Calc III - Ch 14 - Extra Practice

Part 1

14.1

#1b. The *wave heights* h in the open sea depend on the speed v of the wind and the length of time t that the wind has been blowing at that speed. Values of the function $h = f(v,t)$ are recorded in feet in the following table:

Duration (hours)

Wind speed (knots) v \ t	5	10	15	20	30	40	50
10	2	2	2	2	2	2	2
15	4	4	5	5	5	5	5
20	5	7	8	8	9	9	9
30	9	13	16	17	18	19	19
40	14	21	25	28	31	33	33
50	19	29	36	40	45	48	50
60	24	37	47	54	62	67	69

(i) What is the value of $f(40,15)$? What is its meaning?
(ii) What is the meaning of the function $h = f(30,t)$? Describe the behavior of this function.
(iii) What is the meaning of the function $h = f(v,30)$? Describe the behavior of this function.

(i) $f(\underset{v}{40},\underset{t}{15}) = \boxed{25\text{ ft}}$
The wave ht = 25 ft when a 40 knot wind has been blowing for 15 hrs.

(ii) $h = f(30,t)$ shows how wave height varies with change in time a 30 knot wind has been blowing. (as duration increases, wave ht increases).

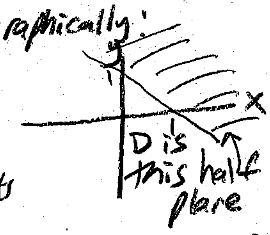
(iii) $h = f(v,30)$ shows how wave height varies with change in wind speed for winds that have been blowing for 30 hours. (as wind speed increases, wave ht increases)

#2b. Let $f(x,y) = \ln(x+y-1)$.
(i) Evaluate $f(1,1)$ and $f(e,1)$.
(ii) Find the domain of f.
(iii) Find the range of f.

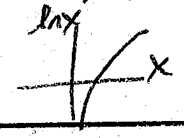
(i) $f(1,1) = \ln(1+1-1) = \ln(1) = \boxed{0}$
$f(e,1) = \ln(e+1-1) = \ln(e) = \boxed{1}$

(ii) ln cannot be zero or <0 so $x+y-1>0$
or $x+y \geq 1$ so
$\boxed{D: \{(x,y) \mid x+y>1\}}$

graphically:



(iii) ln can produce outputs from $-\infty$ to ∞:



so $\boxed{R: \{z \mid -\infty < z < \infty\}}$

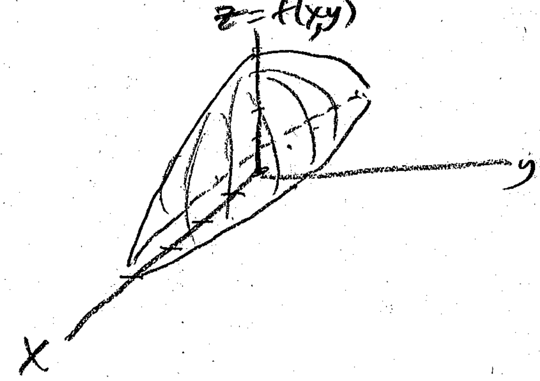
#3b. Sketch the graph of the function $f(x,y) = \sqrt{16 - x^2 - 16y^2}$. $z = \sqrt{16-x^2-16y^2}$

$z^2 = 16 - x^2 - 16y^2$, $x^2 + 16y^2 + z^2 = 16$ is an ellipsoid

$\frac{x^2}{16} + \frac{y^2}{1} + \frac{z^2}{16} = 1$

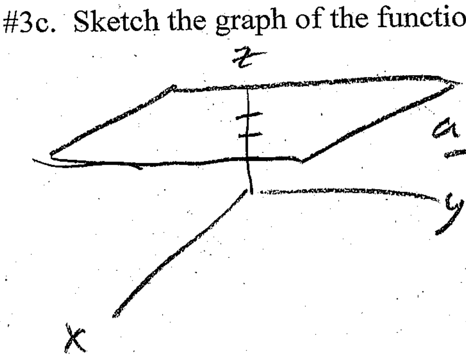
but $z = \sqrt{16-x^2-16y^2}$ uses only $+z$ part;

$z = f(x,y)$



#3c. Sketch the graph of the function $f(x,y) = 3$.

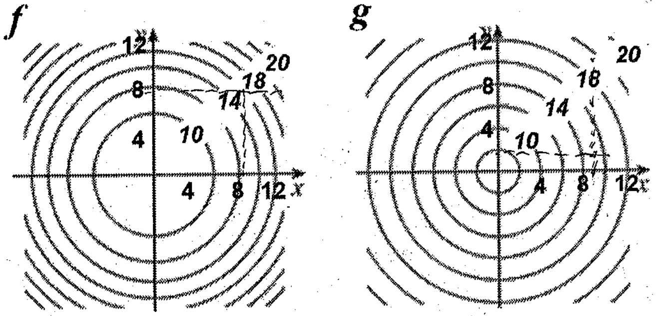
$z = 3$



a plane at $z=3$

#4b. Contour maps for function f and g are shown. Use these to estimate the values of $f(8,8)$ and $g(9,2)$. One of these is a cone and the other a paraboloid – which is which?

f g



$f(8,8) \approx \boxed{16}$

$g(9,2) \approx \boxed{15}$ (halfway between 14 & 16)

$\boxed{g \text{ is the cone}}$ and f is the paraboloid, because height is changing steadily for g (cone).

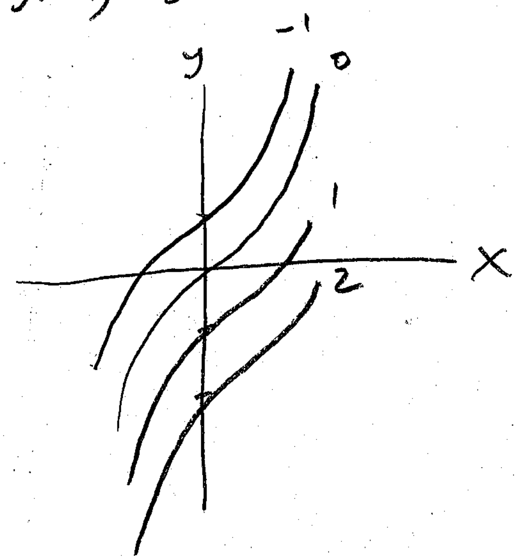
#5b. Draw a contour map of the function showing several level curves: $f(x,y) = x^3 - y$.

