Calc III - Ch 14 Part 2 - Extra Practice

14.6

#1b. Find the directional derivative of f at the given point in the direction indicated by the angle θ

$f(x,y)=x^2y^3-y^4,\quad (2,1),\quad \theta=\frac{\pi}{4}$

$\vec{u}=\langle \cos\frac{\pi}{4}, \sin\frac{\pi}{4}\rangle=\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\rangle=\frac{\sqrt{2}}{2}\langle 1,1\rangle$

$\nabla f=\langle 2xy^3, 3x^2y^2-4y^3\rangle\Big|_{(2,1)}$

$=\langle 2(2)(1)^3, 3(2)^2(1)^2-4(1)^3\rangle=\langle 4,8\rangle$

$D_uf=\nabla f\cdot\vec{u}=\langle 4,8\rangle\cdot\frac{\sqrt{2}}{2}\langle 1,1\rangle$

$=\frac{\sqrt{2}}{2}[(4)(1)+(8)(1)]$

$=\boxed{6\sqrt{2}}$

#2b. (i) Find the gradient of f.
(ii) Evaluate the gradient at the point P.
(iii) Find the rate of change of f at P in the direction of the vector $\vec{u}$.

$f(x,y,z)=xe^{2yz},\quad P(3,0,2),\quad \vec{u}=\left\langle \frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right\rangle$

(i) (ii) $\frac{\partial f}{\partial x}=e^{2yz}=e^{2(0)(2)}=1$

$\frac{\partial f}{\partial y}=xe^{2yz}(2z)=(3)e^{2(0)(2)}2(2)=12$

$\frac{\partial f}{\partial z}=xe^{2yz}(2y)=(3)e^{2(0)(2)}2(0)=0$

(i) $\boxed{\nabla f=\langle e^{2yz}, 2xze^{2yz}, 2xye^{2yz}\rangle}$

(ii) $\boxed{\nabla f\big|_{(3,0,2)}=\langle 1, 12, 0\rangle}$

(iii) $\vec{u}$ is a unit vector, so

$D_uf=\nabla f\cdot\vec{u}$

$=\langle 1,12,0\rangle\cdot\frac{1}{3}\langle 2,-2,1\rangle$

$=\frac{1}{3}((1)(2)+(12)(-2)+(0)(1))$

$=\boxed{\frac{-22}{3}}$

#3b. Find the directional derivative of the function at the given point in the direction of the vector $\vec{v}$

$g(p,q)=p^4-p^2q^3,\quad (2,1),\quad \vec{v}=\langle 1, 3\rangle$

$\vec{u}=\frac{\langle 1,3\rangle}{\sqrt{1^2+3^2}}=\frac{1}{\sqrt{10}}\langle 1,3\rangle$

$\nabla g=\langle 4p^3-2pq^3, -3p^2q^2\rangle\Big|_{(2,1)}$

$=\langle 4(2)^3-2(2)(1)^3, -3(2)^2(1)^2\rangle$

$=\langle 28,-12\rangle$

$D_uf=\langle 28,-12\rangle\cdot\frac{1}{\sqrt{10}}\langle 1,3\rangle$

$=\frac{1}{\sqrt{10}}((28)(1)+(-12)(3))$

$=\boxed{\frac{-8}{\sqrt{10}}}$

#4b. Find the directional derivative of the function at the given point in the direction of the vector $\vec{v}$

$g(x,y,z)=(x+2y+3z)^{3/2},\quad (1,1,2),\quad \vec{v}=\langle 0, 2, -1\rangle$

$\vec{u}=\frac{\langle 0,2,-1\rangle}{\sqrt{0^2+2^2+1^2}}=\frac{1}{\sqrt{5}}\langle 0,2,-1\rangle$

$\frac{\partial g}{\partial x}=\frac{3}{2}(x+2y+3z)^{1/2}(1)=\frac{3}{2}\sqrt{1+2(1)+3(2)}=\frac{9}{2}$

$\frac{\partial g}{\partial y}=\frac{3}{2}(x+2y+3z)^{1/2}(2)=\frac{18}{2}=9$

$\frac{\partial g}{\partial z}=\frac{3}{2}(x+2y+3z)^{1/2}(3)=\frac{9}{2}(3)=\frac{27}{2}$

$\nabla g=\langle \frac{9}{2}, 9, \frac{27}{2}\rangle$

$D_uf=\langle \frac{9}{2}, 9, \frac{27}{2}\rangle\cdot\frac{1}{\sqrt{5}}\langle 0,2,-1\rangle$

$=\frac{1}{\sqrt{5}}((\frac{9}{2})(0)+(9)(2)+(\frac{27}{2})(-1))$

$=\boxed{\frac{9}{2\sqrt{5}}}$

#5b. Find the directional derivative of $f(x,y,z) = xy + yz + zx$ at $P(1,-1,3)$ in the direction of $Q(2,4,5)$.

