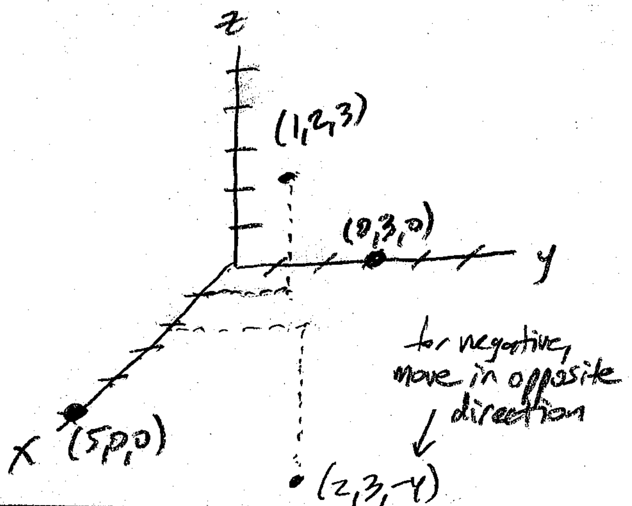
**Calc III - Ch 12 - Extra Practice** $\boxed{\text{SOLUTIONS}}$

#1b. On a single set of coordinate axes, sketch the points (1,2,3), (5,0,0), (0,3,0), and (2,3,-4).



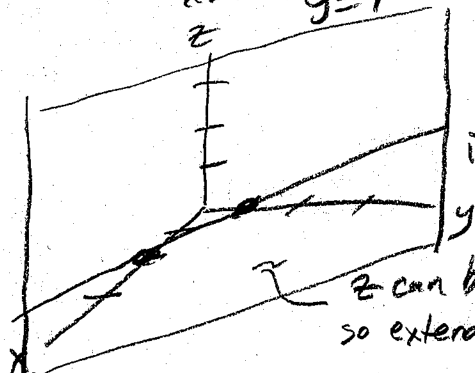

for negative, move in opposite direction

#2b. Describe and sketch the surface in $\mathbb{R}^3$ represented by the equation $2x+4y=4$.

a plane : find intercepts

$2x=4 \quad 4y=4$

$x=2 \quad y=1$




draw line in xy plane

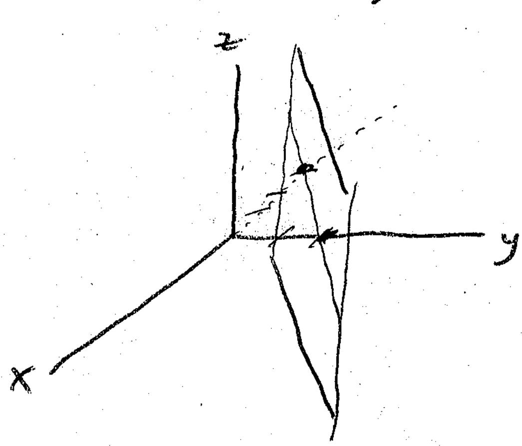
z can be anything so extend up/down

#2c. Describe and sketch the surface in $\mathbb{R}^3$ represented by the equation $-2x+3y=6$.

a plane : find intercepts

$-2x=6 \quad 3y=6$

$x=-3 \quad y=2$

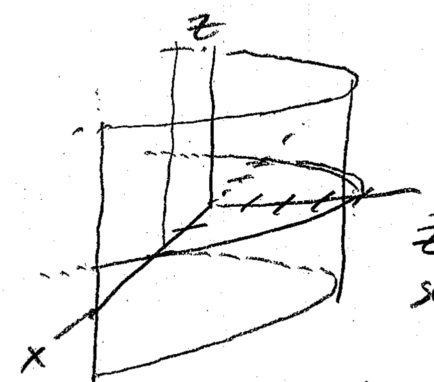



#2d. Describe and sketch the surface in $\mathbb{R}^3$ represented by the equation $2x^2+2y=8$.

Solve for equation in xy plane: $x^2+y=4$

$y=4-x^2$

is a parabola on xy plane




z can be anything so extend

a parabolic cylinder

#2e. Describe and sketch the surface in $\mathbb{R}^3$ represented by the equation $y^2+4z^2-2y-16z+13=0$.

(complete the square)

$y^2-2y+4z^2-16z=-13$

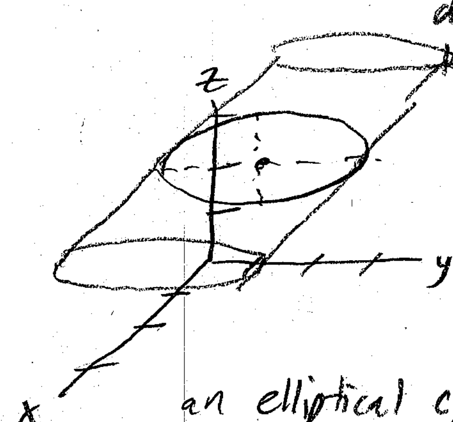
$(y^2-2y+1)+4(z^2-4z+4)=-13+1+16$

$(y-1)^2+4(z-2)^2=4$

$\frac{(y-1)^2}{4}+\frac{(z-2)^2}{1}=1$

an ellipse on yz-plane with center (1,2)

dist. from center to edges: 2 in y, 1 in z




an elliptical cylinder

#3b. Find the lengths of the sides of the triangle $PQR$. Is it a right triangle? Is it an isosceles triangle? $P(2,-1,0)$, $Q(4,1,1)$, $R(4,-5,4)$.

$|PQ| = \sqrt{(4-2)^2+(1+1)^2+(1-0)^2} = \sqrt{9} = 3$

$|PR| = \sqrt{(4-2)^2+(-5+1)^2+(4-0)^2} = \sqrt{36} = 6$

$|RQ| = \sqrt{(4-0)^2+(1+5)^2+(1-4)^2} = \sqrt{61}$

no sides congruent, $\boxed{\text{not isosceles}}$

right? $a^2+b^2 \overset{?}{=} c^2$

$(3)^2+(6)^2 \overset{?}{=} (\sqrt{61})^2$

$9+36 \overset{?}{=} 61$

$45 \neq 61$

$\boxed{\text{not a right triangle}}$

#5b. Show that the equation represents a sphere, and find its center and radius: $4x^2+4y^2+4z^2-8x+16y=1$

(complete the squares)

$4x^2+8x \; +4y^2+16y \; +4z^2 = 1$

$4(x^2+2x+\underline{1}) + 4(y^2+4y+\underline{4}) + 4(z-0)^2 = 1+\underline{4}+\underline{16}$

$4(x+1)^2+4(y+2)^2+4(z-0)^2 = 21$

$(x+1)^2+(y+2)^2+4(z-0)^2 = \frac{21}{4}$

is a sphere

with center: $(-1,-2,0)$

radius: $\sqrt{\frac{21}{4}} = \frac{\sqrt{21}}{2}$

#4b. Find an equation of the sphere with center $(2,-6,4)$ and radius 5. What is the intersection of this sphere with the $yz$-plane?