$x^3 - y = 1$, $y = x^3 - 1$

$x^3 - y = 2$, $y = x^3 - 2$

$x^3 - y = 0$, $y = x^3$

$x^3 - y = -1$, $y = x^3 + 1$



#6b. Describe the level surfaces of the function

$f(x,y,z) =$ [redacted] $x^2 - y^2 + z^2$

$x^2 - y^2 + z^2 = 1$ } hyperboloids of 1 sheet

$x^2 - y^2 + z^2 = 2$

$x^2 - y^2 + z^2 = -1 \rightarrow -x^2 + y^2 - z^2 = 1$ }

$x^2 - y^2 + z^2 = -2 \rightarrow -x^2 + y^2 - z^2 = 2$

hyperboloids of two sheets

$x^2 - y^2 + z^2 = 0 \rightarrow$ $x^2 + z^2 = y^2$ a cone

for $f(x,y,z) = 0$ $\boxed{\text{a cone}}$

for $f(x,y,z) > 0$ $\boxed{\text{hyperboloids of one sheet}}$

for $f(x,y,z) < 0$ $\boxed{\text{hyperboloids of two sheets}}$

14.2

#1b. Find the limit, if it exists, or show that the limit does not exist $\lim_{(x,y)\to(1,2)}\left(5x^3 - x^2y^2\right)$.

Polynomials are continuous everywhere, so we can just plug in...

$\lim_{(x,y)\to(1,2)} 5(1)^3-(1)^2(2)^2$

$5-4$

$\boxed{1}$

#2b. Find the limit, if it exists, or show that the limit does not exist $\lim_{(x,y)\to(0,0)}\frac{xy\cos y}{3x^2+y^2}$. $\frac{0}{0}\in$

try paths...

y-axis $(x=0)$

$\lim_{(x,y)\to(0,0)}\frac{(0)y\cos y}{3(0)^2+y^2} = \lim_{y\to 0}\frac{0}{y^2} = \boxed{0}$

x-axis $(y=0)$

$\lim_{(x,y)\to(0,0)}\frac{x(0)\cos(0)}{3x^2+(0)^2} = \lim_{x\to 0}\frac{0}{3x^2} = \boxed{0}$

lines $(y=mx+b$ through $(0,0)$

so $(0)=m(0)+b,\ b=0$

$y=mx)$

$\lim_{(x,y)\to(0,0)}\frac{x(mx)\cos(mx)}{3x^2+(mx)^2} = \lim_{x\to 0}\frac{mx^2\cos(mx)}{x^2(3+m^2)}$

$=\lim_{x\to 0}\frac{m\cos(mx)}{3+m^2} = \frac{m(1)}{3+m^2}\ \boxed{\neq 0}$

so limit $\boxed{DNE}$

#3b. Find the limit, if it exists, or show that the limit does not exist $\lim_{(x,y)\to(0,0)}\frac{xy}{\sqrt{x^2+y^2}}$. $\frac{0}{0}\in$

y-axis $(x=0)$:

$\lim_{(x,y)\to(0,0)}\frac{(0)y}{\sqrt{0^2+y^2}} = \lim_{y\to 0}\frac{0}{y} = \boxed{0}$

x-axis $(y=0)$:

$\lim_{(x,y)\to(0,0)}\frac{x(0)}{\sqrt{x^2+0^2}} = \lim_{x\to 0}\frac{0}{x} = \boxed{0}$

lines $(y=mx+b$ through $(0,0)$ $(0)=m(0)+b,\ b=0)$

$y=mx$

$\lim_{(x,y)\to(0,0)}\frac{x(mx)}{\sqrt{x^2+(mx)^2}} = \lim_{x\to 0}\frac{mx^2}{\sqrt{x^2(1+m^2)}}$

$=\lim_{x\to 0}\frac{x^2 m}{\sqrt{x^2}\sqrt{1+m^2}} = \lim_{x\to 0}\frac{\cancel{x}(xm)}{\cancel{x}\sqrt{1+m^2}}$

$=\lim_{x\to 0}\frac{mx}{\sqrt{1+m^2}} = \boxed{0}$

parabolas $(y=ax^2)$

$\lim_{(x,y)\to(0,0)}\frac{x(ax^2)}{\sqrt{x^2+(ax^2)^2}} = \lim_{x\to 0}\frac{ax^3}{\sqrt{x^2}\sqrt{1+a^2x^2}}$

$=\lim_{x\to 0}\frac{x(ax^2)}{x\sqrt{1+a^2x^2}} = \lim_{x\to 0}\frac{ax^2}{\sqrt{1+a^2x^2}} = \boxed{0}$

.... probably limit $=0$, do Squeeze Theorem!

$\frac{xy}{\sqrt{x^2+y^2}}$ $\quad 0 \le \frac{x}{\sqrt{x^2+y^2}} \le 1$

so $\lim_{(x,y)\to(0,0)} 0 \le \lim_{(x,y)\to(0,0)}\frac{x|y|}{\sqrt{x^2+y^2}} \le \lim_{(x,y)\to(0,0)}|y|$

$=0 \qquad\qquad =0$

so $\lim_{(x,y)\to(0,0)}\frac{xy}{\sqrt{x^2+y^2}}\ \boxed{=0}$

(for all paths)

14.3

#1b. The *wave heights h* in the open sea depend on the speed v of the wind and the length of time t that the wind has been blowing at that speed. Values of the function $h = f(v,t)$ are recorded in feet in the following table:

Duration (hours)

v \ t	5	10	15	20	30	40	50
10	2	2	2	2	2	2	2
15	4	4	5	5	5	5	5
20	5	7	8	8	9	9	9
30	9	13	16	17	18	19	19
40	14	21	25	28	31	33	33
50	19	29	36	40	45	48	50
60	24	37	47	54	62	67	69

Wind speed (knots)

(i) Estimate the values of $f_t(40,15)$ *and* $f_v(40,15)$. What are the practice interpretations of these values?
(ii) What are the meanings of the partial derivatives $\frac{\partial h}{\partial t}$ and $\frac{\partial h}{\partial v}$?
(iii) What appear to be the value of the following limit? $\lim_{t\to\infty} \frac{\partial h}{\partial t}$

(i) $f_t(40,15) \approx \frac{28-21}{20-10} = \frac{7}{10} = \boxed{0.7 \text{ ft/hr}}$

For every increase of 1 hr in duration of wind of 40 knots, wave height increases by 0.7 ft, on average.