$\vec{v} = \langle 2-1, 4+1, 5-3 \rangle = \langle 1,5,2 \rangle$

$\vec{u} = \frac{1}{\sqrt{1^2+5^2+2^2}}\langle 1,5,2 \rangle = \frac{1}{\sqrt{30}}\langle 1,5,2 \rangle$

$\nabla f = \langle y+z, x+z, y+x \rangle \Big|_{(1,-1,3)}$

$= \langle -1+3, 1+3, -1+1 \rangle = \langle 2,4,0 \rangle$

$D_u f = \langle 2,4,0 \rangle \cdot \frac{1}{\sqrt{30}}\langle 1,5,2 \rangle$

$= \frac{1}{\sqrt{30}}\left((2)(1)+(4)(5)+(0)(2)\right)$

$= \frac{1}{\sqrt{30}}(22) = \boxed{\frac{22}{\sqrt{30}}}$

#6b. Find the maximum rate of change of f at the given point and the direction in which it occurs.
$f(x,y) = \sin(xy)$, $(1,0)$

$\frac{\partial f}{\partial x} = \cos(xy)(y)\Big|_{(1,0)} = \cos(0)(0) = 0$

$\frac{\partial f}{\partial y} = \cos(xy)(x)\Big|_{(1,0)} = \cos(0)(1) = 1$

$\nabla f = \langle 0,1 \rangle$

max change $|\nabla f| = \sqrt{0^2+1^2} = \boxed{1}$

direction: $\boxed{\langle 0,1 \rangle}$

#6c. Find the maximum rate of change of f at the given point and the direction in which it occurs.
$f(x,y,z) = \sqrt{x^2+y^2+z^2}$, $(3,6,-2)$

$\frac{\partial f}{\partial x} = \frac{1}{2}(x^2+y^2+z^2)^{-1/2}(2x) = \frac{(3)}{\sqrt{3^2+6^2+2^2}} = \frac{3}{7}$

$\frac{\partial f}{\partial y} = \frac{1}{2}(x^2+y^2+z^2)^{-1/2}(2y) = \frac{(6)}{\sqrt{49}} = \frac{6}{7}$

$\frac{\partial f}{\partial z} = \frac{1}{2}(x^2+y^2+z^2)^{-1/2}(2z) = \frac{-2}{\sqrt{49}} = \frac{-2}{7}$

$\nabla f = \langle \frac{3}{7}, \frac{6}{7}, \frac{-2}{7} \rangle$

max rate $= |\nabla f| = \sqrt{(\frac{3}{7})^2+(\frac{6}{7})^2+(\frac{-2}{7})^2} = \sqrt{\frac{49}{49}}$

$= \boxed{1}$

in direction of ∇f: $\boxed{\langle \frac{3}{7}, \frac{6}{7}, \frac{-2}{7} \rangle \text{ or } \langle 3,6,-2 \rangle}$

#7b. Find all points at which the direction of fastest change of the function $f(x,y) = x^2 + 3xy + y^2$ is $\langle 2,1 \rangle$.

$\nabla f = \langle 2x+3y, 3x+2y \rangle = \langle 2,1 \rangle$

all points where $\begin{cases} 2x+3y=2 \\ 3x+2y=1 \end{cases}$

$\left[\begin{array}{cc|c} 2 & 3 & 2 \\ 3 & 2 & 1 \end{array}\right] \text{rref} \left[\begin{array}{cc|c} 1 & 0 & -1/5 \\ 0 & 1 & 4/5 \end{array}\right]$

$\boxed{\left(\frac{-1}{5}, \frac{4}{5}\right)}$

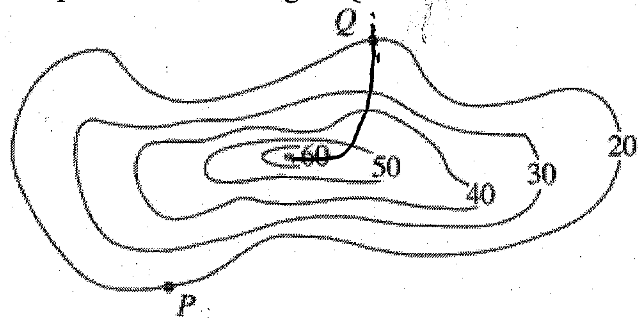
#8b. (hints)

(i) south = $\langle 0,-1\rangle$, find the directional derivative in this direction.

(ii) northwest = $\langle -1,1\rangle$, but remember to make this a unit vector before finding the directional derivative.

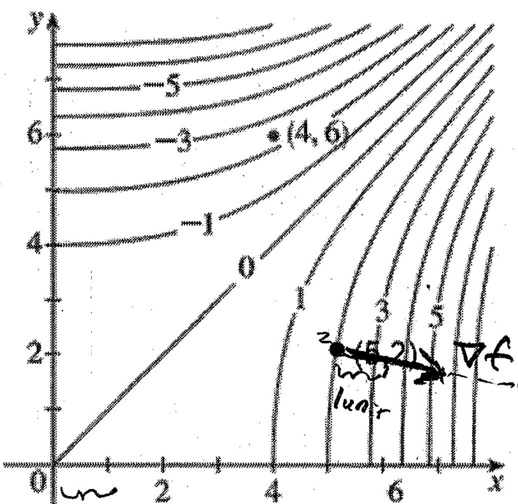
(iii) max in direction of gradient. Rate is the magnitude of the gradient. To find angle above horizontal remember that the gradient is a slope, so you can make a triangle with gradient up for every 1 unit horizontally.