$\boxed{(x-2)^2+(y+6)^2+(z-4)^2 = 25}$

$yz$ plane: $x=0$

intersection is solution of a system:

$\begin{cases} x=0 \\ (x-2)^2+(y+6)^2+(z-4)^2=25 \end{cases}$

$(0-2)^2+(y+6)^2+(z-4)^2=25$

$4+(y+6)^2+(z-4)^2=25$

$(y+6)^2+(z-4)^2=21$

is a circle in $yz$-plane
centered at $(-6,4)$
$y \quad z$
with a radius of $\sqrt{21}$

(try the 2 system equations in geogebra 3D grapher to see this)

#6b. Find an equation of a sphere if one of its diameters has endpoints $(5,2,0)$ and $(3,8,8)$.

center = midpt = $\left(\frac{5+3}{2}, \frac{2+8}{2}, \frac{0+8}{2}\right)$

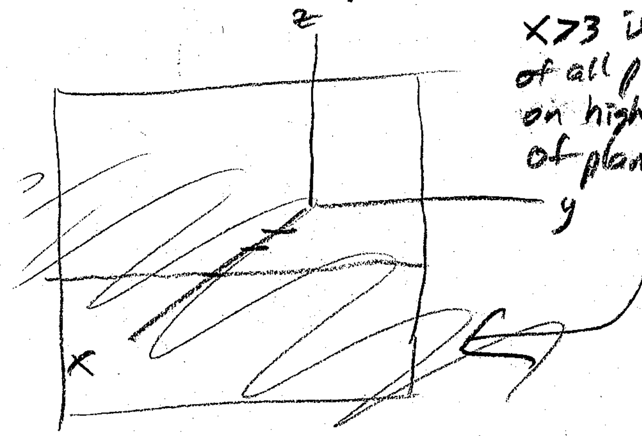
$= (4,5,4)$

radius = ½ length = $\frac{1}{2}\sqrt{(5-3)^2+(2-8)^2+(0-8)^2} = \frac{\sqrt{104}}{2}$

$\boxed{(x-4)^2+(y-5)^2+(z-4)^2 = \frac{104}{4}}$

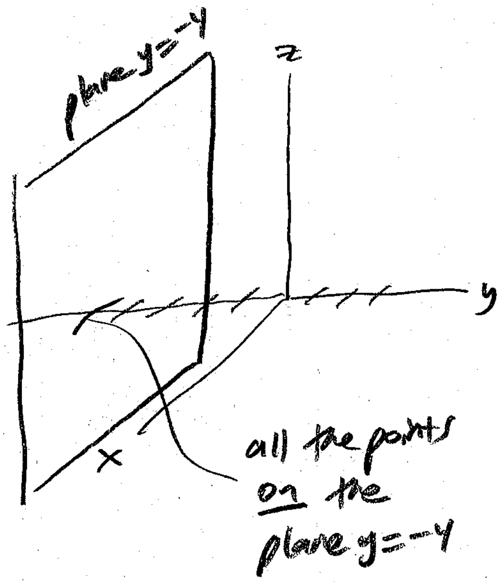
#7b. Describe in words the region of $\mathbb{R}^3$ represented by the equation or inequality: $x>3$.

$x=3$ is a plane!




$x>3$ is the set of all points on higher $x$ side of plane $x=3$

#7c. Describe in words the region of $\mathbb{R}^3$ represented by the equation or inequality: $y = -4$.




#7d. Describe in words the region of $\mathbb{R}^3$ represented by the equation or inequality: $x^2 + y^2 + z^2 > 2z$.

make equal 1st...

$x^2+y^2+z^2=2z$

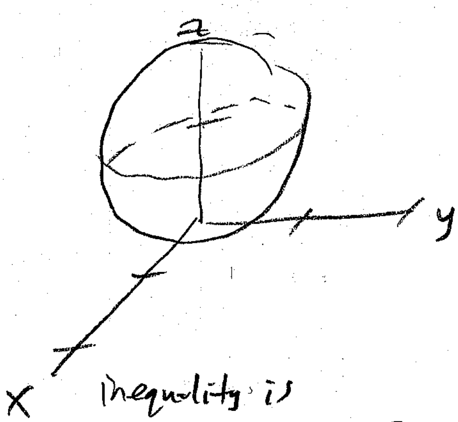
$x^2+y^2+z^2-2z=0$

(complete the square)

$x^2+y^2+(z^2-2z+1) = 0+1$

$(x-0)^2+(y-0)^2+(z-1)^2=1$

is a radius 1 sphere centered at $(0,0,1)$




inequality is

$\underbrace{(x-0)^2+(y-0)^2+(z-1)^2}>1$

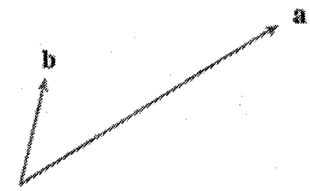
gen'l: $(x-h)^2+(y-k)^2+(z-l)^2=r^2$

so this means points with radius $>1$

all points <u>outside</u> this sphere

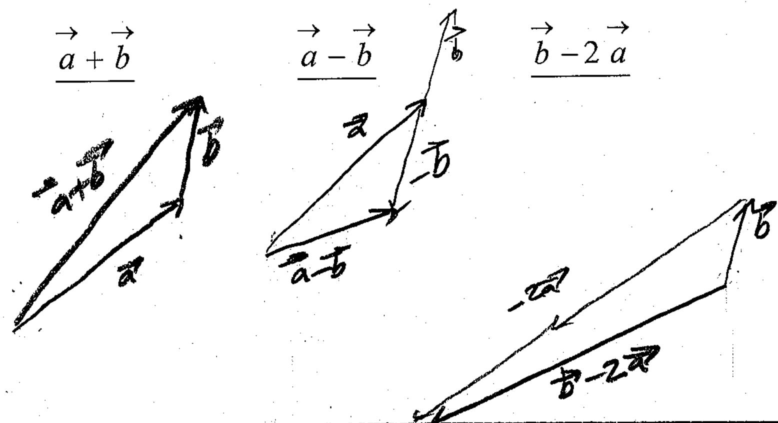
## 12.2

#1b. For the given vectors $\vec{a}$ and $\vec{b}$:




sketch each of the following:

$\vec{a}+\vec{b}$ $\quad$ $\vec{a}-\vec{b}$ $\quad$ $\vec{b}-2\vec{a}$

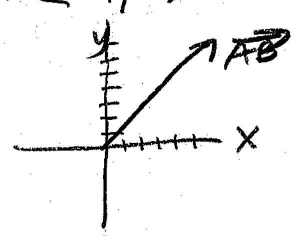



#2b. Find the vector $\vec{AB}$ if $A(-2,-2)$, $B(5,3)$.

Then sketch $\vec{AB}$ starting at the origin.

$\vec{AB} = \langle 5-(-2), 3-(-2)\rangle$

$= \langle 7, 5\rangle$

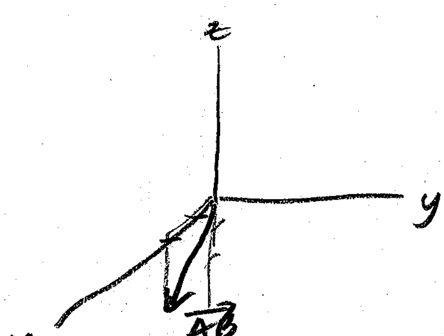



#2c. Find the vector $\vec{AB}$ if $A(0,3,1)$, $B(2,3,-1)$.

Then sketch $\vec{AB}$ starting at the origin.

$\vec{AB} = \langle 2-0, 3-3, -1-1\rangle$

$= \langle 2, 0, -2\rangle$



#3b. Find a unit vector that has the same direction as the given vector $-3\vec{i}+7\vec{j}$.