$f_v(40,15) \approx \frac{36-16}{50-30} = \frac{20}{20} = \boxed{1 \text{ ft/knot}}$

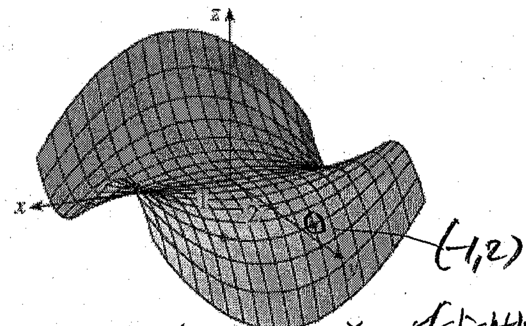
For every increase of 1 knot in windspeed (blowing for 15 hrs), wave height increases by 1 ft, on average.

(ii) $\frac{\partial h}{\partial t}$ is the change in wave ht per unit change in duration (with fixed wind speed) and $\frac{\partial h}{\partial v}$ is the change in wave ht per unit change in windspeed (with fixed duration).

(iii) as t increases, change in wave ht get smaller so

$\lim_{t\to\infty} \frac{\partial h}{\partial t} = \boxed{0}$

#2b. Determine the signs of the partial derivatives $f_x(-1,2)$, $f_y(-1,2)$, and $f_{yy}(-1,2)$ for the function f whose graph is shown.



(-1,2)

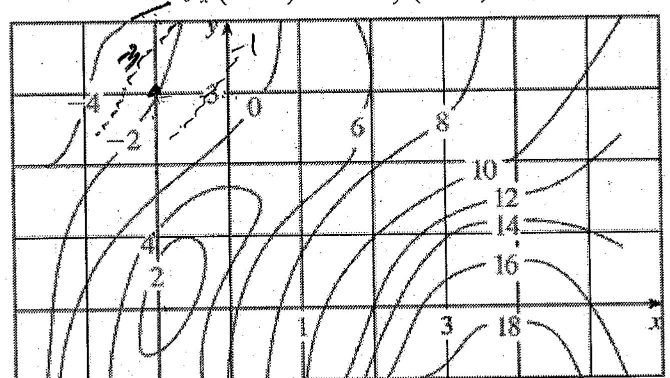
$f_x(-1,2) < 0$ (or ≈ 0) x (slightly decreasing)

$f_y(-1,2) < 0$ (decreasing)

$f_{yy}(-1,2) < 0$ (concave down) in y direction

y

#3b. A contour map is given for a function f. Use it to estimate $f_x(-1,3)$ and $f_y(-1,3)$



$f_x(-1,3) \approx \frac{-0.7-(-2)}{1} = \frac{1.3}{1} = \boxed{1.3}$

$f_y(-1,3) \approx \frac{-3.7-(-2)}{1} = \frac{-1.7}{1} = \boxed{-1.7}$

#4b. Find both first partial derivatives of the function $f(x,y) =$ [redacted] $2xy^2 - 3x$

$2xy^2 - 3x$

$f_x = 2(1)y^2 - 3(1) = \boxed{2y^2 - 3}$

$2xy^2 - 3x$ (any)

$f_y = (2x)(2y) - 0 = \boxed{4xy}$

#5b. Find both first partial derivatives of the function $z = x^y$

$\frac{\partial z}{\partial x} = y(x^{y-1})\frac{\partial}{\partial x}[x]$ (chain rule)

$= y\,x^{y-1}(1) = \boxed{yx^{y-1}}$

$\frac{\partial z}{\partial y}$ (x is a constant, so x^y is like 2^y)

$\frac{d}{dy}[2^y] = 2^y \cdot \ln(2)$

here: $\frac{\partial z}{\partial y} = \boxed{x^y \ln(x)}$

#6b. Find both first partial derivatives of the function $f(x,y) = \int_x^y e^{t^2}\,dt$

f_x (y constant): $\frac{\partial}{\partial x}\int_x^{y} e^{t^2}dt$ (y constant)

$= \frac{\partial}{\partial x}\left(-\int_y^x e^{t^2}dt\right)$

$= \boxed{-e^{x^2}}$

f_y (x constant): $\frac{\partial}{\partial y}\int_x^y e^{t^2}dt = \boxed{e^{y^2}}$

#7b. Find both first partial derivatives of the function $p = ze^{xyz}$

$\frac{\partial p}{\partial x}$: ($ze^{xyz}$, x variable) $\frac{\partial p}{\partial x} = ze^{xyz}\frac{\partial}{\partial x}[xyz]$

$= \boxed{ze^{xyz}(yz)}$

$\frac{\partial p}{\partial y}$: ($ze^{xyz}$, y variable) $\frac{\partial p}{\partial y} = ze^{xyz}\frac{\partial}{\partial y}[xyz]$

$= \boxed{ze^{xyz}(xz)}$

$\frac{\partial p}{\partial z}$: ($ze^{xyz}$, variables) requires product rule

$\frac{\partial p}{\partial z} = z\frac{\partial}{\partial z}[e^{xyz}] + e^{xyz}\frac{\partial}{\partial z}[z]$

$= ze^{xyz}\frac{\partial}{\partial z}[xyz] + e^{xyz}(1)$

$= ze^{xyz}(xy) + e^{xyz}$

$= \boxed{e^{xyz}(xyz+1)}$

#8b. Find $f_z\left(0,0,\frac{\pi}{4}\right)$ for

$f(x,y,z) = \sqrt{\sin^2 x + \sin^2 y + \sin^2 z}$

$f_z(x,y,z) = \frac{1}{2}(\sin^2 x + \sin^2 y + \sin^2 z)^{-1/2}\frac{\partial}{\partial z}[\sin^2 x + \sin^2 y + \sin^2 z]$