#9b. For the given contour map draw the curve of steepest ascent starting at Q.



path must be perpendicular to contour lines

#10b. Sketch the gradient vector $\nabla f(5,2)$ for the function f whose level curves are shown.



direction of ∇f (max + change in $f(x,y)$)

1 unit $\quad |\nabla f| = \frac{f_2 - f_1}{\Delta dir} = \frac{4-2}{1} = 2$

= length

#11b. Find an equation of the tangent plane to the given surface at the specified point.

$x^2 - 2y^2 + z^2 + yz = 2, \quad (2,1,-1)$

$f = x^2 - 2y^2 + z^2 + yz - 2$

$f_x = 2x\Big|_{(2,1,-1)} = 2(2) = \underline{\underline{4}}; \quad f_y = -4y + z\Big|_{(2,1,-1)} = -4(1) + (-1) = \underline{\underline{-5}}$

$f_z = 2z + y\Big|_{(2,1,-1)} = 2(-1) + (1) = \underline{\underline{-1}}$

$\vec{n} = \nabla f = \langle 4, -5, -1\rangle, \quad \vec{P_0} = \langle 2, 1, -1\rangle$

$ax + by + cz = \vec{n} \cdot \vec{P_0}$

$4x - 5y - z = \langle 4, -5, -1\rangle \cdot \langle 2, 1, -1\rangle$

$= (4)(2) + (-5)(1) + (-1)(-1)$

$\boxed{4x - 5y - z = 4}$

14.7

#1b. Suppose (1,1) is a critical point of a function f with continuous second derivatives. What can you say about f at (1,1)?

(i) $f_{xx}(1,1)=4,\quad f_{xy}(1,1)=1,\quad f_{yy}(1,1)=2$

(ii) $f_{xx}(1,1)=4,\quad f_{xy}(1,1)=3,\quad f_{yy}(1,1)=2$

$D = f_{xx}f_{yy} - f_{xy}^2$

(i) $D=(4)(2)-(1)^2=7>0$

local max or min

to distinguish look at one of the concavities (f_{xx} or f_{yy}):

$f_{xx}=4>0$, concave up

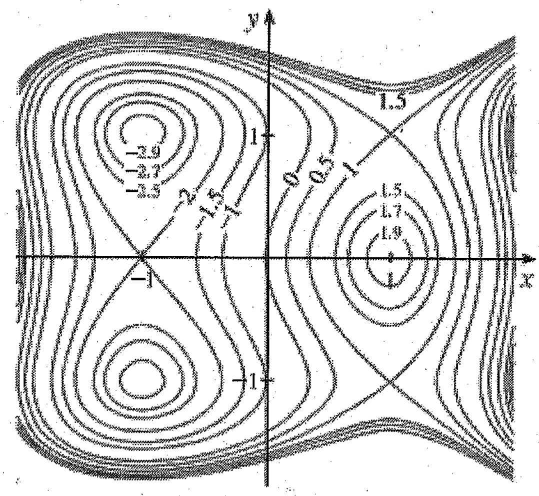
$\boxed{\text{so } f \text{ has a local min at } (1,1)}$

(ii) $D=(4)(2)-(3)^2=-1<0$

$\boxed{\text{so } f \text{ has a saddle pt at } (1,1)}$

#2b. Use the level curves in the figure to predict the location of the critical points of f and whether f has a saddle point or local maximum or minimum at each critical point. Then use the Second Derivatives Test to confirm your predictions.

$f(x,y)=3x-x^3-2y^2+y^4$



f appears to have local minima at $(-1,1)$ and $(-1,-1)$

local maxima at $(1,0)$

saddle points at $(-1,0)$, $(1,1)$ and $(1,-1)$

$f_x=3-3x^2,\quad f_y=-4y+4y^3$

critical pts when $f_x=0$ and $f_y=0$

$\begin{cases} 3-3x^2=0 & 3x^2=3 \quad \underline{x=1,\ x=-1} \\ -4y+4y^3=0 & 4y(-1+y^2)=0 \quad \underline{y=1,\ y=-1,\ y=0} \end{cases}$

(check every combination)

$f_{xx}=-6x,\quad f_{xy}=0,\quad f_{yy}=-4+12y^2$

$D=f_{xx}f_{yy}-f_{xy}^2=(-6x)(-4+12y^2)-(0)^2$

$=(-6x)(12y^2-4)$

(x,y)	D	
$(1,1)$	$(-6(1))(12(1)^2-4)=(-6)(8)=-48$	$\boxed{\text{Saddle}}$
$(1,-1)$	$(-6(1))(12(-1)^2-4)=(-6)(8)=-48$	$\boxed{\text{Saddle}}$
$(1,0)$	$(-6(1))(12(0)^2-4)=(-6)(-4)=24$	$\boxed{\text{local max}}$
$(-1,1)$	$(-6(-1))(12(1)^2-4)=(6)(8)=48$	$\boxed{\text{local min}}$
$(-1,-1)$	$(-6(-1))(12(-1)^2-4)=(6)(8)=48$	$\boxed{\text{local min}}$
$(-1,0)$	$(-6(-1))(12(0)^2-4)=(6)(-4)=-48$	$\boxed{\text{saddle}}$

$f_{xx}=-6(1)=-6$
concave down

$f_{xx}=-6(-1)=6$
concave up

#3b. Find the local maximum and minimum values and saddle point(s) of the function.

