Calc III - Ch 16 - Required Practice Name: Solutions

Ch16 Test Review (for test day 1)

#1. Find $div\,\vec{F}$ and $curl\,\vec{F}$

(a) $\vec{F}(x,y,z)=\langle x^2yz,\ \ xy^2z,\ \ xyz^2\rangle$

$div\,\vec{F}=\frac{\partial}{\partial x}[x^2yz]+\frac{\partial}{\partial y}[xy^2z]+\frac{\partial}{\partial z}[xyz^2]$

$=2xyz+2xyz+2xyz$

$=\boxed{6xyz}$

$curl\,\vec{F}=\begin{array}{ccc} + & - & + \end{array}$
$\begin{vmatrix} \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2yz & xy^2z & xyz^2 \end{vmatrix}$

$=\boxed{\langle xz^2-xy^2,\ -(yz^2-x^2y),\ y^2z-x^2z\rangle}$

(b) $\vec{F}(x,y,z)=\langle 1,\ \ x+yz,\ \ xy-\sqrt{z}\rangle$

$div\,\vec{F}=\frac{\partial}{\partial x}[1]+\frac{\partial}{\partial y}[x+yz]+\frac{\partial}{\partial z}[xy-z^{1/2}]$

$=0+z+(-\frac{1}{2}z^{-1/2})$

$=\boxed{z-\frac{1}{2\sqrt{z}}}$

$curl\,\vec{F}=\begin{array}{ccc} + & - & + \end{array}$
$\begin{vmatrix} \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 1 & x+yz & xy-z^{1/2} \end{vmatrix}$

$=\langle x-y,\ -(y-0),\ 1-0\rangle$

$=\boxed{\langle x-y,\ -y,\ 1\rangle}$

#2. Determine whether or not the field is conservative. If it is, find the potential function such that $\nabla f = \vec{F}$.

(a) $\vec{F}(x,y) = \langle \overset{P}{e^x \cos y}, \overset{Q}{e^x \sin y} \rangle$ conservative? $\frac{\partial P}{\partial y} = -e^x \sin y$, $\frac{\partial Q}{\partial x} = e^x \sin y$

Not conservative (even $-$sign is enough)

So no potential function f.

(b) $\vec{F}(x,y) = \langle \overset{P}{e^x \sin y}, \overset{Q}{e^x \cos y} \rangle$ conservative? $\frac{\partial P}{\partial y} = e^x \cos y$, $\frac{\partial Q}{\partial x} = e^x \cos y$ yes

$f_x = e^x \sin y$

$f = \int e^x \sin y \, dx = e^x \sin y + g(y)$

$f_y = e^x \cos y + g'(y) \overset{\text{must}}{=} e^x \cos y \rightarrow g'(y) = 0$

$g(y) = \int 0 \, dy = C$

So $\boxed{f = e^x \sin y + C}$ (or you can leave off the constant if you wish)

(c) $\vec{F}(x,y) = \langle \overset{P}{3x^2 + 2y^2}, \overset{Q}{4xy + 3} \rangle$ conservative? $\frac{\partial P}{\partial y} = 4y$, $\frac{\partial Q}{\partial x} = 4y$ = yes

$f_x = 3x^2 + 2y^2$

$f = \int (3x^2 + 2y^2) dx = x^3 + 2y^2 x + g(y)$

$f_y = 0 + 4yx + g'(y) \overset{\text{must}}{=} 4xy + 3 \Rightarrow g'(y) = 3$

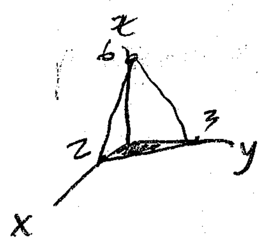
$g(y) = \int 3 dy = 3y + C$

So $\boxed{f = x^3 + 2xy^2 + 3y + C}$

#3. Set up the integral to find the surface area (do not evaluate the integral). (integrand = 1)