$= \frac{1}{2}(\sin^2 x + \sin^2 y + \sin^2 z)^{-1/2}\left(2(\sin z)\cdot\frac{\partial}{\partial z}[\sin z]\right)$

$2\sin z \cdot \cos z$

$= \frac{\sin z \cos z}{\sqrt{\sin^2 x + \sin^2 y + \sin^2 z}}$

$f_z(0,0,\frac{\pi}{4}) = \frac{\sin \pi/4 \cos \pi/4}{\sqrt{\sin^2 0 + \sin^2 0 + \sin^2\frac{\pi}{4}}}$

$= \frac{\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)}{\sqrt{0+0+\left(\frac{\sqrt{2}}{2}\right)^2}} = \frac{\left(\frac{2}{4}\right)}{\frac{\sqrt{2}}{2}}$

$= \frac{1}{2}\cdot\frac{2}{\sqrt{2}} = \boxed{\frac{1}{\sqrt{2}}}$ (or $\frac{\sqrt{2}}{2}$)

#9b. Verify that the conclusion of Clariaut's Theorem hold (that $u_{xy} = u_{yx}$) $u = \ln\left(\sqrt{x^2+y^2}\right)$

$u_x = \frac{1}{\sqrt{x^2+y^2}}\frac{\partial}{\partial x}[(x^2+y^2)^{1/2}]$

$= \frac{1}{\sqrt{x^2+y^2}}\frac{1}{2}(x^2+y^2)^{-1/2}\frac{\partial}{\partial x}(x^2+y^2)$

$= \frac{1}{\sqrt{x^2+y^2}}\frac{1}{2}\frac{1}{\sqrt{x^2+y^2}}(2x) = \frac{x}{x^2+y^2}$

$u_{xy} = \frac{(x^2+y^2)\frac{\partial}{\partial y}[x] - (x)\frac{\partial}{\partial y}[x^2+y^2]}{(x^2+y^2)^2}$

$= \frac{-x(2y)}{(x^2+y^2)^2} = \frac{-2xy}{(x^2+y^2)^2}$

$u_y = \frac{1}{\sqrt{x^2+y^2}}\frac{\partial}{\partial y}[(x^2+y^2)^{1/2}]$

$= \frac{1}{\sqrt{x^2+y^2}}\frac{1}{2}(x^2+y^2)^{-1/2}\frac{\partial}{\partial y}[x^2+y^2]$

$= \frac{1}{\sqrt{x^2+y^2}}\frac{1}{2}\frac{1}{\sqrt{x^2+y^2}}(2y) = \frac{y}{x^2+y^2}$

$u_{yx} = \frac{(x^2+y^2)\frac{\partial}{\partial x}[y] - (y)\frac{\partial}{\partial x}[x^2+y^2]}{(x^2+y^2)^2}$

$= \frac{-y(2x)}{(x^2+y^2)^2} = \frac{-2xy}{(x^2+y^2)^2} = u_{xy}$ ✓

14.4

#1b. Find an equation of the tangent plane to the given surface at the specified point
$z=3(x-1)^2+2(y+3)^2+7,\ (2,-2,12)$.

$z-z_0=f_x(x-x_0)+f_y(y-y_0)$

$f_x=6(x-1)'(1)\Big|_{x=2}=6(2-1)=6$

$f_y=4(y+3)'(1)\Big|_{y=-2}=4(-2+3)=4$

$\boxed{z-12=(6)(x-2)+(4)(y+2)}$

or

$z-12=6x-12+4y+8$

$\boxed{6x+4y-z=-8}$

#2b. Find an equation of the tangent plane to the given surface at the specified point
$z=e^{x^2-y^2},\ (1,-1,1)$.

$z-z_0=f_x(x-x_0)+f_y(y-y_0)$

$f_x=e^{x^2-y^2}\frac{\partial}{\partial x}[x^2-y^2]=e^{x^2-y^2}(2x)\Big|_{(1,-1,1)}$

$=e^{1^2-(-1)^2}2(1)=e^0 2=2$

$f_y=e^{x^2-y^2}\frac{\partial}{\partial y}[x^2-y^2]=e^{x^2-y^2}(-2y)\Big|_{(1,-1,1)}$

$=e^{1^2-(-1)^2}-2(-1)=e^0 2=2$

$\boxed{z-1=(2)(x-1)+(2)(y+1)}$

or

$z-1=2x-2+2y+2$

$\boxed{2x+2y-z=-1}$

#3b. Explain why the function is differentiable at the given point. Then find the linearization $L(x,y)$ of the function at that point.

$f(x,y)=\dfrac{x}{x+y},\ (2,1)$.

rational functions are continuous everywhere in their domain & $(2,1)$ is in domain

$L(x,y)=f(a,b)+f_x(x-a)+f_b(y-b)$

$f(2,1)=\frac{2}{2+1}=\frac{2}{3}$

$f_x=\dfrac{(x+y)\frac{\partial}{\partial x}[x]-x\frac{\partial}{\partial x}[x+y]}{(x+y)^2}=\dfrac{(x+y)(1)-x(1)}{(x+y)^2}\Big|_{(2,1)}$

$=\dfrac{(2+1)(1)-(2)(1)}{(2+1)^2}=\frac{1}{9}$

$f_y=\dfrac{(x+y)\frac{\partial}{\partial y}[x]-x\frac{\partial}{\partial y}[x+y]}{(x+y)^2}=\dfrac{(x+y)(0)-x(1)}{(x+y)^2}\Big|_{(2,1)}$

$=\dfrac{(0)-(2)(1)}{(2+1)^2}=\frac{-2}{9}$

$\boxed{L(x,y)=\frac{2}{3}+\frac{1}{9}(x-2)-\frac{2}{9}(y-1)}$

#4b. Find the differential of the function.
$v=y\cos xy$.

$dz=f_x dx+f_y dy$

$f_x=y(-\sin(xy)\frac{\partial}{\partial x}[xy])=y(-\sin(xy)y)$

$f_y=y\frac{\partial}{\partial y}[\cos(xy)]+\cos(xy)\frac{\partial}{\partial y}[y]$

$=y(-\sin(xy)\frac{\partial}{\partial y}[xy])+\cos(xy)(1)$

$=-y\sin(xy)(x)+\cos(xy)$

$\boxed{dz=(-y^2\sin(xy))dx+(\cos(xy)-xy\sin(xy))dy}$

#5b. Verify the linear approximation at (0,0).

