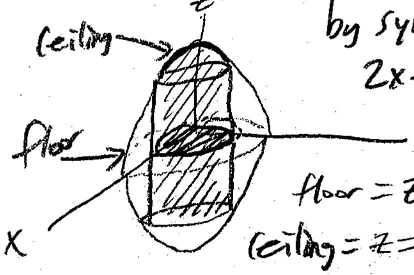
$$-\tfrac12\left[\tfrac23 u^{3/2}\right]_1^{1/2} - \left(\tfrac13\left(\tfrac{1}{2\sqrt2}\right) - 0\right)$$

$$-\tfrac13\left(\left(\tfrac12\right)^{3/2} - (1)^{3/2}\right) - \frac{1}{6\sqrt2}$$

$$-\tfrac13\left(\frac{1}{2\sqrt2} - 1\right) - \frac{1}{6\sqrt2} = \boxed{(2\pi)\left(\frac13 - \frac{1}{3\sqrt2}\right)}$$

#8b. Use polar coordinates to find the volume of the given solid: inside both the cylinder $x^2+y^2=4$ and the ellipsoid $4x^2+4y^2+z^2=64$.

$\frac{x^2}{16}+\frac{y^2}{16}+\frac{z^2}{64}$



by symmetry = 2x top half

floor = z = 0

ceiling = $z=\sqrt{64-4x^2-4y^2}$

domain region established by intersection of ellipsoid and cylinder

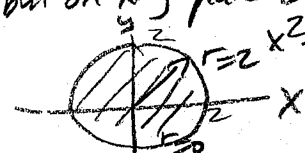
$\begin{cases} x^2+y^2=4 \\ 4x^2+4y^2+z^2=64 \end{cases} \rightarrow 4x^2+4y^2=16$

$16+z^2=64$

$z^2=48$

$z=\sqrt{48}$

but on x-y plane is the cylinder!



$x^2+y^2=4$

height = ceiling − floor = $\sqrt{64-4x^2-4y^2}-0$

in polar: $=\sqrt{64-4r^2}$

$$V=2\int_0^{2\pi}\int_0^2\sqrt{64-4r^2}\,r\,dr\,d\theta$$

$$=2\int_0^{2\pi}d\theta\int_0^2 r\sqrt{64-4r^2}\,dr$$

$u=64-4r^2$

$\frac{du}{dr}=-8r$

$r\,dr=-\frac{1}{8}du$

$r=0\rightarrow u=64$

$r=2\rightarrow u=48$

$$-\frac{1}{8}\int_{64}^{48}u^{1/2}\,du$$

$$2[\theta]_0^{2\pi}\left(-\frac{1}{8}\right)\left[\frac{2}{3}u^{3/2}\right]_{64}^{48}$$

$$=-\frac{1}{6}(2\pi-0)\left((48)^{3/2}-(64)^{3/2}\right)$$

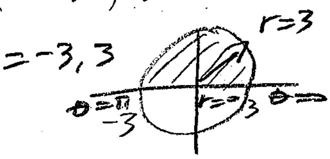
$$=\frac{\pi}{3}\left((\sqrt{64})^3-(\sqrt{48})^3\right)$$

$$=\boxed{\frac{\pi}{3}\left[512-48\sqrt{48}\right]}$$

#9b. Evaluate the iterated integral by converting to polar coordinates: $\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}}\sin\left(x^2+y^2\right)dy\,dx$

$y=0$ to $y=\sqrt{9-x^2}$ $(x^2+y^2=9)$

$x=-3,3$



$\sin(x^2+y^2)\rightarrow\sin(r^2)$

$$\int_0^{\pi}\int_0^3\sin(r^2)\,r\,dr\,d\theta$$

$$\int_0^{\pi}d\theta\int_0^3\sin(r^2)\,r\,dr$$

$u=r^2$ $\quad \frac{du}{dr}=2r$ $\quad r\,dr=\frac{1}{2}du$

$r=0\rightarrow u=0$

$r=3\rightarrow u=9$

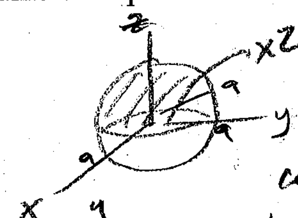
$$\frac{1}{2}\int_0^9\sin(u)\,du=\frac{1}{2}\left[-\cos u\right]_0^9=-\frac{1}{2}(\cos 9-\cos 0)$$

$$=-\frac{1}{2}\cos(9)+\frac{1}{2}$$

$$(\pi-0)\left(\frac{1}{2}-\frac{1}{2}\cos(9)\right)$$

$$\boxed{\frac{\pi}{2}\left[1-\cos(9)\right]}$$

Extra #10. Use polar coordinates to find the volume of a sphere of radius a.