(a) the part of the plane $3x+2y+z=6$ that lies in the first octant. $z=6-3x-2y$

$$A=\iint_D f(x,y,z)\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA$$



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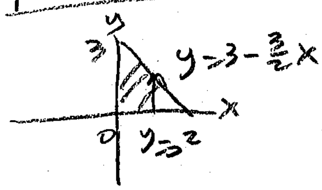


$$\boxed{\int_0^2\int_0^{3-\frac{3}{2}x}(1)\sqrt{1+(-3)^2+(-2)^2}\,dy\,dx}$$

$\vec{r}(x,y)=\langle x,y,6-3x-2y\rangle$

$\frac{\partial z}{\partial x}=-3,\quad \frac{\partial z}{\partial y}=-2$

(b) the part of the plane with vector equation $\vec{r}(u,v)=\langle 1+v,\ u-2v,\ 3-5u+v\rangle$ that is given by $0\le u\le 1,\ \ 0\le v\le 1$.

$$A=\iint_D f(\vec{r}(u,v))\,|\vec{r}_u\times\vec{r}_v|\,dA$$

$\vec{r}_u=\langle 0,1,-5\rangle,\quad \vec{r}_v=\langle 1,-2,1\rangle$

$$\vec{r}_u\times\vec{r}_v=\begin{vmatrix} + & - & + \\ 0 & 1 & -5 \\ 1 & -2 & 1\end{vmatrix}=\langle 1-10,-(0+5),0-1\rangle=\langle -9,-5,-1\rangle$$

$$|\vec{r}_u\times\vec{r}_v|=\sqrt{9^2+5^2+1^2}=\sqrt{107}$$

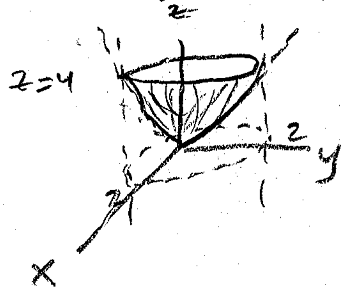
$$\boxed{\int_0^1\int_0^1(1)\sqrt{107}\,du\,dv}$$

(or $dv\,du$)

#3 (continued). Set up the integral to find the surface area (do not evaluate the integral).

(c) the part of the paraboloid $z = x^2 + y^2$ that lies inside the cylinder $x^2 + y^2 = 4$.

$$\iint_D f(x,y,z)\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA$$



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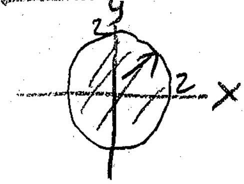


$$\int_0^{2\pi}\int_0^2 (1)\sqrt{1+(2x)^2+(2y)^2}\;r\,dr\,d\theta$$

to polar: $4x^2+4y^2$

$4(x^2+y^2)$

$4r^2$

$$\boxed{\int_0^{2\pi}\int_0^2 (1)\sqrt{1+4r^2}\;r\,dr\,d\theta}$$

$\vec{r}(x,y) = \langle x, y, x^2+y^2\rangle$

use polar:

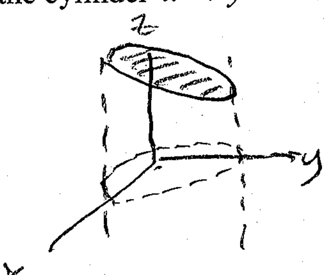
$r=0$ to $r=2$

$\theta=0$ to $\theta=2\pi$

$\frac{\partial z}{\partial x} = 2x$, $\frac{\partial z}{\partial y} = 2y$

(d) the part of the plane $z = 4 - y$ that lies inside the cylinder $x^2 + y^2 = 4$.

$$\iint_D f(x,y,z)\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA$$



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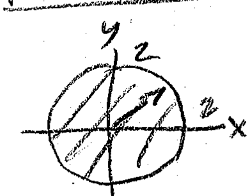


$$\int_0^{2\pi}\int_0^2 (1)\sqrt{1+(0)^2+(-1)^2}\;r\,dr\,d\theta$$

or

$$\int_0^{2\pi}\int_0^2 (1)\sqrt{2}\;r\,dr\,d\theta$$

$\vec{r}(x,y) = \langle x, y, 4-y\rangle$

use polar:

$r=0$ to $r=2$

$\theta=0$ to $\theta=2\pi$

$\frac{\partial z}{\partial x} = 0$, $\frac{\partial z}{\partial y} = -1$

#4. Set up (but do not evaluate) the line integral.