$\vec{a}=\langle -3,7\rangle$

$|\vec{a}|=\sqrt{3^2+7^2}=\sqrt{58}$

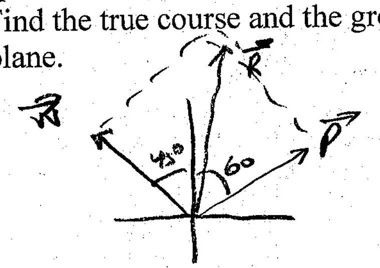
$\vec{u}=\frac{\vec{a}}{|\vec{a}|}=\boxed{\frac{1}{\sqrt{58}}\langle -3,7\rangle \text{ or } \left\langle \frac{-3}{\sqrt{58}}, \frac{7}{\sqrt{58}}\right\rangle}$

#3c. Find a unit vector that has the same direction as the given vector $\langle -4,2,4\rangle$.

$|\vec{a}|=\sqrt{4^2+2^2+4^2}=\sqrt{36}=6$

$\vec{u}=\frac{\vec{a}}{|\vec{a}|}=\boxed{\frac{1}{6}\langle -4,2,4\rangle \text{ or } \left\langle -\frac{4}{6}, \frac{2}{6}, \frac{4}{6}\right\rangle \text{ or } \left\langle -\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right\rangle}$

#4b. Suppose that a wind is blowing from the direction N45°W at a speed of 50 km/hr (this mean that the direction the wind pushes the plane is 45° west of the northerly direction). A pilot is steering a plane in the direction N60°E at an airspeed (speed in still air) of 250 km/hr. The true course of the plane is the direction of the resultant of the velocity vectors of the plane and the wind. The ground speed of the plan is the magnitude of the resultant. Find the true course and the ground speed of the plane.




wind $\vec{W}=\langle 50\cos 135°, 50\sin 135°\rangle$

plane $\vec{P}=\langle 250\cos 40°, 250\sin 40°\rangle$

resultant $\vec{R}=\vec{W}+\vec{P}$

$\vec{R}=\langle 50\cos 135°, 50\sin 135°\rangle+\langle 250\cos 40°, 250\sin 40°\rangle$

$=\langle 156.156, 196.052\rangle$

38.54° 51.46°

$\tan\theta=\frac{196.052}{156.156}$ $\theta=\tan^{-1}\left(\frac{196.052}{156.156}\right)=51.46°$

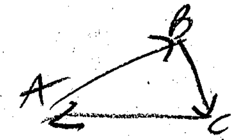
bearing (direction) $=\boxed{N\,38.54°E}$

Speed $=\sqrt{156.156^2+196.052^2}=\boxed{250.64 \text{ km/hr}}$

#5b. (For an example, please look at the example worked in the lesson notes.)

#6b. (This problem is setup and solved to the point where there is a system to solve in the lesson notes. From this point, look at the 'star' example to see how to solve the system.

#7. If $A$, $B$, and $C$ are the vertices of a triangle, find $\vec{AB}+\vec{BC}+\vec{CA}$.




$\vec{AB}+\vec{BC}+\vec{CA}=\boxed{\vec{0}}$ (back to start)

**12.3**

#1b. Which of the following expressions are meaningful and which are meaningless?

| $\vec{a}\bullet\left(\vec{b}+\vec{c}\right)$ | $\vec{a}\bullet\vec{b}+\vec{c}$ | $\left\|\vec{a}\right\|\bullet\left(\vec{b}\bullet\vec{c}\right)$ |
|---|---|---|
| vector • vector | 1st (multiply before add) | scalar • scalar |
| scalar | scalar + vector | only for vectors |
| meaningful | meaningless | meaningless |

#2b. Find $\vec{a}\bullet\vec{b}$ if $\vec{a}=\langle -2,3\rangle$, $\vec{b}=\langle 3,5\rangle$

$(-2)(3)+(3)(5)$

$-6+15$

$\boxed{9}$

#3b. Find $\vec{a}\bullet\vec{b}$ if $\vec{a}=\langle s,2t,3\rangle$, $\vec{b}=\langle t^2,5,s^3\rangle$

$(s)(t^2)+(2t)(5)+(3)(s^3)$

$\boxed{st^2+10t+3s^3}$

#4b. Find $\vec{a}\bullet\vec{b}$ if $\vec{a}=4\vec{j}-3\vec{k}$, $\vec{b}=2\vec{i}+4\vec{j}+6\vec{k}$

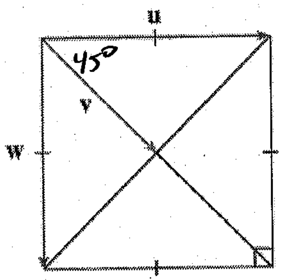
$\langle 0,4,-3\rangle$ $\langle 2,4,6\rangle$

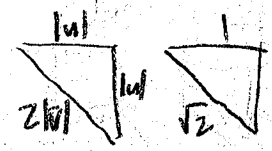
$(0)(2)+(4)(4)+(-3)(6)$

$0+16-18$

$\boxed{-2}$

#5b. If $\vec{u}$ is a unit vector, find $\vec{u} \bullet \vec{v}$ and $\vec{u} \bullet \vec{w}$.




$$\frac{2|v|}{\sqrt{2}} = \frac{|u|}{1} = \frac{1}{1}$$

$$|\vec{v}| = \frac{\sqrt{2}}{2}$$

$$\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos\theta = (1)\left(\frac{\sqrt{2}}{2}\right)\cos 45^\circ$$

$$= \frac{\sqrt{2}}{2}\frac{\sqrt{2}}{2} = \boxed{\frac{1}{2}}$$

$$\vec{u}\cdot\vec{w} = |\vec{u}||\vec{w}|\cos\theta = (1)(1)\cos 90^\circ$$

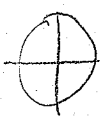
$$= (1)(1)\,0$$

$$= \boxed{0}$$

#6b. Find the angle between the vectors (exact and decimal): $\vec{a} = \langle \sqrt{3}, 1 \rangle$, $\vec{b} = \langle 0, 5 \rangle$

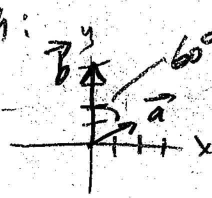
$$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta \qquad \cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$$

$$\cos\theta = \frac{\langle\sqrt{3},1\rangle\cdot\langle 0,5\rangle}{\sqrt{(\sqrt{3})^2+1^2}\sqrt{0^2+5^2}} = \frac{5}{(2)(5)} = \frac{1}{2}$$



$$\theta = \cos^{-1}\left(\frac{1}{2}\right) = \boxed{60^\circ \text{ or } \frac{\pi}{3}}$$

check w/sketch:



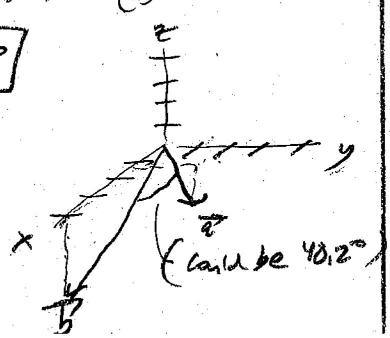

60° seems reasonable (might be other angles from unit circle)