$x^2+y^2+z^2=a^2$

floor: $z=0$

ceiling: $z=\sqrt{a^2-x^2-y^2}$

height = ceiling − floor

$\sqrt{a^2-x^2-y^2}-0$

$=\sqrt{a^2-r^2}$

$$2\int_0^{2\pi}\int_0^a\sqrt{a^2-r^2}\,r\,dr\,d\theta$$

$u=a^2-r^2$

$\frac{du}{dr}=-2r$

$r\,dr=-\frac{1}{2}du$

$r=0\rightarrow u=a^2$

$r=a\rightarrow u=0$

$$2\int_0^{2\pi}d\theta\left(-\frac{1}{2}\right)\int_{a^2}^{0}u^{1/2}\,du$$

$$2[\theta]_0^{2\pi}\left(-\frac{1}{2}\right)\left[\frac{2}{3}u^{3/2}\right]_{a^2}^{0}$$

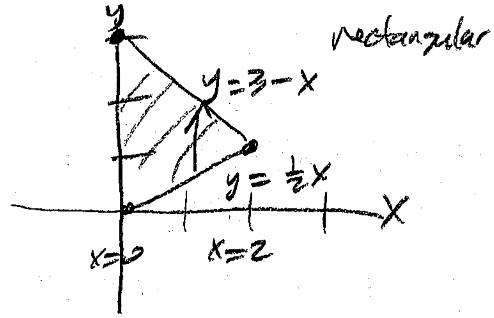
$$-\left[2\pi-0\right]\left[\frac{2}{3}(0)^{3/2}-\frac{2}{3}(a^2)^{3/2}\right]$$

$$-2\pi\left(0-\frac{2}{3}a^3\right)=\boxed{\frac{4}{3}\pi a^3}$$

(which is why $V_{sphere}=\frac{4}{3}\pi r^3$)

15.5

#1b. Find the mass and center of mass of the lamina that occupies the region D and has the density function ρ.
D is the triangular region with vertices (0,0), (2,1), (0,3); $\rho(x,y)=x+y$.



$$m=\iint \rho(x,y)\,dA=\int_0^2\int_{\frac{1}{2}x}^{3-x}(x+y)\,dy\,dx$$

$$\int_{\frac{1}{2}x}^{3-x}(x+y)\,dy=\left[xy+\tfrac{1}{2}y^2\right]_{\frac{1}{2}x}^{3-x}=\left(x(3-x)+\tfrac{1}{2}\underbrace{(3-x)^2}_{(9-6x+x^2)}\right)-\left(x(\tfrac{1}{2}x)+\tfrac{1}{2}(\tfrac{1}{2}x)^2\right)$$

$$=3x-x^2+\tfrac{9}{2}-3x+\tfrac{1}{2}x^2-\tfrac{1}{2}x^2-\tfrac{1}{8}x^2=-\tfrac{9}{8}x^2+\tfrac{9}{2}$$

$$\int_0^2\left(-\tfrac{9}{8}x^2+\tfrac{9}{2}\right)dx=\left[-\tfrac{9}{24}x^3+\tfrac{9}{2}x\right]_0^2=\left(-\tfrac{9}{24}(2)^3+\tfrac{9}{2}(2)\right)-(0+0)=\boxed{6}$$

$$M_x=\iint y\rho(x,y)\,dA=\int_0^2\int_{\frac{1}{2}x}^{3-x}y(x+y)\,dy\,dx$$

$$\int_{\frac{1}{2}x}^{3-x}(xy+y^2)\,dy=\left[\tfrac{1}{2}xy^2+\tfrac{1}{3}y^3\right]_{\frac{1}{2}x}^{3-x}=\left(\tfrac{1}{2}x\underbrace{(3-x)^2}_{9-6x+x^2}+\tfrac{1}{3}\underbrace{(3-x)^3}_{27-27x+9x^2-x^3}\right)-\left(\tfrac{1}{2}x(\tfrac{1}{2}x)^2+\tfrac{1}{3}(\tfrac{1}{2}x)^3\right)$$

$$=\tfrac{9}{2}x-3x^2+\tfrac{1}{2}x^3+9-9x+3x^2-\tfrac{1}{3}x^3-\tfrac{1}{8}x^3-\tfrac{1}{24}x^3=-\tfrac{9}{2}x+9$$

$$\int_0^2\left(-\tfrac{9}{2}x+9\right)dx=\left[-\tfrac{9}{4}x^2+9x\right]_0^2=\left(-\tfrac{9}{4}(2)^2+9(2)\right)-(0+0)=(9)$$

$$M_y=\iint x\rho(x,y)\,dA=\int_0^2\int_{\frac{1}{2}x}^{3-x}x(x+y)\,dy\,dx$$

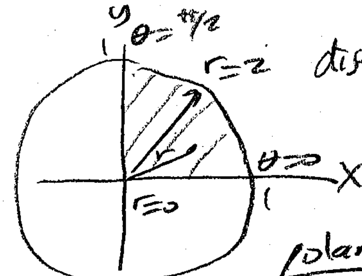
$$\int_{\frac{1}{2}x}^{3-x}(x^2+xy)\,dy=\left[x^2y+\tfrac{1}{2}xy^2\right]_{\frac{1}{2}x}^{3-x}=\left[x^2(3-x)+\tfrac{1}{2}x\underbrace{(3-x)^2}_{9-6x+x^2}\right]-\left[x^2(\tfrac{1}{2}x)+\tfrac{1}{2}x(\tfrac{1}{2}x)^2\right]$$