$f(x,y) = x^4 + y^4 - 4xy + 2$

$f_x = 4x^3 - 4y \qquad f_y = 4y^3 - 4x$

critical point when $f_x = 0$ and $f_y = 0$

$\begin{cases} 4x^3 - 4y = 0 \rightarrow 4y = 4x^3, \ y = x^3 \\ 4y^3 - 4x = 0 \quad 4(x^3)^3 - 4x = 0 \end{cases}$

$4x^9 - 4x = 0$

$4x(x^8 - 1) = 0$

$x = 0 \qquad x^8 = 1 \qquad y = x^3$

$x = 1, x = -1$

so points are $(0,0)(1,1)(-1,-1)$

$f_{xx} = 12x^2 \quad f_{xy} = -4 \quad f_{yy} = 12y^2$

$D = f_{xx}f_{yy} - f_{xy}^2 = (12x^2)(12y^2) - (-4)^2$

$D = 144x^2y^2 - 16$

(x,y)	$D = 144x^2y^2 - 16$
$(0,0)$	$D = 144(0)^2(0)^2 - 16 = -16$ $\boxed{\text{saddle point}}$
$(1,1)$	$D = 144(1)^2(1)^2 - 16 = 128$ max or min use $f_{xx} = 12(1)^2 = 12 > 0$ concave up $\boxed{\text{local min}}$
$(-1,-1)$	$D = 144(-1)^2(-1)^2 - 16 = 128$ max or min use $f_{xx} = 12(-1)^2 = 12 > 0$ concave up $\boxed{\text{local min}}$

#4b. Find the absolute maximum and minimum values of f on the set D.

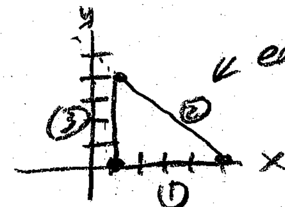
$f(x,y) = 3 + xy - x - 2y$

D is the closed triangular region with vertices (1,0), (5,0), and (1,4).

local extrema; critical pts?

$f_x = y - 1 \qquad f_y = x - 2$

$\begin{cases} y - 1 = 0 \\ x - 2 = 0 \end{cases} \quad \begin{matrix} y = 1 \\ x = 2 \end{matrix} \quad \boxed{(2,1)}$ critical pt



← edges

edge 1: $y = 0 \quad f(x,0) = 3 + x(0) - x - 2(0)$

$f(x) = 3 - x$ a line

so no critical pts, max/min on ends

$\boxed{(1,0)(5,0)}$

edge 2: $y = -x + 5$

$f(x, -x+5) = 3 + x(-x+5) - x - 2(-x+5)$

$f(x) = 3 - x^2 + 5x - x + 2x - 10$

$f(x) = -x^2 + 6x - 7$ quadratic

critical pt when $f'(x) = 0$ at $2x = 6$, $\underline{x = 3}$

$-2x + 6 = 0$

$y = -(3) + 5 = 2$

so include $\boxed{(3,2) \text{ and ends } (5,0), (1,4)}$

edge 3: $x = 1 \quad f(1,y) = 3 + (1)y - (1) - 2y$

$f(y) = 3 + y - 1 - 2y = 2 - y$

a line, no critical pts

max/min on ends

$\boxed{(1,4)(1,0)}$

points checklist

(x,y)	$f(x,y) = 3 + xy - x - 2y$
$(2,1)$	$3 + (2)(1) - (2) - 2(1) = 1$
$(1,0)$	$3 + (1)(0) - (1) - 2(0) = 2$
$(5,0)$	$3 + (5)(0) - (5) - 2(0) = -2$
$(1,4)$	$3 + (1)(4) - (1) - 2(4) = -2$
$(3,2)$	$3 + (3)(2) - (3) - 2(2) = 2$

absolute max of 2 at $(1,0)$ and $(3,2)$

absolute min of -2 at $(5,0)$ and $(1,4)$

#5b. Find the points on the cone $z^2 = x^2 + y^2$ that are closest to the point $(4,2,0)$.

points on cone is (x,y,z) $z^2 = x^2+y^2$

$z = \pm\sqrt{x^2+y^2}$

distance is...

$d = \sqrt{(x-4)^2+(y-2)^2+(z-0)^2}$

$d = \sqrt{(x-4)^2+(y-2)^2+(\pm\sqrt{x^2+y^2}-0)^2}$

$d = \sqrt{(x-4)^2+(y-2)^2+x^2+y^2}$

make $f = d^2$ (and minimize d^2)

$f = d^2 = (x-4)^2+(y-2)^2+x^2+y^2$

critical pts: $\begin{cases} f_x = 2(x-4)+2x = 0 \\ f_y = 2(y-2)+2y = 0 \end{cases}$

$\begin{cases} 2x-8+2x=0 \\ 2y-4+2y=0 \end{cases}$ $4x=8,\ x=2$; $4y=4\ \ y=1$

$\boxed{(2,1)}$

verify min w/D:

$f_{xx} = 4 \quad f_{xy} = 0 \quad f_{yy} = 4$

$D = f_{xx}f_{yy} - f_{xy}^2 = (4)(4)-(0)^2 = 16$

max or min ✓

$f_{xx} = 4$, concave up

so this is a minimum d^2

this occurs at point $(2,1)$

$z^2 = x^2+y^2 = (2)^2+(1)^2 = 5$, $z = \pm\sqrt{5}$

$\boxed{\text{so occurs at points } (2,1,\sqrt{5}) \text{ and } (2,1,-\sqrt{5})}$

and the min distance is:

$d^2 = f(2,1) = (2-4)^2+(1-2)^2+2^2+1^2$

$d^2 = 10$

$(d = \sqrt{10})$

#6b. Find three positive numbers whose sum is 12 and whose sum of squares is as small as possible.

objective function: $f = x^2+y^2+z^2$ (min)

constraint: $x+y+z = 12$

$z = 12-x-y$

$f = x^2+y^2+(12-x-y)^2 = x^2+y^2+(12-x-y)(12-x-y)$

$f = x^2+y^2+144-12x-12y-12x+x^2+xy-12y+xy+y^2$

$f = 2x^2+2y^2+144-24x-24y+2xy$

critical points: when $f_x = 0$ & $f_y = 0$

$\begin{cases} f_x = 4x-24+2y = 0 \\ f_y = 4y-24+2x = 0 \end{cases}$ $\begin{cases} 4x+2y = 24 \\ 2x+4y = 24 \end{cases}$

$\left[\begin{array}{cc|c} 4 & 2 & 24 \\ 2 & 4 & 24 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & 4 \end{array}\right]$

$x=4,\ y=4;\ z = 12-x-y = 12-4-4 = 4$

verify min

$f_{xx} = 4 \quad f_{xy} = 2 \quad f_{yy} = 4$

$D = f_{xx}f_{yy} - f_{xy}^2 = (4)(4)-(2)^2 = 12$

max or min

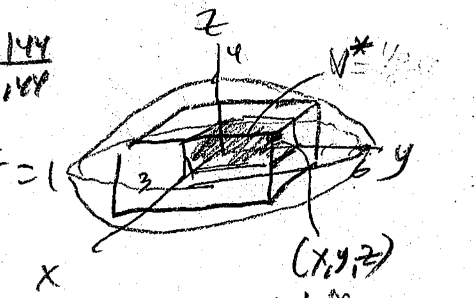
$f_{xx} = 4$ concave up, so min

min sum of squares when $\boxed{x = y = z = 4}$

#7b. Find the dimensions of the rectangular parallelepiped with faces parallel to the coordinate plane that can be inscribed in the ellipsoid $16x^2+4y^2+9z^2=144$ which will maximize the volume inside the parallelepiped.

$$\frac{16x^2}{144}+\frac{4y^2}{144}+\frac{9z^2}{144}=\frac{144}{144}$$

$$\frac{x^2}{9}+\frac{y^2}{36}+\frac{z^2}{16}=1$$



must satisfy $16x^2+4y^2+9z^2=144$

Let $V^* = \frac{1}{8}$ of actual volume $= xyz$

objective function — constraint

maximize $f = V^* = xyz$

$16x^2+4y^2+9z^2=144$

$9z^2 = 144-16x^2-4y^2$

$z^2 = \dfrac{144-16x^2-4y^2}{9}$

$z = \pm\sqrt{\dfrac{144-16x^2-4y^2}{9}}$ (use + case for 1st quadrant)

$$f = xy\left(\tfrac{1}{3}\right)\sqrt{144-16x^2-4y^2}$$

$$f_x = \left(\tfrac{1}{3}xy\right)\frac{\partial}{\partial x}\left[(144-16x^2-4y^2)^{1/2}\right] + \sqrt{144-16x^2-4y^2}\,\frac{\partial}{\partial x}\left[\tfrac{1}{3}xy\right]$$

$$= \tfrac{1}{3}xy\left(\tfrac{1}{2}(144-16x^2-4y^2)^{-1/2}(-32x)\right) + \sqrt{144-16x^2-4y^2}\left(\tfrac{1}{3}y\right)$$

$$= \frac{-16x^2y}{3\sqrt{144-16x^2-4y^2}} + \frac{y\sqrt{144-16x^2-4y^2}}{3}\left(\frac{\sqrt{144-16x^2-4y^2}}{\sqrt{144-16x^2-4y^2}}\right)$$ ← to get common denominator

$$= \frac{-16x^2y + y(144-16x^2-4y^2)}{3\sqrt{144-16x^2-4y^2}} = \frac{-16x^2y+144y-16x^2y-4y^3}{3\sqrt{144-16x^2-4y^2}} = \frac{-32x^2y+144y-4y^3}{3\sqrt{144-16x^2-4y^2}}=0$$

when numerator $=0$: $\boxed{-32x^2y+144y-4y^3=0}$

$$f_y = \left(\tfrac{1}{3}xy\right)\frac{\partial}{\partial y}\left[(144-16x^2-4y^2)^{1/2}\right] + \sqrt{144-16x^2-4y^2}\,\frac{\partial}{\partial y}\left[\tfrac{1}{3}xy\right]$$