#7b. Find the angle between the vectors (exact and decimal): $\vec{a} = \vec{i} + 2\vec{j} - 2\vec{k}$, $\vec{b} = 4\vec{i} - 3\vec{k}$

$\langle 1,2,-2\rangle \qquad \langle 4,0,-3\rangle$

$$\cos\theta = \frac{\langle 1,2,-2\rangle\cdot\langle 4,0,-3\rangle}{\sqrt{1^2+2^2+2^2}\sqrt{4^2+0^2+3^2}} = \frac{4+6}{(3)(5)} = \frac{10}{15} = \frac{2}{3}$$

$$\boxed{\theta = \cos^{-1}\left(\frac{2}{3}\right) = 48.2^\circ}$$




#8b. Determine whether the given vectors are orthogonal, parallel, or neither:

$\vec{a} = \langle -3, 9, 6 \rangle$, $\vec{b} = \langle 4, -12, -8 \rangle$

$$\langle -3,9,6\rangle\cdot\langle 4,-12,-8\rangle = -12 - 108 - 48 = -168 \neq 0$$

not orthogonal

is one a scaled version of other?

$$\frac{4}{-3} \stackrel{?}{=} \frac{-12}{9} \stackrel{?}{=} \frac{-8}{6}$$

$-1.3 = -1.3 = -1.3$ yes, so $\boxed{\text{parallel}}$

$\vec{a} = \vec{i} - \vec{j} + 2\vec{k}$, $\vec{b} = 2\vec{i} - \vec{j} + \vec{k}$

$$\langle 1,-1,2\rangle\cdot\langle 2,-1,1\rangle = 2+1+2 = 5 \neq 0$$

not orthogonal

$$\frac{2}{1} \stackrel{?}{=} \frac{-1}{-1} \stackrel{?}{=} \frac{1}{2}$$

no, not parallel so $\boxed{\text{neither}}$

$\vec{a} = \langle x, y, z \rangle$, $\vec{b} = \langle -y, x, 0 \rangle$

$$\langle x,y,z\rangle\cdot\langle -y,x,0\rangle = -xy + xy + 0$$

$$= 0$$

so $\boxed{\text{orthogonal}}$

(for all x, y, z)

#9b. Hint: vectors are orthogonal when their dot product is zero.

#10b. Hint: Two ways to solve:

1) graph in 3D and just figure it out visually.

2) postulate a general vector $\langle a,b,c \rangle$ and use dot product equals zero with the given vectors to find a,b,c.

#11b. Find the work done by a force $\vec{F} = 8\vec{i} - 6\vec{j} + 0\vec{k}$ that moves an object from the point $(0,10,8)$ to the point $(6,12,20)$ along a straight line. The distance is measured in meters and the force in Newtons.

$\vec{D} = \langle 6-0,\ 12-10,\ 20-8 \rangle$
$= \langle 6, 2, 12 \rangle$
$\text{work} = \vec{F} \cdot \vec{D} = \langle 8, -6, 0 \rangle \cdot \langle 6, 2, 12 \rangle$
$= 48 - 12 + 0$
$= \boxed{36\ \text{N-m}}$

**12.4**

#1b. Find the cross product $\vec{a} \times \vec{b}$ and verify that it is orthogonal to both $\vec{a}$ and $\vec{b}$.

$\vec{a} = \langle 6, 0, -2 \rangle,\ \vec{b} = \langle 0, 8, 0 \rangle$

$\vec{a} \times \vec{b} = \begin{vmatrix} + & - & + \\ 6 & 0 & -2 \\ 0 & 8 & 0 \end{vmatrix} = \langle 0-(-16), -(0-0), 48-0 \rangle$
$= \langle 16, 0, 48 \rangle$

$(\vec{a} \times \vec{b}) \cdot \vec{a} = \langle 16, 0, 48 \rangle \cdot \langle 6, 0, -2 \rangle$
$= (16)(6) + (0)(0) + (48)(-2)$
$= 96 + 0 - 96 = 0$ (orthogonal ✓)

$(\vec{a} \times \vec{b}) \cdot \vec{b} = \langle 16, 0, 48 \rangle \cdot \langle 0, 8, 0 \rangle$
$= (16)(0) + (0)(8) + (48)(0)$
$= 0 + 0 + 0 = 0$ (orthogonal ✓)

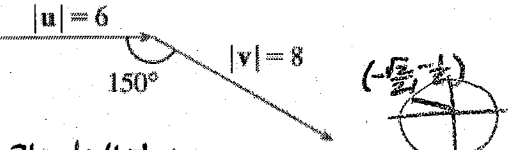
#2b. Which of the following expressions are meaningful and which are meaningless? If meaningful, state whether the result is a vector or a scalar.

$(\vec{a} \bullet \vec{b}) \times \vec{c}$ scalar x vector $\boxed{\text{meaningless}}$

$(\vec{a} \bullet \vec{b}) \times (\vec{c} \bullet \vec{d})$ scalar x scalar $\boxed{\text{meaningless}}$

$(\vec{a} \times \vec{b}) \bullet (\vec{c} \times \vec{d})$ vector • vector
$= \boxed{\text{scalar}}$ (meaningful)

#3b. Find $|\vec{u} \times \vec{v}|$ and determine whether $\vec{u} \times \vec{v}$ is directed into the page or out of the page.




$|\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}|\sin\theta$
$= |(6)(8)\sin 150°| = |(6)(8)(-\tfrac{1}{2})| = |-24|$
$= \boxed{24}$

$\boxed{\text{Into the page}}$ (right hand rule)

#4b. If $\vec{a} = \langle 2,3,1 \rangle$ and $\vec{b} = \langle 1,-2,4 \rangle$, find $\vec{a} \times \vec{b}$ and $\vec{b} \times \vec{a}$.

$\vec{a} \times \vec{b} = \begin{vmatrix} + & - & + \\ 2 & 3 & 1 \\ 1 & -2 & 4 \end{vmatrix} = \langle 12-(-2), -(8-1), -4-3 \rangle$
$= \boxed{\langle 14, -7, -7 \rangle}$

$\vec{b} \times \vec{a} = \begin{vmatrix} + & - & + \\ 1 & -2 & 4 \\ 2 & 3 & 1 \end{vmatrix} = \langle -2-12, -(1-8), 3-(-4) \rangle$
$= \boxed{\langle -14, 7, 7 \rangle}$

#5b. Show that $(\vec{a} \times \vec{b}) \bullet \vec{b} = 0$ for all vectors $\vec{a}$ and $\vec{b}$ in $\mathbb{R}^3$.