$$=3x^2-x^3+\tfrac{9}{2}x-3x^2+\tfrac{1}{2}x^3-\tfrac{1}{2}x^3-\tfrac{1}{8}x^3=-\tfrac{9}{8}x^3+\tfrac{9}{2}x$$

$$\int_0^2\left(-\tfrac{9}{8}x^3+\tfrac{9}{2}x\right)dx=\left[-\tfrac{9}{32}x^4+\tfrac{9}{4}x^2\right]_0^2=\left(-\tfrac{9}{32}(2)^4+\tfrac{9}{4}(2^2)\right)-(0+0)=\left(\tfrac{9}{2}\right)$$

$$\text{C.O.M}=(\bar{x},\bar{y})=\left(\frac{M_y}{m},\frac{M_x}{m}\right)=\left(\frac{9/2}{6},\frac{9}{6}\right)=\boxed{\left(\frac{3}{4},\frac{3}{2}\right)}$$

#2b. A lamina occupies the part of the disk $x^2 + y^2 \le 1$ in the first quadrant. Find its center of mass if the density at any point is proportional to *square* of its distance from the *origin*.



distance from origin $= r$

so $\rho(r,\theta) = k(r^2)$

polar

$$m = \iint \rho(r,\theta)\,dA = \int_0^{\pi/2}\int_0^1 (kr^2)\,r\,dr\,d\theta = k\int_0^{\pi/2} d\theta \int_0^1 r^3\,dr$$

$$= k\left[\theta\right]_0^{\pi/2}\left[\tfrac{1}{4}r^4\right]_0^1 = k\left(\tfrac{\pi}{2} - 0\right)\left(\tfrac{1}{4}(1)^4 - \tfrac{1}{4}(0)^4\right) = \left(\tfrac{\pi}{8}k\right)$$

$$M_x = \iint \underset{(r\sin\theta)}{y}\,\rho(r,\theta)\,dA = \int_0^{\pi/2}\int_0^1 (r\sin\theta)(kr^2)\,r\,dr\,d\theta = k\int_0^{\pi/2}\sin\theta\,d\theta \int_0^1 r^4\,dr$$

$$= k\left[-\cos\theta\right]_0^{\pi/2}\left[\tfrac{1}{5}r^5\right]_0^1 = k\left(-\overset{0}{\cos\tfrac{\pi}{2}} + \overset{1}{\cos 0}\right)\left[\tfrac{1}{5}(1)^5 - \tfrac{1}{5}(0)^5\right] = \left(\tfrac{1}{5}k\right)$$

$$M_y = \iint \underset{(r\cos\theta)}{x}\,\rho(r,\theta)\,dA = \int_0^{\pi/2}\int_0^1 (r\cos\theta)(kr^2)\,r\,dr\,d\theta = k\int_0^{\pi/2}\cos\theta\,d\theta \int_0^1 r^4\,dr$$

$$= k\left[\sin\theta\right]_0^{\pi/2}\left[\tfrac{1}{5}r^5\right]_0^1 = k\left(\overset{1}{\sin\tfrac{\pi}{2}} - \overset{0}{\sin 0}\right)\left(\tfrac{1}{5}(1)^5 - 0\right) = \left(\tfrac{1}{5}k\right)$$

$$\text{C.O.M.} = (\bar{x},\bar{y}) = \left(\frac{M_y}{m}, \frac{M_x}{m}\right) = \left(\frac{\frac{1}{5}k}{\frac{\pi}{8}k}, \frac{\frac{1}{5}k}{\frac{\pi}{8}k}\right) = \boxed{\left(\frac{8}{5\pi}, \frac{8}{5\pi}\right)}$$

#2c. A lamina occupies the region inside the circle $x^2+y^2=2y$ but outside the circle $x^2+y^2=1$. Find the center of mass if the density at any point is *inversely* proportional to its distance from the origin.

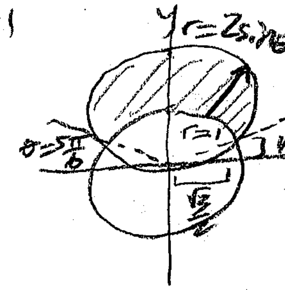
$x^2+y^2-2y+1=0+1$

$x^2+(y-1)^2=1$

polar

$\rho(r,\theta)=\frac{k}{r}$

intersections:



$\begin{cases} x^2+y^2=2y \\ x^2+y^2=1 \end{cases}$

$1=2y$

$y=\frac12$

$x^2+(\frac12)^2=1$

$x^2=\frac34$

$x=\frac{\sqrt3}{2}$

$x^2+y^2=2y$

$r^2=2r\sin\theta$

$r^2-2r\sin\theta=0$

$r(r-2\sin\theta)=0$

$r=2\sin\theta$

$$m=\iint \rho(r,\theta)\,dA=\int_{\pi/6}^{5\pi/6}\int_1^{2\sin\theta} k\frac{1}{r}\,r\,dr\,d\theta$$

$$\int_1^{2\sin\theta} k\,dr=\left[k\theta\right]_1^{2\sin\theta}=k(2\sin\theta-1)$$

$$k\int_{\pi/6}^{5\pi/6}(2\sin\theta-1)\,d\theta=k\left[-2\cos\theta-\theta\right]_{\pi/6}^{5\pi/6}=k\left\{\left(-2\cos\frac{5\pi}{6}-\frac{5\pi}{6}\right)-\left(-2\cos\frac{\pi}{6}-\frac{\pi}{6}\right)\right\}$$

$$=k\left[-2\left(-\frac{\sqrt3}{2}\right)-\frac{5\pi}{6}+2\left(\frac{\sqrt3}{2}\right)+\frac{\pi}{6}\right]=k\left(2\sqrt3-\frac{2\pi}{3}\right)$$

$$M_x=\iint \underbrace{y}_{r\sin\theta}\rho(r,\theta)\,dA=\int_{\pi/6}^{5\pi/6}\int_1^{2\sin\theta} r\sin\theta\left(k\frac1r\right)r\,dr\,d\theta=k\int_{\pi/6}^{5\pi/6}\int_1^{2\sin\theta} r\sin\theta\,dr\,d\theta$$

$$\int_1^{2\sin\theta} r\sin\theta\,dr=\frac12\sin\theta\left[r^2\right]_1^{2\sin\theta}=\frac12\sin\theta\left[4\sin^2\theta-1\right]=2\sin^3\theta-\frac12\sin\theta$$

$$k\left(2\int_{\pi/6}^{5\pi/6}\sin^3\theta\,d\theta-\frac12\int_{\pi/6}^{5\pi/6}\sin\theta\,d\theta\right)$$

$$=k\left(2\int_{\pi/6}^{5\pi/6}\sin^2\theta\sin\theta\,d\theta-\frac12\left[-\cos\theta\right]_{\pi/6}^{5\pi/6}\right)$$

$$=k\left(2\int_{\pi/6}^{5\pi/6}(1-\cos^2\theta)\sin\theta\,d\theta-\frac12\left(-\cos5\pi/6-(-\cos\pi/6)\right)\right)$$

$$=k\left(2\int_{\pi/6}^{5\pi/6}\sin\theta\,d\theta-2\int_{\pi/6}^{5\pi/6}\cos^2\theta\sin\theta\,d\theta-\frac12\left(-\left(-\frac{\sqrt3}{2}\right)+\left(\frac{\sqrt3}{2}\right)\right)\right)$$

$u=\cos\theta$

$\frac{du}{d\theta}=-\sin\theta$, $\sin\theta\,d\theta=-du$

$\theta=\pi/6\to u=\cos\pi/6=\frac{\sqrt3}{2}$

$\theta=5\pi/6\to u=\cos5\pi/6=-\frac{\sqrt3}{2}$

$$=k\left(2\left[-\cos\theta\right]_{\pi/6}^{5\pi/6}+2\int_{\frac{\sqrt3}{2}}^{-\frac{\sqrt3}{2}}u^2\,du-\frac{\sqrt3}{4}-\frac{\sqrt3}{4}\right)=k\left(2\left[-\cos5\pi/6-(-\cos\pi/6)\right]+\frac23\left[u^3\right]_{\sqrt3/2}^{-\sqrt3/2}-\frac{\sqrt3}{2}\right)$$

$$=k\left[2\left(-\left(-\frac{\sqrt3}{2}\right)+\frac{\sqrt3}{2}\right)+\frac23\left(-\frac{3\sqrt3}{8}-\frac{3\sqrt3}{8}\right)-\frac{\sqrt3}{2}\right]=k\left[\sqrt3\right]=\sqrt3k$$

2c (continued) ...

$$M_y = \iint \underbrace{x}_{r\cos\theta} \rho(r,\theta)\, dA = \int_{\pi/6}^{5\pi/6} \int_1^{2\sin\theta} r\cos\theta \left(k\frac{1}{r}\right) r\, dr\, d\theta = k \int_{\pi/6}^{5\pi/6} \int_1^{2\sin\theta} r\cos\theta\, dr\, d\theta$$

$$\int_1^{2\sin\theta} r\cos\theta\, dr = \frac{1}{2}\cos\theta \left[r^2\right]_1^{2\sin\theta} = \frac{1}{2}\cos\theta\left(4\sin^2\theta - 1\right) = 2\cos\theta\sin^2\theta - \frac{1}{2}\cos\theta$$

$$k\left(2\int_{\pi/6}^{5\pi/6} \underline{\cos\theta \sin^2\theta\, d\theta} - \frac{1}{2}\int_{\pi/6}^{5\pi/6} \cos\theta\, d\theta\right)$$