$$= \tfrac{1}{3}xy\left[\tfrac{1}{2}(144-16x^2-4y^2)^{-1/2}(-8y)\right] + \sqrt{144-16x^2-4y^2}\left(\tfrac{1}{3}x\right)$$

$$= \frac{-4xy^2}{3\sqrt{144-16x^2-4y^2}} + \frac{x\sqrt{144-16x^2-4y^2}}{3}\left(\frac{\sqrt{144-16x^2-4y^2}}{\sqrt{144-16x^2-4y^2}}\right)$$

$$= \frac{-4xy^2 + x(144-16x^2-4y^2)}{3\sqrt{144-16x^2-4y^2}} = \frac{-4xy^2+144x-16x^3-4xy^2}{3\sqrt{144-16x^2-4y^2}} = \frac{-8xy^2+144x-16x^3}{3\sqrt{144-16x^2-4y^2}}=0$$

when numerator $=0$: $\boxed{-8xy^2+144x-16x^3=0}$

System: $\begin{cases} -32x^2y+144y-4y^3=0 \\ -8xy^2+144x-16x^3=0 \end{cases}$

(continued...)

#7b (continued)

$$\begin{cases} -32x^2y + 144y - 4y^3 = 0 \\ -8xy^2 + 144x - 16x^3 = 0 \end{cases} \longrightarrow \begin{aligned} 8xy^2 &= 144x - 16x^3 \\ y^2 &= \frac{144x - 16x^3}{8x} = \frac{8x(18-2x^2)}{8x} = 18 - 2x^2 \end{aligned}$$

$-32x^2y + 144y - 4y^3 = 0 \xleftarrow{\text{into}} y = \sqrt{18-2x^2}$ (we + case (1st quadrant))

$-32x^2(\sqrt{18-2x^2}) + 144(\sqrt{18-2x^2}) - 4(\sqrt{18-2x^2})^3 = 0$

$\sqrt{18-2x^2}\,[-32x^2 + 144 - 4(\sqrt{18-2x^2})^2] = 0$

$\sqrt{18-2x^2}\,[-32x^2 + 144 - 4(18-2x^2)] = 0$

$\sqrt{18-2x^2}\,(-32x^2 + 144 - 72 + 8x^2) = 0$

$\sqrt{18-2x^2}\,(-24x^2 + 72) = 0 \longrightarrow$ when $18 - 2x^2 = 0$ or $-24x^2 + 72 = 0$

$2x^2 = 18$	$24x^2 = 72$
$x^2 = 9$	$x^2 = 3$
$x = \pm 3$	$x = \pm\sqrt{3}$
$(x = 3)$	$(x = \sqrt{3})$

other dimensions when...

$x = 3$	$x = \sqrt{3}$
$y = \sqrt{18-2x^2}$	$y = \sqrt{18-2x^2}$
$y = \sqrt{18-2(3)^2}$	$y = \sqrt{18-2(\sqrt{3})^2}$
$y = 0$	$y = \sqrt{12} = \sqrt{4}\sqrt{3} = 2\sqrt{3}$
$z = \sqrt{\frac{144-16x^2-4y^2}{9}}$	$z = \sqrt{\frac{144-16x^2-4y^2}{9}}$
$z = \sqrt{\frac{144-16(3)^2-4(0)^2}{9}}$	$z = \sqrt{\frac{144-16(\sqrt{3})^2-4(\sqrt{12})^2}{9}}$
$z = 0$	$z = \sqrt{\frac{48}{9}} = \frac{\sqrt{48}}{3} = \frac{\sqrt{16}\sqrt{3}}{3}$
$(3, 0, 0)$	$= \frac{4\sqrt{3}}{3}$

this can't be the dimension (volume would be zero)

So $\boxed{\begin{aligned} x &= \sqrt{3} \\ y &= 2\sqrt{3} \\ z &= \frac{4\sqrt{3}}{3} \end{aligned}}$

Note: the required practice problem is quite a bit easier than this :)
(took me only 1 page to solve)

14.8

#1b. Use Lagrange multipliers to find the maximum and minimum values of the function subject to the given constraint.
$f(x,y) = x^2 y; \quad x^2 + 2y^2 = 6$

$\nabla f = \langle 2xy, x^2 \rangle \quad \nabla g = \langle 2x, 4y \rangle$

$\nabla f = \lambda \nabla g$

$\langle 2xy, x^2 \rangle = \lambda \langle 2x, 4y \rangle$

$\begin{cases} 2xy = \lambda 2x & \lambda = \frac{2xy}{2x} = y \\ x^2 = \lambda 4y & x^2 = (y)4y \\ x^2 + 2y^2 = 6 \end{cases}$

$\begin{cases} x^2 = 4y^2 \\ x^2 + 2y^2 = 6 \end{cases}$ $\quad 4y^2 + 2y^2 = 6$

$6y^2 = 6$

$y^2 = 1$

$y = \pm 1$

$y = 1$: $x^2 = 4y^2$, $x^2 = 4(1)^2 = 4$, $x = 2$, $x = -2$; $\lambda = y$, $\lambda = y$; $\lambda = 1$, $\lambda = 1$; $\boxed{(2,1) \quad (-2,1)}$