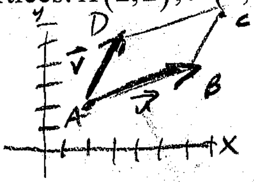
$\vec{a} = \langle a_1, a_2, a_3 \rangle$
$\vec{b} = \langle b_1, b_2, b_3 \rangle$

$\vec{a} \times \vec{b} = \begin{vmatrix} + & - & + \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

$= \langle a_2b_3 - a_3b_2, -(a_1b_3 - a_3b_1), a_1b_2 - a_2b_1 \rangle$

$(\vec{a} \times \vec{b}) \cdot \vec{b} =$
$\langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle \cdot \langle b_1, b_2, b_3 \rangle$
$= (a_2b_3 - a_3b_2)(b_1) + (a_3b_1 - a_1b_3)(b_2) + (a_1b_2 - a_2b_1)(b_3)$
$= a_2b_1b_3 - a_3b_1b_2 + a_3b_1b_2 - a_1b_2b_3 + a_1b_2b_3 - a_2b_1b_3$
$= 0$ ✓

#6b. Find the area of the parallelogram with vertices: $A(2,2)$, $B(6,3)$, $C(7,6)$, and $D(3,5)$.




make vectors 3D by adding $z=0$

then area $|\vec{u} \times \vec{v}|$

$\vec{u} = \vec{AB} = \langle 6-2,\ 3-2,\ 0 \rangle$ add this

$= \langle 4, 1, 0 \rangle$

$\vec{v} = \vec{AB} = \langle 3-2, 5-2, 0 \rangle$

$= \langle 1, 3, 0 \rangle$

area $= |\vec{u} \times \vec{v}| = \begin{vmatrix} + & - & + \\ 4 & 1 & 0 \\ 1 & 3 & 0 \end{vmatrix}$

$= |\langle 0-0, -(0-0), 12-1 \rangle|$

$= |\langle 0, 0, 11 \rangle|$

$= \boxed{11}$

#7b. (i) Find a nonzero vector orthogonal to the plane through the given points: $P(2,1,5)$, $Q(-1,3,4)$, $R(3,0,6)$.
(ii) Find the area of triangle $PQR$.

(i) $\vec{PQ} = \langle -1-2, 3-1, 4-5 \rangle$

$= \langle -3, 2, -1 \rangle$

$\vec{PR} = \langle 3-2, 0-1, 6-5 \rangle$

$= \langle 1, -1, 1 \rangle$

$\vec{PQ} \times \vec{PR} = \begin{vmatrix} + & - & + \\ -3 & 2 & -1 \\ 1 & -1 & 1 \end{vmatrix}$

$= \langle 2-1, -(-3+1), 3-2 \rangle$

$= \boxed{\langle 1, 2, 1 \rangle}$

(ii) $A = \frac{1}{2} |\vec{PQ} \times \vec{PR}|$

$= \frac{1}{2} \sqrt{1^2 + 2^2 + 1^2}$

$= \boxed{\frac{1}{2}\sqrt{6}}$

#8b. Find the volume of the parallelepiped with adjacent edges, $PQ$, $PR$, and $PS$.
$P(3,0,1)$, $Q(-1,2,5)$, $R(5,1,-1)$, $S(0,4,2)$.

volume $= |\vec{PQ} \cdot (\vec{PR} \times \vec{PS})|$

$\vec{PQ} = \langle -1-3, 2-0, 5-1 \rangle = \langle -4, 2, 4 \rangle$

$\vec{PR} = \langle 5-3, 1-0, -1-1 \rangle = \langle 2, 1, -2 \rangle$

$\vec{PS} = \langle 0-3, 4-0, 2-1 \rangle = \langle -3, 4, 1 \rangle$

$\vec{PR} \times \vec{PS} = \begin{vmatrix} + & - & + \\ 2 & 1 & -2 \\ -3 & 4 & 1 \end{vmatrix} = \langle 1+8, -(2-6), 8+3 \rangle$

$= \langle 9, 4, 11 \rangle$

volume $= |\vec{PQ} \cdot (\vec{PR} \times \vec{PS})|$

$= |\langle -4, 2, 4 \rangle \cdot \langle 9, 4, 11 \rangle|$

$= |(-4)(9) + (2)(4) + (4)(11)|$

$= |-36 + 8 + 44|$

$= |16| = \boxed{16}$

#9b. Use the scalar triple product to verify that the vectors
$\vec{u} = \langle 2, 10, -4 \rangle$, $\vec{v} = \left\langle \frac{5}{2}, \frac{9}{2}, -2 \right\rangle$, $\vec{w} = \langle 6, -2, 0 \rangle$ are coplanar.

if $\vec{u} \cdot (\vec{v} \times \vec{w}) = 0$, coplanar

$\vec{v} \times \vec{w} = \begin{vmatrix} + & - & + \\ 5/2 & 9/2 & -2 \\ 6 & -2 & 0 \end{vmatrix}$

$= \langle 0-4, -(0+12), -5-27 \rangle$

$= \langle -4, -12, -32 \rangle$

$\vec{u} \cdot (\vec{v} \times \vec{w}) = \langle 2, 10, -4 \rangle \cdot \langle -4, -12, -32 \rangle$

$= (2)(-4) + (10)(-12) + (-4)(-32)$

$= -8 - 120 + 128$

$= \boxed{0 \rightarrow \text{coplanar}}$

## 12.5

#1b. Find a vector equation and parametric equations for the line though the point $(2,-5,2)$ and parallel to the vector $\left\langle 1,3,-\frac{2}{3}\right\rangle$.

$\vec{r} = \vec{r_0} + t\vec{v}$

$\vec{r} = \langle 2,-5,2\rangle + t\langle 1,3,-\frac{2}{3}\rangle$

$\vec{r} = \langle 2,-5,2\rangle + \langle t, 3t, -\frac{2}{3}t\rangle$

$\boxed{\vec{r} = \langle 2+t, -5+3t, 2-\frac{2}{3}t\rangle}$

parametric eqns:

$x = 2+t$

$y = -5+3t$

$z = 2 - \frac{2}{3}t$

#2b. Find parametric equations and symmetric equations for the line though the points $(5,1,-3)$ and $(2,4,5)$.

$\vec{r} = \vec{r_0} + t\vec{v}$

$\vec{v} = \langle 2-5, 4-1, 5-(-3)\rangle$

$\vec{v} = \langle -3, 3, 8\rangle$

$\vec{r_0} = \langle 5,1,-3\rangle$ (can choose either point)

$\vec{r} = \langle 5,1,-3\rangle + t\langle -3,3,8\rangle$

$\vec{r} = \langle 5-3t, 1+3t, -3+8t\rangle$

parametric eqns...

$x = 5-3t$

$y = 1+3t$

$z = -3+8t$

Symmetric eqns (solve for t)...

$x = 5-3t \qquad y = 1+3t \qquad z = -3+8t$

$3t = x-5 \qquad 3t = y-1 \qquad 8t = z+3$

$t = \boxed{\dfrac{x-5}{3} = \dfrac{y-1}{3} = \dfrac{z+3}{8}}$

#3b. Find parametric equations and symmetric equations for the line though $(3,1,4)$ and perpendicular to both $\langle 2,1,0\rangle$ and $\langle 4,2,1\rangle$.

$\hookrightarrow \vec{v} = \langle 2,1,0\rangle \times \langle 4,2,1\rangle$

$= \begin{vmatrix} + & - & + \\ 2 & 1 & 0 \\ 4 & 2 & 1 \end{vmatrix} = \langle 1-0, -(2-0), 4-4\rangle$

$= \langle 1,-2,0\rangle$

$\vec{r} = \vec{r_0} + t\vec{v} = \langle 3,1,4\rangle + t\langle 1,-2,0\rangle$

$= \langle 3+t, 1-2t, 4\rangle$

parametric...

$x = 3+t$

$y = 1-2t$

$z = 4$

Symmetric ... (solve for t)

$x = 3+t \qquad y = 1-2t \qquad z = 4$

$2t = y-1$

$t = \boxed{x-3 = \dfrac{y-1}{2},\ z = 4}$

if no t just report the equation as is.