$\sqrt{y+\cos^2 x} \approx 1+\frac{1}{2}y$.

$L(x,y) = f(a,b) + f_x(x-a) + f_y(y-b)$

$f(x,y) = \sqrt{y+\cos^2 x} = (y+(\cos x)^2)^{1/2}$

$f(0,0) = \sqrt{0+\cos^2 0} = \sqrt{0+1^2} = 1$

$f_x = \frac{1}{2}(y+(\cos x)^2)^{-1/2} \frac{\partial}{\partial x}[y+(\cos x)^2]$

$= \frac{1}{2\sqrt{y+\cos^2 x}}\left[0+2\cos x \frac{\partial}{\partial x}[\cos x]\right]$

$= \frac{1}{2\sqrt{y+\cos^2 x}}(2\cos x(-\sin x))$

$= \frac{-\sin x\cos x}{\sqrt{y+\cos^2 x}}\Big|_{(0,0)} = \frac{-\sin 0\cos 0}{\sqrt{0+\cos^2 0}} = \frac{0}{1} = 0$

$f_y = \frac{1}{2}(y+(\cos x)^2)^{-1/2} \frac{\partial}{\partial y}[y+(\cos x)^2]$

$= \frac{1}{2\sqrt{y+\cos^2 x}}[1+0]$

$= \frac{1}{2\sqrt{y+\cos^2 x}}\Big|_{(0,0)} = \frac{1}{2\sqrt{0+\cos^2 0}} = \frac{1}{2}$

$L(x,y) = 1 + 0(x-0) + \frac{1}{2}(y-0)$

$= 1 + \frac{1}{2}y$ ✓

#6b. If $z = x^2 - xy + 3y^2$ and (x, y) changes from $(3,-1)$ to $(2.96,-0.95)$, compare the values of Δz and dz.

$\Delta z = z(2.96,-0.95) - z(3,-1)$

$= ((2.96)^2 - (2.96)(-.95) + 3(-.95)^2)$

$- ((3)^2 - (3)(-1) + 3(-1)^2)$

$= 14.2811 - 15$

$\boxed{\Delta z = -0.7189}$

$dz = f_x dx + f_y dy$

$f_x = 2x - (x\frac{\partial}{\partial x}[y] + y\frac{\partial}{\partial x}[x]) + 0$

$= 2x - x(0) - y(1)$

$= 2x - y\Big|_{(3,-1)} = 2(3) - (-1) = 7$

$f_y = 0 - (x\frac{\partial}{\partial y}[y] + y\frac{\partial}{\partial y}[x]) + 6y$

$= -x(1) - y(0) + 6y$

$= -x + 6y\Big|_{(3,-1)} = -(3) + 6(-1) = -9$

$dx = 2.96 - 3 = -0.04$

$dy = -0.95 - (-1) = 0.05$

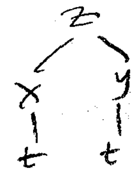
$dz = (7)(-0.04) + (-9)(0.05)$

$\boxed{dz = -0.73}$

14.5

#1b. Use the Chain Rule to find $\frac{dz}{dt}$

$z=\sqrt{1+x^2+y^2}$, $x=\ln t$, $y=\cos t$.



$$\frac{dz}{dt}=\frac{\partial z}{\partial x}\frac{dx}{dt}+\frac{\partial z}{\partial y}\frac{dy}{dt}$$

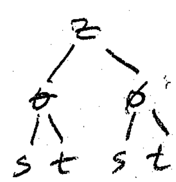
$$\frac{\partial z}{\partial x}=\frac{1}{2}(1+x^2+y^2)^{-1/2}\frac{\partial}{\partial x}(1+x^2+y^2)=\frac{2x}{2\sqrt{1+x^2+y^2}}=\frac{x}{\sqrt{1+x^2+y^2}}$$

$$\frac{\partial z}{\partial y}=\frac{1}{2}(1+x^2+y^2)^{-1/2}\frac{\partial}{\partial y}(1+x^2+y^2)=\frac{2y}{2\sqrt{1+x^2+y^2}}=\frac{y}{\sqrt{1+x^2+y^2}}$$

$$\boxed{\frac{\partial z}{\partial t}=\left(\frac{x}{\sqrt{1+x^2+y^2}}\right)\left(\frac{1}{t}\right)+\left(\frac{y}{\sqrt{1+x^2+y^2}}\right)(-\sin t)}$$

#2b. Use the Chain Rule to find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$

$z=\sin\theta\cos\phi$, $\theta=st^2$, $\phi=s^2t$.



$$\frac{\partial z}{\partial s}=\frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial s}+\frac{\partial z}{\partial \phi}\frac{\partial \phi}{\partial s}$$

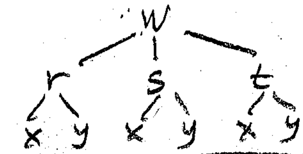
$$\boxed{\frac{\partial z}{\partial s}=(\cos\theta\cos\phi)(t^2)+(-\sin\theta\sin\phi)(2st)}$$

$$\frac{\partial z}{\partial t}=\frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial t}+\frac{\partial z}{\partial \phi}\frac{\partial \phi}{\partial t}$$

$$\boxed{\frac{\partial z}{\partial t}=(\cos\theta\cos\phi)(2st)+(-\sin\theta\sin\phi)(s^2)}$$

#3b. Use a tree diagram to write out the Chain Rule for the given case. Assume all functions are differentiable.