$u = \sin\theta$ $\quad \cos\theta\, d\theta = du$

$\frac{du}{d\theta} = \cos\theta$ $\quad \theta = \pi/6 \to u = \sin \pi/6 = \frac{1}{2}$

$\theta = 5\pi/6 \to u = \sin 5\pi/6 = \frac{1}{2}$

$$k\left(2 \underbrace{\int_{1/2}^{1/2} u^2\, du}_{(=0)} - \frac{1}{2}\int_{\pi/6}^{5\pi/6} \cos\theta\, d\theta\right) = k\left(2(0) - \frac{1}{2}\left[\sin\theta\right]_{\pi/6}^{5\pi/6}\right.$$

$$k\left(\frac{1}{2}\left(\sin 5\pi/6 - \sin \pi/6\right)\right) = k\,\frac{1}{2}\left(\frac{1}{2} - \frac{1}{2}\right) = (0)$$

$$\text{C.O.M.} = (\bar{x}, \bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right) = \left(\frac{0}{k\left(2\sqrt{3} - \frac{2\pi}{3}\right)}, \frac{\sqrt{3}k}{k\left(2\sqrt{3} - \frac{2\pi}{3}\right)}\right)$$

$$= \boxed{\left(0, \frac{\sqrt{3}}{2\sqrt{3} - \frac{2\pi}{3}}\right)}$$

... whew!

#3b. Xavier and Yolanda both has classes that end at noon and they agree to meet every day after class. They arrive at the coffee shop independently. Xavier's arrival time is X and Yolanda's arrival time is Y, where X and Y are measured in minutes after noon. The individual probability density functions are:

$$f_1(x) = \begin{cases} e^{-x} & x \geq 0 \\ 0 & x < 0 \end{cases} \qquad f_2(y) = \begin{cases} \frac{1}{50}y & 0 \leq y \leq 10 \\ 0 & otherwise \end{cases}$$

(Xavier arrives sometime after noon and is more likely to arrive promptly than late. Yolanda always arrives by 12:10 PM and is more like to arrive late than promptly.)

After Yolanda arrives, she'll wait for up to half an hour for Xavier, but he won't wait for her.

Find the probability that they meet.

$P(X \text{ AND } Y) = P(X \cap Y) = P(X) \cdot P(Y|X)$

but problem state "independently"

so $P(Y|X) = P(Y)$

& $P(X \cap Y) = P(X) \cdot P(Y)$

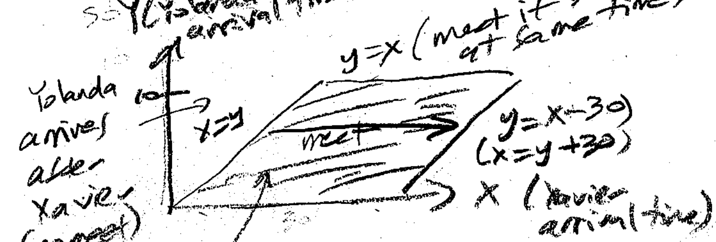
$\therefore$ joint probability is $P(X,Y) = f_1(x) \cdot f_2(x)$

$$P(x,y) = \begin{cases} e^{-x} \frac{1}{50} y & x \geq 0 \text{ \& } 0 \leq y \leq 10 \\ 0 & \text{otherwise} \end{cases}$$

will they meet?

"Yolanda waits up to $\frac{1}{2}$ hr for Xavier but he won't wait for her"

so



use rectangular to integrate PDF from $x=y$ to $x=y+30$ then $y=0$ to 10

$$P(\text{meet}) = \int_0^{10} \int_y^{y+30} \frac{1}{50} e^{-x} y \, dx\, dy$$

$$\frac{1}{50}\int_y^{y+30} y e^{-x} dx = \frac{1}{50} y \left[-e^{-x}\right]_y^{y+30}$$

$$\frac{1}{50} y\left[-e^{-(y+30)} - (-e^{-y})\right] = \frac{1}{50} y\left(e^{-y} - e^{-y}e^{-30}\right)$$

$$= \frac{1}{50} y e^{-y}(1 - e^{-30})$$

$$\frac{1}{50}(1-e^{-30}) \int_0^{10} y e^{-y} dy$$

by part) $u = y$ $\quad dv = e^{-y} dy$

$\frac{du}{dy} = 1$ $\quad \int dv = \int e^{-y} dy$

$du = dy$ $\quad v = -e^{-y}$

$uv - \int v\, du$

$-ye^{-y} + \int e^{-y} dy$

$-ye^{-y} - e^{-y}$

$$\frac{1}{50}(1-e^{-30})\left[-ye^{-y} - e^{-y}\right]_0^{10}$$

$$\frac{1}{50}(1-e^{-30})\left((-(10)e^{-10} - e^{-10}) - (-(0)e^{-0} - e^{-0})\right)$$

$$\frac{1}{50}(1)\left(-4.99399227 \cdot 10^{-4} - (0 - 1)\right)$$

$$\approx .01999 \approx \boxed{.02}$$ (2% probability they meet)

Ch 15 Part 1 Test Review

#1. Evaluate $\int_1^3\int_0^1 (1+4xy)\,dx\,dy$.