$y = -1$: $x^2 = 4y^2$, $x^2 = 4(-1)^2 = 4$, $x = 2$, $x = -2$; $\lambda = y$, $\lambda = y$; $\lambda = -1$, $\lambda = -1$; $\boxed{(2,-1) \quad (-2,-1)}$

(x,y)	$f(x,y) = x^2 y$
$(2,1)$	$(2)^2(1) = 4$
$(-2,1)$	$(-2)^2(1) = 4$
$(2,-1)$	$(2)^2(-1) = -4$
$(-2,-1)$	$(-2)^2(-1) = -4$

$\boxed{f_{max} = 4 \text{ at } (2,1) \;\&\; (-2,1) \quad f_{min} = -4 \text{ at } (2,-1) \;\&\; (-2,-1)}$

#2b. Use Lagrange multipliers to find the maximum and minimum values of the function subject to the given constraint.
$f(x,y,z,t) = x + y + z + t; \quad x^2 + y^2 + z^2 + t^2 = 1$

$\nabla f = \langle 1,1,1,1 \rangle \quad \nabla g = \langle 2x, 2y, 2z, 2t \rangle$

$\nabla f = \lambda \nabla g$

$\langle 1,1,1,1 \rangle = \lambda \langle 2x, 2y, 2z, 2t \rangle$

$\begin{cases} 2x\lambda = 1 & \lambda = \frac{1}{2x} = \frac{1}{2y} = \frac{1}{2z} = \frac{1}{2t} \\ 2y\lambda = 1 & x = y = z = t \\ 2z\lambda = 1 \\ 2t\lambda = 1 \\ x^2+y^2+z^2+t^2 = 1 & \Rightarrow t^2+t^2+t^2+t^2 = 1 \end{cases}$

$4t^2 = 1$

$t^2 = \frac{1}{4}$

$t = \pm\sqrt{\frac{1}{4}} = \pm\frac{1}{2}$

$t = \frac{1}{2}$: $\lambda = \frac{1}{2t} = \frac{1}{2(\frac{1}{2})} = 1$; $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$

$t = -\frac{1}{2}$: $\lambda = \frac{1}{2(-\frac{1}{2})} = -1$; $\left(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right)$

(x,y,z,t)	$f(x,y,z,t) = x+y+z+t$
$(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$	2
$(-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2})$	-2

$\boxed{f_{max} = 2 \text{ at } \left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right) \quad f_{min} = -2 \text{ at } \left(-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right)}$

#3b. Consider the problem of maximizing the function $f(x,y)=x$ subject to the constraint $y^2+x^4-x^3=0$

(i) Try using Lagrange multipliers to solve the problem.

(ii) Show that the minimum value is $f(0,0)=0$ but the Lagrange condition $\nabla f(0,0)=\lambda\nabla g(0,0)$ is not satisfied for any value of λ.

(iii) Explain why Lagrange multipliers fail to find the minimum value in this case.

(i) $\nabla f=\langle 1,0\rangle \quad \nabla g=\langle 4x^3-3x^2, 2y\rangle$

$\nabla f=\lambda\nabla g$

$\langle 1,0\rangle=\lambda\langle 4x^3-3x^2, 2y\rangle$

$\begin{cases}\lambda(4x^3-3x^2)=1\\ 2y=0 \quad\quad y=0\\ y^2+x^4-x^3=0\end{cases}$

$x^4-x^3=0$

$x^3(x-1)=0$

$\underline{x=0}$ $\quad\quad$ $\underline{x=1}$

$y=0$ $\quad\quad$ $y=0$

$\lambda(4(0)^3-3(0)^2)=1$ $\quad\quad$ $\lambda(4(1)^3-3(1)^2)=1$

$\lambda(0)=1$ $\quad\quad$ $\lambda(1)=1$

not possible $\quad\quad$ $\underline{\lambda=1}$

$x=1,\ y=0,\ \lambda=1$

$(1,0)$ max? $f(x,y)=x$

try another point on constraint

$x=2?$

$y^2+(2)^4-(2)^3=0$

$y^2=-8$ not possible

$x=\frac{1}{2}$

$y^2+(\frac{1}{2})^4-(\frac{1}{2})^3=0$

$y^2=.0625$

$y=\frac{1}{4} \quad f(\frac{1}{2},\frac{1}{4})=\frac{1}{2}$

suggests max at $(1,0)$

(ii) $f(0,0)=0$

$\nabla f(0,0)=\langle 1,0\rangle$

$\nabla g(0,0)=\langle 4(0)^3-3(0)^3, 2(0)\rangle=\langle 0,0\rangle$

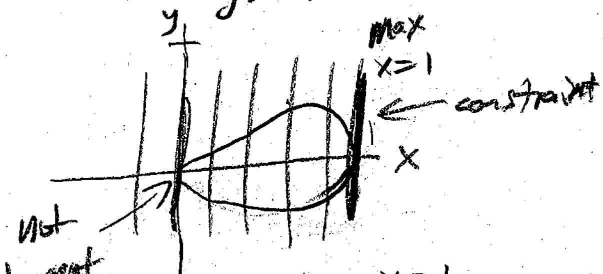
$\langle 1,0\rangle=\lambda\langle 0,0\rangle$

$\begin{cases}0\lambda=1\\ 0\lambda=0\end{cases}$ no possible λ makes this system true