$w=f(r,s,t)$, $r=r(x,y)$, $s=s(x,y)$, $t=t(x,y)$



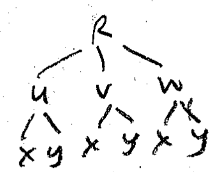
$$\boxed{\frac{\partial w}{\partial x}=\frac{\partial w}{\partial r}\frac{\partial r}{\partial x}+\frac{\partial w}{\partial s}\frac{\partial s}{\partial x}+\frac{\partial w}{\partial t}\frac{\partial t}{\partial x}}$$

$$\boxed{\frac{\partial w}{\partial y}=\frac{\partial w}{\partial r}\frac{\partial r}{\partial y}+\frac{\partial w}{\partial s}\frac{\partial s}{\partial y}+\frac{\partial w}{\partial t}\frac{\partial t}{\partial y}}$$

#4b. Use the Chain Rule to find the indicated partial derivatives

$R=\ln(u^2+v^2+w^2)$, $u=x+2y$, $v=2x-y$, $w=2xy$

$\frac{\partial R}{\partial x}, \frac{\partial R}{\partial y}$ *when* $x=y=1$



$x=1, y=1$

$u=x+2y = (1)+2(1)$, $u=3$

$v=2x-y = 2(1)-1$, $v=1$

$w=2xy = 2(1)(1)$, $w=2$

$$\frac{\partial R}{\partial x}=\frac{\partial R}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial R}{\partial v}\frac{\partial v}{\partial x}+\frac{\partial R}{\partial w}\frac{\partial w}{\partial x}$$

$$=\left(\frac{2u}{u^2+v^2+w^2}\right)(1)+\left(\frac{2v}{u^2+v^2+w^2}\right)(2)+\left(\frac{2w}{u^2+v^2+w^2}\right)(2y)$$

$$=\left(\frac{2(3)}{3^2+1^2+2^2}\right)(1)+\left(\frac{2(1)}{3^2+1^2+2^2}\right)(2)+\left(\frac{2(2)}{3^2+1^2+2^2}\right)(2(1))$$

$$=\frac{6}{14}+\frac{4}{14}+\frac{8}{14}=\boxed{\frac{18}{14}}$$

$$\frac{\partial R}{\partial y}=\frac{\partial R}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial R}{\partial v}\frac{\partial v}{\partial y}+\frac{\partial R}{\partial w}\frac{\partial w}{\partial y}$$

$$=\left(\frac{2u}{u^2+v^2+w^2}\right)(2)+\left(\frac{2v}{u^2+v^2+w^2}\right)(-1)+\left(\frac{2w}{u^2+v^2+w^2}\right)(2x)$$

$$=\left(\frac{2(3)}{3^2+1^2+2^2}\right)(2)+\left(\frac{2(1)}{3^2+1^2+2^2}\right)(-1)+\left(\frac{2(2)}{3^2+1^2+2^2}\right)(2(1))$$

$$=\frac{12}{14}-\frac{2}{14}+\frac{8}{14}=\boxed{\frac{18}{14}}$$

#5b. Use $\frac{dy}{dx} = -\frac{\left(\frac{\partial F}{\partial x}\right)}{\left(\frac{\partial F}{\partial y}\right)} = -\frac{F_x}{F_y}$ to find $\frac{dy}{dx}$.

$y^5 + x^2y^3 = 1 + ye^{x^2}$.

$$F = y^5 + x^2y^3 - 1 - ye^{x^2} \ (=0)$$

$$F_x = 0 + 2xy^3 - 0 - ye^{x^2}(2x)$$

$$= 2xy^3 - 2xye^{x^2}$$

$$F_y = 5y^4 + 3x^2y^2 - 0 - e^{x^2}$$

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = \frac{-(2xy^3 - 2xye^{x^2})}{5y^4 + 3x^2y^2 - e^{x^2}}$$

$$= \frac{2xye^{x^2} - 2xy^3}{5y^4 + 3x^2y^2 - e^{x^2}}$$

$$= \boxed{\frac{2xy(e^{x^2} - y^2)}{5y^4 + 3x^2y^2 - e^{x^2}}}$$

#6b. The pressure of 1 mole of an ideal gas is increasing at a rate of 0.05 kPa/s and the temperature is increasing at a rate of 0.15 K/s. Use the equation $P = 8.31\frac{T}{V}$ to find the rate of change of the volume when the pressure is 20 kPa and the temperature is 320 K.

$$V = 8.31\frac{T}{P} = 8.31\frac{(320)}{20}$$

Find $\frac{dV}{dt} = \frac{\partial V}{\partial T}\frac{\partial T}{\partial t} + \frac{\partial V}{\partial P}\frac{\partial P}{\partial t}$

$$\frac{\partial V}{\partial T} = \frac{8.31}{P} = \frac{8.31}{20} \qquad (V = 8.31TP^{-1})$$

$$\frac{\partial V}{\partial P} = -8.31TP^{-2} = \frac{-8.31T}{P^2} = \frac{-8.31(320)}{(20)^2}$$

$$\frac{\partial V}{\partial t} = \left(\frac{8.31}{20}\right)(0.15) + \left(\frac{-8.31(320)}{(20)^2}\right)(0.05)$$

$$= \boxed{-0.27 \text{ L/s}}$$

#Extra 7. If $z = f(x-y)$, show that $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0$.

Let $t = x - y$, $z = f(t)$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial t}\frac{\partial t}{\partial x} = \frac{\partial z}{\partial t}(1)$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial t}\frac{\partial t}{\partial y} = \frac{\partial z}{\partial t}(-1)$$

$$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = \frac{\partial z}{\partial t}(1) + \frac{\partial z}{\partial t}(-1)$$

$$= 0 \checkmark$$

Ch 14 Part 1 Test Review

#1. Draw a contour map of the function showing several level curves: $f(x,y)=(y-2x)^2$.

$(y-2x)^2=0$

$y-2x=0$

$y=2x$

$(y-2x)^2=1$

$y-2x=\pm 1$

$y=2x\pm 1$

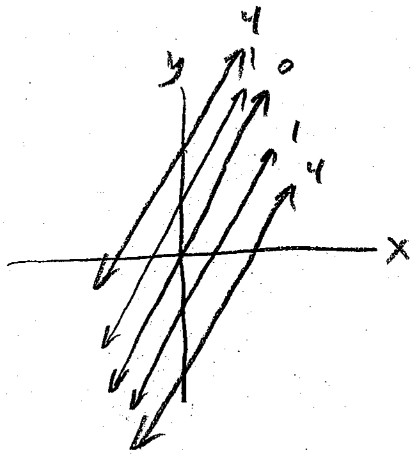
$(y-2x)^2=4$

$y-2x=\pm 2$

$y=2x\pm 2$

$(y-2x)^2=-1$

(not possible)



#2. Draw a contour map of the function showing several level curves: $f(x,y)=x^3-y$.