$\int_0^1 (1+4xy)\,dx = [x+2yx^2]_0^1$

$=((1)+2y(1)^2)-((0)+2y(0)^2)$

$=1+2y$

$\int_1^3 (1+2y)\,dy = [y+y^2]_1^3$

$=((3)+(3)^2)-((1)+(1)^2)$

$=3+9-1-1$

$=\boxed{10}$

#2. Evaluate $\int_1^4\int_1^2 \left(\frac{x}{y}+\frac{y}{x}\right) dy\,dx$.

$\int_1^2 (x\frac{1}{y}+\frac{1}{x}y)\,dy = [x\ln y+\frac{1}{2x}y^2]_1^2$

$(x\ln 2+\frac{1}{2x}(2)^2)-(\cancel{x\ln 1}^{0}+\frac{1}{2x}(1)^2)$

$x\ln 2+\frac{2}{x}-\frac{1}{2x}=x\ln 2+\frac{4-1}{2x}$

$\int_1^4 ((\ln 2)x+\frac{3}{2}\frac{1}{x})\,dx$

$=[\frac{\ln 2}{2}x^2+\frac{3}{2}\ln x]_1^4$

$=(\frac{\ln 2}{2}(4)^2+\frac{3}{2}\ln 4)-(\frac{\ln 2}{2}(1)^2+\frac{3}{2}\cancel{\ln 1}^{0})$

$=\boxed{8\ln 2+\frac{3}{2}\ln 4-\frac{1}{2}\ln 2}$

or

$\boxed{\frac{15}{2}\ln 2+\frac{3}{2}\ln 4}$ $=\frac{15}{2}\ln 2+\frac{3}{2}\ln(2^2)$

$=\frac{15}{2}\ln 2+3\ln 2$

or

$\boxed{\frac{21}{2}\ln 2}$ $=$

#3. Evaluate $\int_0^1\int_{2x}^2 (x-y)\,dy\,dx$.

$\int_{2x}^2 (x-y)\,dy = [xy-\frac{1}{2}y^2]_{2x}^2$

$(x(2)-\frac{1}{2}(2)^2)-(x(2x)-\frac{1}{2}(2x)^2)$

$2x-2-2x^2+2x^2=2x-2$

$\int_0^1 (2x-2)\,dx=[x^2-2x]_0^1$

$((1)^2-2(1))-(0-0)$

$1-2$

$=\boxed{-1}$

#4. Evaluate $\int_0^{\pi/2}\int_0^{\cos\theta} e^{\sin\theta}\,dr\,d\theta$.

$\int_0^{\cos\theta} \underbrace{e^{\sin\theta}}_{\text{constant for } r}\,dr = e^{\sin\theta}[r]_0^{\cos\theta}$

$e^{\sin\theta}(\cos\theta-0)=\cos\theta\, e^{\sin\theta}$

$\int_0^{\pi/2} \cos\theta\, e^{\sin\theta}\,d\theta$

$u=\sin\theta$

$\frac{du}{d\theta}=\cos\theta$

$\cos\theta\, d\theta=du$

$\theta=\frac{\pi}{2}\rightarrow u=\sin\frac{\pi}{2}=1$

$\theta=0\rightarrow u=\sin 0=0$

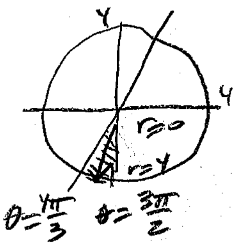
$\int_0^1 e^u\,du$

$[e^u]_0^1$

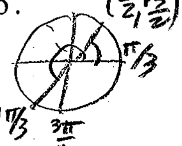
$=e^1-e^0=\boxed{e-1}$

#5. Set up the double-integral which evaluates the following, including limits of integration (but do not evaluate the integral). $\iint_D \sqrt{9x^2+9y^2}\, dA$

where D is the region in the third quadrant bounded by the $y=\sqrt{3}x$, $x=0$, and $x^2+y^2=16$. $\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)$



$\frac{y}{x}=\sqrt{3}=\frac{\sqrt{3}}{1}=\frac{\sqrt{3}/2}{1/2}$



$\theta=\frac{4\pi}{3}$ $\theta=\frac{3\pi}{2}$

integrand:

$\sqrt{9x^2+9y^2}=\sqrt{9}\sqrt{x^2+y^2}$
$=3\sqrt{r^2}=3r$

$$\boxed{\int_{4\pi/3}^{3\pi/2}\int_0^4 (3r)\,r\,dr\,d\theta}$$

#6. Set up the double-integral which evaluates the following, including limits of integration (but do not evaluate the integral). $\iint_D \cos\left(e^{2x-y}\right)dA$ where D is the triangular region with vertices $(1,1)$, $(4,1)$, and $(1,2)$.