#4b. Is the line through $(4,1,-1)$ and $(2,5,3)$ perpendicular to the line through $(-3,2,0)$ and $(5,1,4)$?

direction vectors: $\vec{v_1} = \langle 2-4, 5-1, 3+1\rangle$

$= \langle -2,4,4\rangle$

$\vec{v_2} = \langle 5+3, 1-2, 4-0\rangle$

$= \langle 8,-1,4\rangle$

$\langle -2,4,4\rangle \cdot \langle 8,-1,4\rangle = (-2)(8) + (4)(-1) + (4)(4)$

$= -16 - 4 + 16$

$= -4$

$\neq 0$

so $\boxed{\text{no}}$ not perpendicular

#5b. Find a vector equation for the line segment from $(10,3,1)$ to $(5,6,-3)$.

$\vec{r} = (1-t)\vec{r_0} + t\vec{r_1}$

$\vec{r} = (1-t)\langle 10,3,1\rangle + t\langle 5,6,-3\rangle$

$\vec{r} = \langle 10-10t, 3-3t, 1-t\rangle + \langle 5t, 6t, -3t\rangle$

$\boxed{\vec{r} = \langle 10-5t, 3+3t, 1-4t\rangle}$

#6b. Find an equation of the plane through the point $(4,0,-3)$ and with normal vector $\langle 0,1,2\rangle$.
a b c

$ax+by+cz=\vec{n}\cdot\vec{r_0}$

$0x+1y+2z=\langle 0,1,2\rangle\cdot\langle 4,0,-3\rangle$
$=(0)(4)+(1)(0)+(2)(-3)$

$\boxed{y+2z=-6}$

note: there are others ways to write this, all equally correct (by multiplying both sides by a scalar)...

$-y-2z=6$

$10y+20z=-60$
etc.

#7b. Find an equation of the plane through the points $(3,-1,2)$, $(8,2,4)$, and $(-1,-2,-3)$.
make 2 vectors from the 3 points
$P(3,-1,2)$ $Q(8,2,4)$ $R(-1,-2,-3)$
$\vec{PQ}=\langle 8-3,2+1,4-2\rangle=\langle 5,3,2\rangle$
$\vec{PR}=\langle -1-3,-2+1,-3-2\rangle=\langle -4,-1,-5\rangle$
(doesn't matter which you pick, or order)
form plane normal from cross product:
$\vec{n}=\vec{PQ}\times\vec{PR}=\begin{vmatrix} + & - & + \\ 5 & 3 & 2 \\ -4 & -1 & -5\end{vmatrix}$

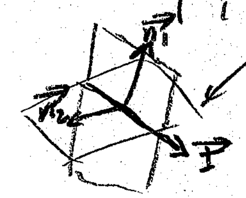
$\vec{n}=\langle -15+2,-(-25+8),-5+12\rangle$
$\vec{n}=\langle -13,17,7\rangle$
then, $ax+by+cz=\vec{n}\cdot\vec{r_0}$ choose any point on plane
$-13x+17y+7z=\langle -13,17,7\rangle\cdot\langle 3,-1,2\rangle$
$=(-13)(3)+(17)(-1)+(7)(2)$
$=-39-17+14$

$\boxed{-13x+17y+7z=-42}$

#8b. Find an equation of the plane that passes through the line of intersection of the planes $y+z=2$ and $x+2y=4$ and is perpendicular to the plane $3x-2y+z=3$.

The 2 planes have normals: $\vec{n_1}=\langle 0,1,1\rangle$
$\vec{n_2}=\langle 1,2,0\rangle$

cross-product forms intersection, $\vec{I}$
$\vec{I}=\vec{n_1}\times\vec{n_2}=\begin{vmatrix} + & - & + \\ 0 & 1 & 1 \\ 1 & 2 & 0\end{vmatrix}=\langle 0-2,-(0-1),0-1\rangle$
$=\langle -2,1,-1\rangle$




$\vec{I}$ (intersection) is a vector in direction of plane intersection


desired plane also contains $\vec{I}$, but is rotated about $\vec{I}$ so that it is ⊥ to the 3rd plane, which has normal $\vec{n_3}=\langle 3,-2,1\rangle$

The normal $\vec{n}$ for the desired plane will be $\vec{n_3}\times\vec{I}$:
$\vec{n}=\vec{n_3}\times\vec{I}=\begin{vmatrix} + & - & + \\ 3 & -2 & 1 \\ -2 & 1 & -1\end{vmatrix}$

$\vec{n}=\langle 2-1,-(-3+2),3-4\rangle$
$\vec{n}=\langle 1,1,-1\rangle$
to get a point for $\vec{r_0}$, treat 2 planes as a system: $\begin{cases} y+z=2 \\ x+2y=4\end{cases}$ $z=2-y$, $x=4-2y$

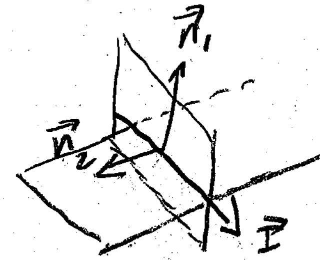
points on intersection have form $(4-2y,\ y,\ 2-y)$ we can choose anything for y, so choose $y=0$:
$(4,0,2)$ is on $\vec{I}$ and on desired plane
so $\vec{r_0}=\langle 4,0,2\rangle$ with $\vec{n}=\langle 1,1,-1\rangle$
now, plane equation: $ax+by+cz=\vec{n}\cdot\vec{r_0}$
$1x+1y-1z=\langle 1,1,-1\rangle\cdot\langle 4,0,2\rangle$
$=(1)(4)+(1)(0)+(-1)(2)$
$=4+0-2$

$\boxed{x+y-z=2}$

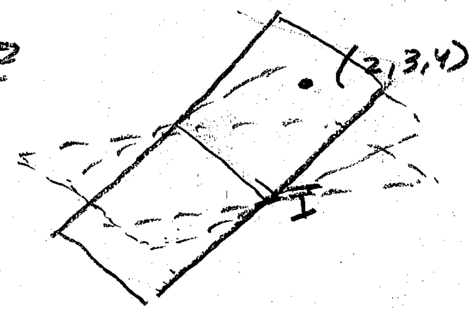
#9b. Find an equation of the plane that passes through the point $(2,3,4)$ and contains the line of intersection of the planes
$x-y-z=4$ and $2x+y+3z=6$

planes have normals: $\vec{n_1}=\langle 1,-1,-1\rangle$
$\vec{n_2}=\langle 2,1,3\rangle$

intersection $\vec{I}=\vec{n_1}\times\vec{n_2}=\begin{vmatrix} + & - & + \\ 1 & -1 & -1 \\ 2 & 1 & 3 \end{vmatrix}=\langle -3+1,-(3+2),1+2\rangle$
$=\langle -2,-5,3\rangle$
is direction of intersection




There are many planes which contain intersection $\vec{I}$
(rotated around $\vec{I}$)
we need the one which also contains $(2,3,4)$

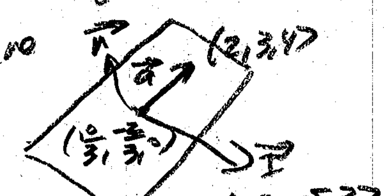



First, find a point on the intersection line by treating the 2 planes as a system: $\begin{cases} x-y-z=4 \\ 2x+y+3z=6 \end{cases}$