$x^3-y=0$

$y=x^3$

$x^3-y=1$

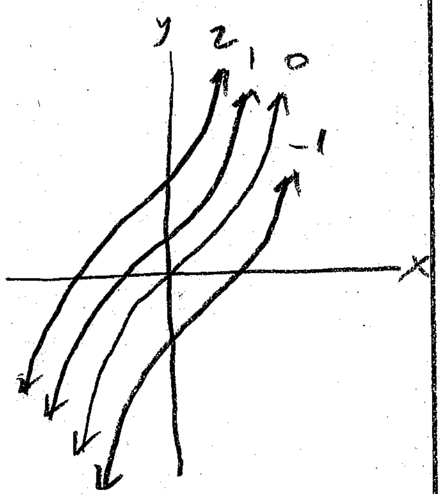
$y=x^3+1$

$x^3-y=2$

$y=x^3+2$

$x^3-y=-1$

$y=x^3-1$



#3. Find the limit, if it exists, or show that the limit does not exist $\lim_{(x,y)\to(0,0)} \frac{y^4}{x^4+3y^4}$. $(0,0)$ not in domain

y-axis $(x=0)$:

$\lim_{(x,y)\to(0,0)} \frac{y^4}{(0)^4+3y^4} = \lim_{y\to 0} \frac{y^4}{3y^4} = \lim_{y\to 0} \frac{1}{3} = \boxed{\frac{1}{3}}$

x-axis $(y=0)$:

$\lim_{(x,y)\to(0,0)} \frac{(0)^4}{x^4+3(0)^4} = \lim_{x\to 0} \frac{0}{x^4} = \boxed{0}$

$\frac{1}{3} \neq 0$

So limit $\boxed{\text{DNE}}$

#4. Find the limit, if it exists, or show that the limit does not exist $\lim_{(x,y)\to(0,0)} \frac{6x^3y}{2x^4+y^4}$. $(0,0)$ not in domain

y-axis $(x=0)$

$\lim_{(x,y)\to(0,0)} \frac{6(0)^3y}{2(0)^4+y^4} = \lim_{y\to 0} \frac{0}{y^4} = \boxed{0}$

x-axis $(y=0)$:

$\lim_{(x,y)\to(0,0)} \frac{6x^3(0)}{2x^4+(0)^4} = \lim_{x\to 0} \frac{0}{2x^4} = \boxed{0}$

lines ($y=mx+b$ through $(0,0)$: $0=m(0)+b,\ b=0$)

$y=mx$

$\lim_{(x,y)\to(0,0)} \frac{6x^3(mx)}{2x^4+(mx)^4} = \lim_{x\to 0} \frac{x^4(6m)}{x^4(2+m^4)}$

$= \lim_{x\to 0} \frac{6m}{2+m^4}$ $\boxed{\neq 0}$

So limit $\boxed{\text{DNE}}$

#5. Find both first partial derivatives of the function $f(x,y) = y^5 - 3xy$

$f_x = 0 - 3(1)y = \boxed{-3y}$

$f_y = 5y^4 - 3x(1) = \boxed{5y^4 - 3x}$

#6. Find both first partial derivatives of the function $f(x,y) = x^4y^3 + 8x^2y$

$f_x = \boxed{4x^3y^3 + 16xy}$

$f_y = x^4(3y^2) + 8x^2(1)$

$= \boxed{3x^4y^2 + 8x^2}$

#7. Find both first partial derivatives of the function $u = te^{(w/t)}$

$\frac{\partial u}{\partial t}$: $u = te^{(\frac{w}{t})}$

2 variable requires product rule

$\frac{\partial u}{\partial t} = t\frac{\partial}{\partial t}[e^{(\frac{w}{t})}] + e^{(\frac{w}{t})}\frac{\partial}{\partial t}[t]$

$= t\left(e^{(\frac{w}{t})}\frac{\partial}{\partial t}[\frac{w}{t}]\right) + e^{\frac{w}{t}}(1)$

$\frac{\partial}{\partial t}[wt^{-1}]$

$= te^{(\frac{w}{t})}(-wt^{-2}) + e^{(\frac{w}{t})}$

$= \frac{-twe^{(w/t)}}{t^2} + e^{(w/t)}$

$= \boxed{-\frac{we^{(w/t)}}{t} + e^{(w/t)}}$

all correct

$\frac{\partial u}{\partial w}$: $u = te^{(w/t)}$

one variable (no product rule needed)

$\frac{\partial u}{\partial w} = t\frac{\partial}{\partial w}[e^{(w/t)}]$

$= te^{(w/t)}\frac{\partial}{\partial w}[\frac{1}{t}w]$

$= te^{(w/t)}(\frac{1}{t})$

$= \boxed{e^{(w/t)}}$

#8. Find all the second partial derivatives of $f(x,y) = x^3y^5 + 2x^4y$

$f_x = 3x^2y^5 + 8x^3y$

$\boxed{f_{xx} = 6xy^5 + 24x^2y}$

$f_{xy} = 3x^2(5y^4) + 8x^3(1)$

$\boxed{f_{xy} = 15x^2y^4 + 8x^3}$

$f_y = 5x^3y^4 + 2x^4$

$f_{yy} = 5x^3(4y^3) + 0$

$\boxed{f_{yy} = 20x^3y^3}$

$f_{yx} = 5(3x^2)y^4 + 8x^3$

$\boxed{f_{yx} = 15x^2y^4 + 8x^3}$

#9. Find an equation of the tangent plane to the given surface at the specified point
$z = 4x^2 - y^2 + 2y, \ (-1, 2, 4)$

$z - z_0 = f_x(x - x_0) + f_y(y - y_0)$

$f_x = \frac{\partial z}{\partial x} = 8x \Big|_{(-1,2,4)} = 8(-1) = -8$

$f_y = \frac{\partial z}{\partial y} = -2y + 2 \Big|_{(-1,2,4)} = -2(2) + 2 = -2$

$\boxed{z - 4 = -8(x+1) - 2(y-2)}$

#10. Find an equation of the tangent plane to the given surface at the specified point
$z = 3(x-1)^2 + 2(y+3)^2 + 7, \ (2, -2, 12)$

$z - z_0 = f_x(x - x_0) + f_y(y - y_0)$

$f_x = \frac{\partial z}{\partial x} = 3(2(x-1)^1(1)) \Big|_{(2,-2,12)} = 3(2)(2-1) = 6$

$f_y = \frac{\partial z}{\partial y} = 2(2(y+3)^1(1)) \Big|_{(2,-2,12)} = 2(2)(-2+3) = 4$

$\boxed{z - 12 = 6(x-2) + 4(y+2)}$

#11. Find the linear approximation of the function $f(x,y)=\ln(x-3y)$ at $(7,2)$ and use it to approximate $f(6.9,\ 2.06)$.