line thru $(4,1)$ & $(1,2)$

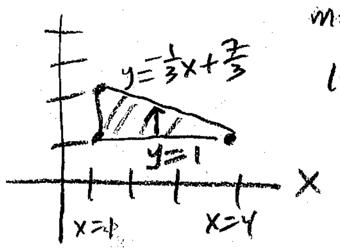
$m=\frac{2-1}{1-4}=\frac{1}{-3}$

$(y-2)=-\frac{1}{3}(x-1)$

$3y-6=-x+1$

$3y=-x+7$

$y=-\frac{1}{3}x+\frac{7}{3}$



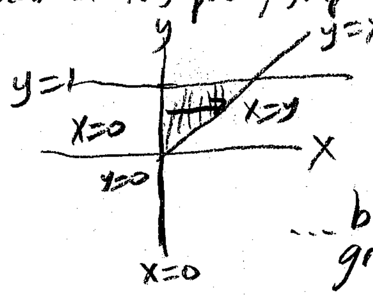
$$\boxed{\int_1^4\int_1^{-\frac{1}{3}x+\frac{7}{3}} \cos\left(e^{2x-y}\right)dy\,dx}$$

#7. Set up the double-integral which evaluates the following, including limits of integration (but do not evaluate the integral).

The volume of the solid enclosed by the paraboloid $z=x^2+3y^2$ and the planes $x=0$, $y=1$, $y=x$, and $z=0$.

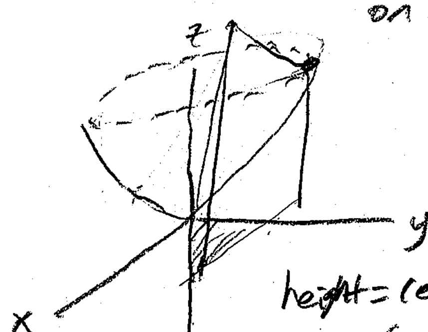
floor: $z=0$, ceiling: $z=x^2+3y^2$

domain: x-y plane, graph other curves...



← I strongly recommend drawing this...

... but graphs not req'd on test



height = ceiling − floor
$=(x^2+3y^2)-(0)$

$$\boxed{\int_0^1\int_0^y (x^2+3y^2)\,dx\,dy}$$

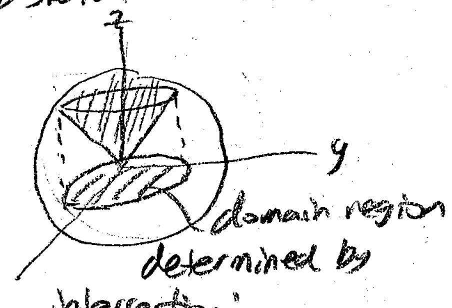
#8. Set up the double-integral which evaluates the following, including limits of integration (but do not evaluate the integral).

The volume of the solid formed by the region above the cone $z=\sqrt{x^2+y^2}$ and below the sphere $x^2+y^2+z^2=1$.

floor: $z=\sqrt{x^2+y^2}$

Ceiling: $x^2+y^2+z^2=1$, $z=\sqrt{1-x^2-y^2}$

need a 3D sketch to see domain:



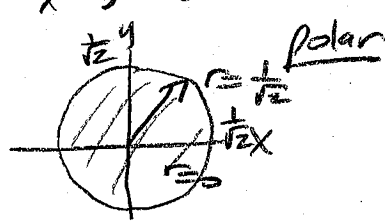
domain region determined by intersection:

$$\begin{cases} z=\sqrt{x^2+y^2} \\ x^2+y^2+z^2=1 \end{cases}$$

$$x^2+y^2+\left(\sqrt{x^2+y^2}\right)^2=1$$

$$2x^2+2y^2=1$$

$$x^2+y^2=\tfrac{1}{2} \quad \left(\text{radius}=\tfrac{1}{\sqrt{2}}\right)$$



polar

height = ceiling − floor

$$=\sqrt{1-x^2-y^2}-\sqrt{x^2+y^2}$$

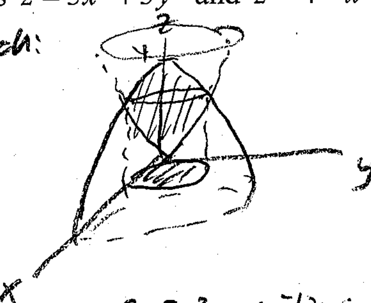
$$=\sqrt{1-r^2}-\sqrt{r^2}=\sqrt{1-r^2}-r$$

$$\boxed{\int_0^{2\pi}\int_0^{\frac{1}{\sqrt{2}}}\left(\sqrt{1-r^2}-r\right)r\,dr\,d\theta}$$

#9. Set up the double-integral which evaluates the following, including limits of integration (but do not evaluate the integral).

The volume of the solid bounded by the paraboloids $z=3x^2+3y^2$ and $z=4-x^2-y^2$.