$\left[\begin{array}{ccc|c} 1 & -1 & -1 & 4 \\ 2 & 1 & 3 & 6 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{ccc|c} 1 & 0 & 2/3 & 10/3 \\ 0 & 1 & 5/3 & -2/3 \end{array}\right]$ $\quad x+\frac{2}{3}z=\frac{10}{3}$
$y+\frac{5}{3}z=-2/3$

so $x=\frac{10}{3}-\frac{2}{3}z$, $y=-\frac{2}{3}-\frac{5}{3}z$ and points on $\vec{I}$ have form $\left(\frac{10}{3}-\frac{2}{3}z,-\frac{2}{3}-\frac{5}{3}z,z\right)$
Can choose any value for $z$, so choose $z=0$: $\left(\frac{10}{3},-2/3,0\right)$ is a point on $\vec{I}$ (and on desired plane)

Now to get $\vec{n}$ to desired plane, find a 2nd vector on plane between points $\left(\frac{10}{3},\frac{-2}{3},0\right)$ and $(2,3,4)$:




$\vec{a}=\langle 2-\frac{10}{3},3+2/3,4-0\rangle=\langle -\frac{4}{3},\frac{11}{3},4\rangle$

then $\vec{n}=\vec{I}\times\vec{a}=\begin{vmatrix} + & - & + \\ -2 & -5 & 3 \\ -4/3 & 11/3 & 4 \end{vmatrix}=\langle -20-11,-(-8+4),-\frac{22}{3}-\frac{20}{3}\rangle$
$\vec{n}=\langle -31,4,-14\rangle$

.. and plane equation is $ax+by+cz=\vec{n}\cdot\vec{P_0}$
$-31x+4y-14z=\langle -31,4,-14\rangle\cdot\langle 2,3,4\rangle$
$=(-31)(2)+(4)(3)+(-14)(4)$
$=-62+12-56$

$\boxed{-31x+4y-14z=-106}$

#10b. Where does the line through $(1,2,3)$ and $(6,3,4)$ intersect the plane $x+2y+z=4$?

line thru pts: $\vec{v} = \langle 6-1, 3-2, 4-3 \rangle$

$\vec{v} = \langle 5,1,1 \rangle$

$\vec{r} = \vec{r_0} + t\vec{v} = \langle 1,2,3 \rangle + t\langle 5,1,1 \rangle$

can choose either pt

$\vec{r} = \langle 1+5t, 2+t, 3+t \rangle$

(x) (y) (z)

now, substitute these expressions into the plane equation...

$x+2y+z=4$

$(1+5t)+2(2+t)+(3+t)=4$

$1+5t+4+2t+3+t=4$

$8t+8=4$

$8t=-4$

$t=-\frac{1}{2}$ this is the parameter value at the point of intersection,

To get the coordinates of the point, plug $t$ into $\vec{r}(t)$:

$\vec{r}(t) = \langle 1+5t, 2+t, 3+t \rangle$

$= \langle 1+5(\frac{1}{2}), 2+(\frac{1}{2}), 3+(\frac{1}{2}) \rangle$

$= \langle \frac{7}{2}, \frac{5}{2}, \frac{7}{2} \rangle$

So intersection point is

$\boxed{\left(\frac{7}{2}, \frac{5}{2}, \frac{7}{2}\right)}$

#11b. Determine whether the plane are parallel, perpendicular, or neither. If neither, find the angle between them. $x+y+z=1$, $x-y+z=1$

$\vec{n_1} = \langle 1,1,1 \rangle$

$\vec{n_2} = \langle 1,-1,1 \rangle$

$\langle 1,1,1 \rangle \cdot \langle 1,-1,1 \rangle = (1)(1)+(1)(-1)+(1)(1)$

$= 1-1+1 = 1 \neq 0$

so not perpendicular

$\begin{vmatrix} + & - & + \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{vmatrix} = \langle 1+1, -(1-1), -1-1 \rangle$

$= \langle 2, 0, -2 \rangle$

$\neq \vec{0}$ so not parallel

angle between planes is the same as the angle between normals: $\vec{n_1} \cdot \vec{n_2} = |\vec{n_1}||\vec{n_2}|\cos\theta$

$\cos\theta = \frac{\vec{n_1} \cdot \vec{n_2}}{|\vec{n_1}||\vec{n_2}|} = \frac{1}{\sqrt{1^2+1^2+1^2}\sqrt{1^2+1^2+1^2}} = \frac{1}{\sqrt{3}\sqrt{3}} = \frac{1}{3}$

$\theta = \cos^{-1}\left(\frac{1}{3}\right) = \boxed{1.23 \text{ radians} \approx 70.5^\circ}$

#12b. Find an equation for the plane consisting of all points that are equidistant from the points $(2,5,5)$ and $(-6,3,1)$. point: $(x,y,z)$

distances: $d_1 = \sqrt{(x-2)^2+(y-5)^2+(z-5)^2}$

$d_2 = \sqrt{(x+6)^2+(y-3)^2+(z-1)^2}$

equidistant: $d_1 = d_2$

$\sqrt{(x-2)^2+(y-5)^2+(z-5)^2} = \sqrt{(x+6)^2+(y-3)^2+(z-1)^2}$

$(x-2)^2+(y-5)^2+(z-5)^2 = (x+6)^2+(y-3)^2+(z-1)^2$

$x^2-4x+4+y^2-10y+25+z^2-10z+25$

$= x^2+12x+36+y^2-6y+9+z^2-2z+1$

$\boxed{-16x-4y-8z=-8}$

or

$-4x-y-2z=-2$

or

$\boxed{4x+y+2z=2}$

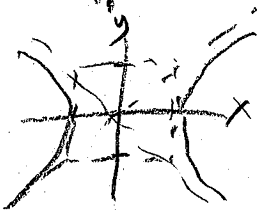
**12.6**

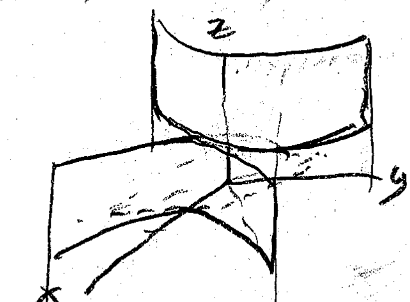
#1b. Describe and sketch the surface: $x^2 - y^2 = 1$.

missing z (cylinder)

$\frac{x^2}{1} - \frac{y^2}{1} = 1$

hyperbola:



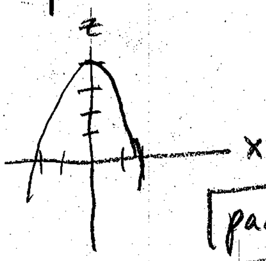

$\boxed{\text{hyperbolic cylinder}}$

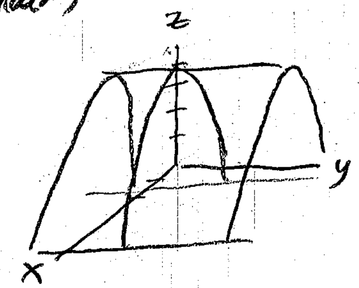
#2b. Describe and sketch the surface: $z = 4 - x^2$.

missing y (cylinder)

$z = 4 - x^2$

parabola:




$\boxed{\text{parabolic cylinder}}$

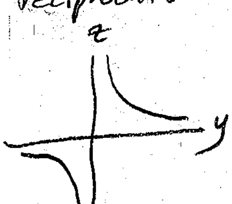
#3b. Describe and sketch the surface: $yz = 4$.