$f(x,y) \approx f(a,b) + f_x(x-a) + f_y(y-b)$

$f(7,2) = \ln(7-3(2)) = \ln(1) = 0$

$f_x = \frac{1}{x-3y}\frac{\partial}{\partial x}[x-3y] = \frac{1}{x-3y}(1) = \frac{1}{x-3y}$

at $(7,2)$ $f_x = \frac{1}{7-3(2)} = 1$

$f_y = \frac{1}{x-3y}\frac{\partial}{\partial y}[x-3y] = \frac{1}{x-3y}(-3) = \frac{-3}{x-3y}$

at $(7,2)$ $f_y = \frac{-3}{(7)-3(2)} = -3$

$f(x,y) \approx 0 + (1)(x-7) - 3(y-2)$

$f(6.9, 2.06) \approx 0 + (1)(6.9-7) - 3(2.06-2)$

$= \boxed{-0.28}$

#12. Find the differential of the function $z = x^3 \ln(y^2)$.

$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dx$

$\frac{\partial z}{\partial x} = (3x^2)\ln(y^2)$

$\frac{\partial z}{\partial y} = (x^3)\frac{1}{y^2}(2y) = \frac{2x^3}{y}$

$\boxed{dz = (3x^2\ln(y^2))dx + \left(\frac{2x^3}{y}\right)dy}$

#13. Find the differential of the function $v = y\cos xy$.

$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$

$\frac{\partial v}{\partial x} = y(-\sin(xy)\frac{\partial}{\partial x}[xy])$

$= -y\sin(xy)y = -y^2\sin(xy)$

$\frac{\partial v}{\partial y}$ requires product rule:

$\frac{\partial v}{\partial y} = y\frac{\partial}{\partial y}[\cos(xy)] + \cos(xy)\frac{\partial}{\partial y}[y]$

$= y(-\sin(xy)\frac{\partial}{\partial y}[xy]) + \cos(xy)(1)$

$= -y\sin(xy)(x) + \cos(xy)$

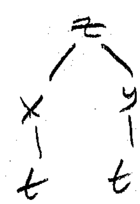
$= -xy\sin(xy) + \cos(xy)$

$\boxed{dv = (-y^2\sin(xy))dx + (-xy\sin(xy) + \cos(xy))dy}$

#14. Use the Chain Rule to find $\frac{dz}{dt}$

$z = \sqrt{1+x^2+y^2}$, $x = \ln t$, $y = \cos t$.

$(1+x^2+y^2)^{1/2}$



$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$

$\frac{\partial z}{\partial x} = \frac{1}{2}(1+x^2+y^2)^{-1/2}\frac{\partial}{\partial x}[1+x^2+y^2]$

$= \frac{1}{2\sqrt{1+x^2+y^2}}(2x) = \frac{x}{\sqrt{1+x^2+y^2}}$

$\frac{\partial z}{\partial y} = \frac{1}{2}(1+x^2+y^2)^{-1/2}\frac{\partial}{\partial y}(1+x^2+y^2)$

$= \frac{1}{2\sqrt{1+x^2+y^2}}(2y) = \frac{y}{\sqrt{1+x^2+y^2}}$

$\frac{dx}{dt} = \frac{1}{t}$

$\frac{dy}{dt} = -\sin t$

$\boxed{\frac{dz}{dt} = \left(\frac{x}{\sqrt{1+x^2+y^2}}\right)\left(\frac{1}{t}\right) + \left(\frac{y}{\sqrt{1+x^2+y^2}}\right)(-\sin t)}$

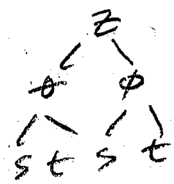
#15. Use the Chain Rule to find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$.

$z = \sin\theta\cos\phi$, $\theta = st^2$, $\phi = s^2 t$.

$\frac{\partial z}{\partial \theta} = \cos\theta\cos\phi \quad \frac{\partial z}{\partial \phi} = -\sin\theta\sin\phi$

$\frac{\partial \theta}{\partial s} = t^2 \quad \frac{\partial \theta}{\partial t} = 2st$

$\frac{\partial \phi}{\partial s} = 2st \quad \frac{\partial \phi}{\partial t} = s^2$



$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial s} + \frac{\partial z}{\partial \phi}\frac{\partial \phi}{\partial s}$

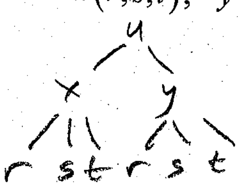
$\boxed{\frac{\partial z}{\partial s} = (\cos\theta\cos\phi)(t^2) + (-\sin\theta\sin\phi)(2st)}$

$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial t} + \frac{\partial z}{\partial \phi}\frac{\partial \phi}{\partial t}$

$\boxed{\frac{\partial z}{\partial t} = (\cos\theta\cos\phi)(2st) + (-\sin\theta\sin\theta)(s^2)}$

#16. Use a tree diagram to write out the Chain Rule for the given case. Assume all functions are differentiable.

$u = f(x, y)$, $x = x(r, s, t)$, $y = y(r, s, t)$.



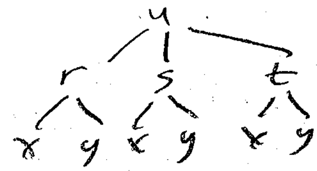
$\boxed{\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}}$

$\boxed{\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s}}$

$\boxed{\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t}}$

#17. Use a tree diagram to write out the Chain Rule for the given case. Assume all functions are differentiable.

$u = f(r, s, t)$, $r = r(x, y)$, $s = s(x, y)$, $t = t(x, y)$



$\boxed{\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial x}}$

$\boxed{\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial y}}$