3D sketch:

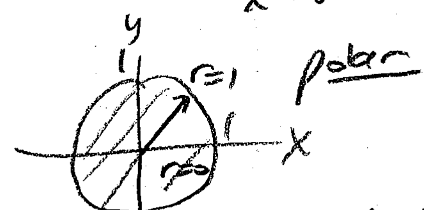


floor: $z=3x^2+3y^2$ ceiling: $z=4-x^2-y^2$

height: $(4-x^2-y^2)-(3x^2+3y^2)$

domain set by intersection:

$$\begin{cases} z=3x^2+3y^2 \\ z=4-x^2-y^2 \end{cases} \qquad \begin{aligned} 3x^2+3y^2&=4-x^2-y^2 \\ 4x^2+4y^2&=4 \\ x^2+y^2&=1 \quad (\text{radius}=1) \end{aligned}$$



polar

$$\text{height}=(4-x^2-y^2)-(3)(x^2+y^2)$$

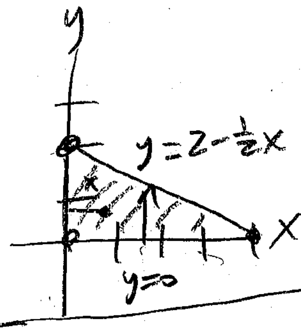
$$4-r^2-3r^2$$

$$4-4r^2$$

$$\boxed{\int_0^{2\pi}\int_0^{1}(4-4r^2)\,r\,dr\,d\theta}$$

For #10, #11, and #12: Find the mass and center of mass for the following lamina. (Set up the integrals including limits of integration, but <u>do not evaluate the integrals</u>. For center of mass, show the formula for computing it using symbols from your earlier work).

#10. The triangular region with vertices (0,0), (4,0), and (0,2) if density is proportional to three times the distance a point is from the y-axis.



$\rho(x,y) = k(3x)$

$= 3kx$

$$\boxed{m = \iint \rho(x,y)\,dA = \int_0^4 \int_0^{2-\frac{1}{2}x} (3kx)\,dy\,dx}$$

$$\boxed{M_x = \iint y\rho(x,y)\,dA = \int_0^4 \int_0^{2-\frac{1}{2}x} y(3kx)\,dy\,dx}$$

$$\boxed{M_y = \iint x\rho(x,y)\,dA = \int_0^4 \int_0^{2-\frac{1}{2}x} x(3kx)\,dy\,dx}$$

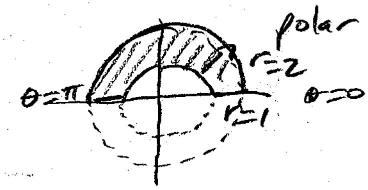
$$C.O.M = (\bar{x},\bar{y}) = \boxed{\left(\frac{M_y}{m}, \frac{M_x}{x}\right)}$$

#11. The boundary of the lamina consists of the semicircles $y=\sqrt{1-x^2}$ and $y=\sqrt{4-x^2}$ together with the portions of the x-axis that join them, if the density at any point is proportional to its distance from the origin.

$y=\sqrt{1-x^2}$

$y^2=1-x^2$

$x^2+y^2=1$



$\rho(r,\theta) = kr$

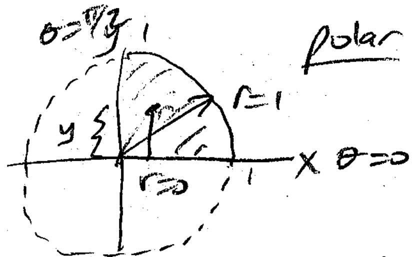
$$\boxed{m = \iint \rho(r,\theta)\,dA = \int_0^{\pi} \int_1^2 (kr)\,r\,dr\,d\theta}$$

$$\boxed{M_x = \iint y\rho(r,\theta)\,dA = \int_0^{\pi} \int_1^2 (r\sin\theta)(kr)\,r\,dr\,d\theta}$$

$$\boxed{M_y = \iint x\rho(r,\theta)\,dA = \int_0^{\pi} \int_1^2 (r\cos\theta)(kr)\,r\,dr\,d\theta}$$

$$C.O.M = (\bar{x},\bar{y}) = \boxed{\left(\frac{M_y}{m}, \frac{M_x}{m}\right)}$$

#12. The part of the disk $x^2 + y^2 \le 1$ in the first quadrant, if the density at any point is proportional to twice the distance from the x-axis.



$$\rho(r,\theta) = 2ky = 2k(r\sin\theta)$$

$$\boxed{m = \iint \rho(r,\theta)\,dA = \int_0^{\pi/2}\int_0^1 (2kr\sin\theta)\,r\,dr\,d\theta}$$

$$\boxed{M_x = \iint y\rho(r,\theta)\,dA = \int_0^{\pi/2}\int_0^1 (r\sin\theta)(2kr\sin\theta)\,r\,dr\,d\theta}$$

$$\boxed{M_y = \iint x\rho(r,\theta)\,dA = \int_0^{\pi/2}\int_0^1 (r\cos\theta)(2kr\sin\theta)\,r\,dr\,d\theta}$$

$$\text{C.O.M} = (\bar{x},\bar{y}) = \boxed{\left(\frac{M_y}{m}, \frac{M_x}{m}\right)}$$