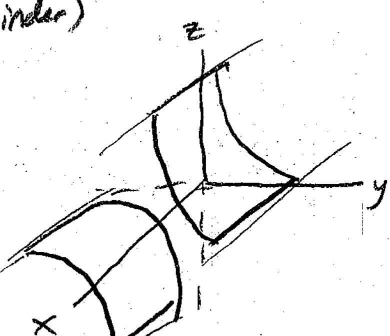
missing x (cylinder)

$yz = 4$

$z = \frac{4}{y}$

reciprocal curve:




$\boxed{\text{cylinder w/reciprocal curve rulings}}$

#4b. Draw at least three traces in each of the coordinate planes and identify the surface for $-x^2 + 4y^2 - z^2 = 4$.

xy (select z)

$z=0$: $4y^2 - x^2 = 4$

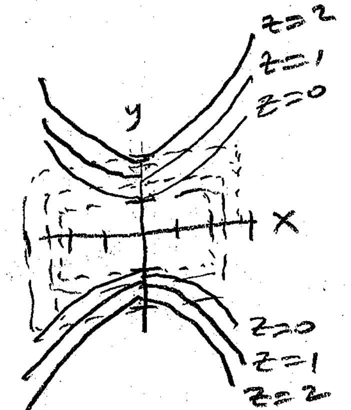
$\frac{y^2}{1} - \frac{x^2}{4} = 1$

$z=1$: $4y^2 - x^2 = 5$

$\frac{y^2}{\frac{5}{4}} - \frac{x^2}{5} = 1$

$z=2$: $4y^2 - x^2 = 8$

$\frac{y^2}{2} - \frac{x^2}{8} = 1$




(hyperbolas)

xz (select y)

$y=0$: $-x^2 - z^2 = 4$

$x^2 + z^2 = -4$

(no curve)

$y=1$: $-x^2 - z^2 = 0$

(a point)

$y=2$: $-x^2 - z^2 = -12$

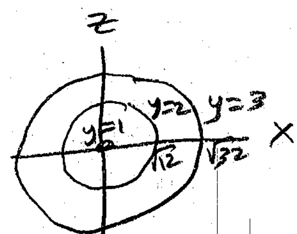
$x^2 + z^2 = 12$

(circle)

$y=3$: $-x^2 - z^2 = -32$

$x^2 + z^2 = 32$

(circle)



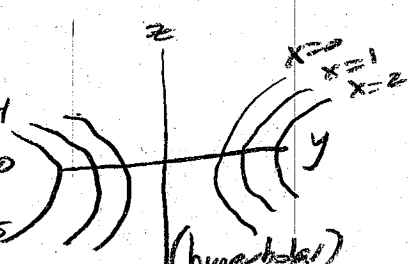

yz (select x)

$x=0$: $4y^2 - z^2 = 4$

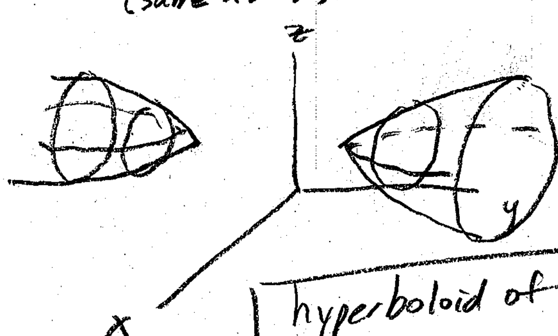
$\frac{y^2}{1} - \frac{z^2}{4} = 0$

$x=1$: $4y^2 - z^2 = 5$

(same as xy)




(hyperbolas)




$\boxed{\text{hyperboloid of two sheets with y-axis direction}}$

#5b. Put the equation in standard form, then name and sketch the surface:

$4y^2+z^2-x-16y-4z+20=0$.

$-x+(4y^2-16y)+(z^2-4z)=-20$

$4(y^2-4y+4)+(z^2-4z+4)=x-20+16+4$

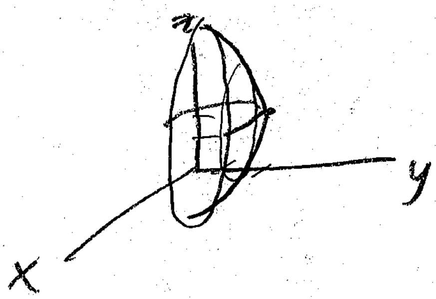
$4(y-2)^2+(z-2)^2=x$

$\frac{x}{4}=\frac{(y-2)^2}{1}+\frac{(z-2)^2}{4}$

$\boxed{\text{elliptical paraboloid}}$

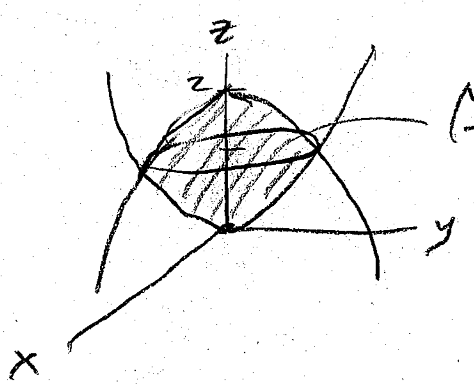
vertex: $(0,2,2)$

ellipses wider in z than in y




#6b. Sketch the region bounded by the paraboloids $z=x^2+y^2$ and $z=2-x^2-y^2$.

circular cross-sections in x-y




(football shaped region)

intersection (as a system):

$\begin{cases} z=x^2+y^2 \\ z=2-x^2-y^2 \end{cases}$

$x^2+y^2=2-x^2-y^2$

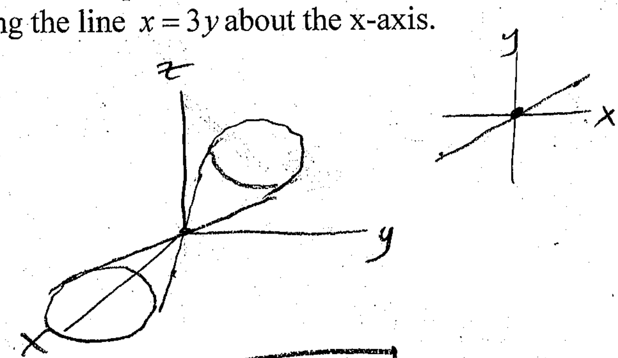
$2x^2+2y^2=2$

$x^2+y^2=1$

(circle of radius 1)

$x^2+y^2=z \rightarrow$ at $z=1$

#7b. Find an equation for the surface obtained by rotating the line $x=3y$ about the x-axis.

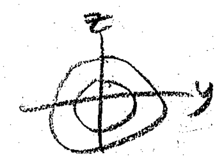



forms $\boxed{\text{a cone}}$

$x=3y$

rotated: $x=3z$

cross-sections are circles




$y^2+z^2=r^2$

$x=3y \rightarrow y=\frac{x}{3}$

$x=3z \rightarrow z=\frac{x}{3}$

$\left(\frac{x}{3}\right)^2+\left(\frac{x}{3}\right)^2=r^2$

$\frac{x^2}{9}+\frac{x^2}{9}=r^2$

$\frac{2}{9}x^2=r^2$

$\boxed{y^2+z^2=\frac{2}{9}x^2}$

or

$\boxed{\frac{y^2}{2}+\frac{z^2}{2}=\frac{x^2}{9}}$

(a cone)