(iii) constraint: $y^2+x^4-x^3=0$

$y^2=x^3-x^4$

$y=\pm\sqrt{x^3-x^4}$



$f(x,y)=x \quad\quad x=1$

$x=2$

$\vdots$

at max, $x=1$ is tangent to the constraint curve but there is no way the objective function $x=$ const (a vertical line) can be tangent to the constraint at the min, $f=0$

Extra #4. The plane $x+y+2z=2$ intersects the paraboloid $z=x^2+y^2$ in an ellipse. Find the points on this ellipse that are nearest to and farthest from the origin.

objective function (distance)

$d=\sqrt{(x-0)^2+(y-0)^2+(z-0)^2}$

$f(x,y,z)=d^2=x^2+y^2+z^2$

$\nabla f=\langle 2x,2y,2z\rangle$

two constraints

$x+y+2z=2$

$g(x,y,z)=x+y+2z-2$

$\nabla g=\langle 1,1,2\rangle$

$z=x^2+y^2$

$h(x,y,z)=x^2+y^2-z$

$\nabla h=\langle 2x,2y,-1\rangle$

$\nabla f=\lambda\nabla g+\mu\nabla h$ ← a 2nd Lagrange multiplier

$\langle 2x,2y,2z\rangle=\lambda\langle 1,1,2\rangle+\mu\langle 2x,2y,-1\rangle$

$$\begin{cases} 2x=\lambda+2x\mu \\ 2y=\lambda+2y\mu \\ 2z=2\lambda-\mu \\ x+y+2z=2 \\ z=x^2+y^2 \end{cases}$$

(1) subtract to eliminate λ:

$2x=\lambda+2x\mu$
$-2y=-\lambda-2y\mu$

$2x-2y=2x\mu-2y\mu$

$2(x-y)=2\mu(x-y)$

two possibilities:

$x=y$ or $x\neq y$, so can divide $(x-y)$

$2=2\mu$

$\mu=1$

if $\mu=1$ rewriting system:

$$\begin{cases} 2x=\lambda+2x \\ 2y=\lambda+2y \\ 2z=2\lambda-1 \\ x+y+2z=2 \\ z=x^2+y^2 \end{cases}$$

$2\lambda=2z+1$

$\lambda=z+\frac{1}{2}$

$2x=\lambda+2x$

$2y=\lambda+2y$

$\lambda=0$

$z+\frac{1}{2}=0$, $z=-\frac{1}{2}$

into $x+y+2z=2$

$z=x^2+y^2$

$-\frac{1}{2}=x^2+y^2$ impossible

therefore $z\neq-\frac{1}{2}$ so $\mu\neq 1$

if $\mu\neq 1$ then other possibility must be true: $\boxed{x=y}$

rewriting $(y=x)$

$$\begin{cases} 2x=\lambda+2x\lambda \\ 2z=2\lambda-\mu \\ 2x+2z=2 \\ z=2x^2 \end{cases}$$

$z=2x^2$ into $2x+2z=2$

$x+z=1$

$x+2x^2=1$

$2x^2+x-1=0$

$x=\frac{-1\pm\sqrt{1^2+4(2)}}{2(2)}$

$=\frac{-1\pm 3}{4}=-1,\ \frac{1}{2}$

$x=-1$ or $x=\frac{1}{2}$

$y=-1$ $\quad y=\frac{1}{2}$

$z=2x^2$ $\quad z=2\left(\frac{1}{2}\right)^2$

$z=2$ $\quad z=\frac{1}{2}$

$(-1,-1,2)$ $\quad \left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$

continued...

Extra #4 (continued)

must still verify λ & μ value for our possible solutions:

$\underline{(-1,-1,2)}$ $\underline{(\frac{1}{2},\frac{1}{2},\frac{1}{2})}$ system has $2x = \lambda + 2x\lambda$

$2z = 2\lambda - \mu$

$\lambda = \frac{2(-1)}{1+2(-1)}$ $\lambda = \frac{2(\frac{1}{2})}{1+2(\frac{1}{2})}$ $2x = \lambda(1+2x)$

$\lambda = \frac{-2}{-1} = 2$ ✓ $\lambda = \frac{1}{1+1} = \frac{1}{2}$ ✓ $\lambda = \frac{2x}{1+2x}$ & $\mu = 2\lambda - 2z$

$\mu = 2(2) - 2(2)$ $\mu = 2(\frac{1}{2}) - 2(\frac{1}{2})$

$\mu = 0$ ✓ $= 0$ ✓

so $(-1,-1,2)$ & $(\frac{1}{2},\frac{1}{2},\frac{1}{2})$ are solutions

now into objective function (distance^2)

(x,y,z)	$f = d^2 = x^2+y^2+z^2$
$(-1,-1,2)$	$(-1)^2+(-1)^2+(2)^2 = 6$
$(\frac{1}{2},\frac{1}{2},\frac{1}{2})$	$(\frac{1}{2})^2+(\frac{1}{2})^2+(\frac{1}{2})^2 = \frac{3}{4}$

point on ellipse closest to origin is $(\frac{1}{2},\frac{1}{2},\frac{1}{2})$
point on ellipse farthest from origin is $(-1,-1,2)$

use 3D software to visualize this ☺