$\int f(\vec{r}) |\vec{r}'| dt$

(a) $\int_C y^3 \, ds$ if C is defined by:

$x = t^3, \; y = t \;\; for \;\; 0 \le t \le 2.$

$\vec{r}(t) = \langle t^3, t \rangle$

$\vec{r}'(t) = \langle 3t^2, 1 \rangle$

$|\vec{r}'(t)| = \sqrt{(3t^2)^2 + (1)^2} = \sqrt{9t^4+1}$

$f(\vec{r}) = (t)^3 = t^3$

$\boxed{\int_0^2 t^3 \sqrt{9t^4+1} \, dt}$

(b) $\int_C xy \, ds$ if C is defined by:

$x = t^2, \; y = 2t \;\; for \;\; 0 \le t \le 1.$

$\int f(\vec{r}) |\vec{r}'| dt$

$\vec{r}(t) = \langle t^2, 2t \rangle$

$\vec{r}'(t) = \langle 2t, 2 \rangle$

$|\vec{r}'(t)| = \sqrt{(2t)^2 + (2)^2} = \sqrt{4t^2+4}$

$f(\vec{r}) = (t^2)(2t) = 2t^3$

$\boxed{\int_0^1 2t^3 \sqrt{4t^2+4} \, dt}$

#5. Set up (**and evaluate**) the line integral. (hint: check to see if the field is conservative)

(a) $\int_C \vec{F} \bullet d\vec{r}$ for $\vec{F} = \langle \overset{P}{x^2}, \overset{Q}{y^2} \rangle$ along curve C defined by the arc of the parabola $y = 2x^2$ from $(-1,2)$ to $(2,8)$.

$\vec{F}$ conservative? $\frac{\partial P}{\partial y} = 0$, $\frac{\partial Q}{\partial x} = 0 \Rightarrow \underline{yes}$

So $\int_C \vec{F} \cdot d\vec{r} = f(\text{end}) - f(\text{start})$

$f_x = x^2$

$f = \int x^2 dx = \frac{1}{3}x^3 + g(y)$

$f_y = 0 + g'(y) \overset{\text{must}}{=} y^2 \rightarrow g'(y) = y^2$

$g(y) = \int y^2 dy = \frac{1}{3}y^3 \ (+c)$

$f = \frac{1}{3}x^3 + \frac{1}{3}y^3$

So $\int_C \vec{F} \cdot d\vec{r} = f(2,8) - f(-1,2) = \left[\frac{1}{3}(2)^3 + \frac{1}{3}(8)^3\right) - \left[\frac{1}{3}(-1)^3 + \frac{1}{3}(2)^3\right]$

$\frac{8}{3} + \frac{512}{3} + \frac{1}{3} - \frac{8}{3} = \boxed{\frac{513}{3}}$

(don't need to use the path)

(b) $\int_C \vec{F} \bullet d\vec{r}$ for $\vec{F} = \langle \overset{P}{xy^2}, \overset{Q}{x^2 y} \rangle$ along curve C defined by $\vec{r}(t) = \left\langle t + \sin\left(\frac{1}{2}\pi t\right), \ t + \cos\left(\frac{1}{2}\pi t\right) \right\rangle$

$0 \le t \le 1$

$\vec{F}$ conservative? $\frac{\partial P}{\partial y} = 2xy$, $\frac{\partial Q}{\partial x} = 2xy \Rightarrow \underline{yes}$

So $\int_C \vec{F} \cdot d\vec{r} = f(\text{end}) - f(\text{start})$

$f_x = xy^2$

$f = \int xy^2 dx = \frac{1}{2}x^2y^2 + g(y)$

$f_y = x^2y + g'(y) \overset{\text{must}}{=} x^2y \rightarrow g'(y) = 0$

$g(y) = \int 0 \, dy = 0 \ (+c)$

$f = \frac{1}{2}x^2y^2$

end: $t = 1$

$\vec{r}(1) = \langle 1 + \sin\frac{\pi}{2}, 1 + \cos\frac{\pi}{2} \rangle$

$= \langle 1+1, 1+0 \rangle = \langle 2, 1 \rangle$

start: $t = 0$

$\vec{r}(0) = \langle 0 + \sin 0, 0 + \cos 0 \rangle$

$= \langle 0+0, 0+1 \rangle = \langle 0, 1 \rangle$

So $\int_C \vec{F} \cdot d\vec{r} = f(\text{end}) - f(\text{start})$

$= f(2,1) - f(0,1)$

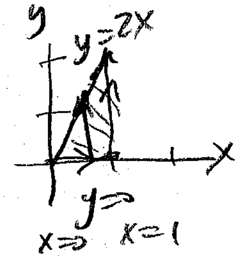
$= \frac{1}{2}(2)^2(1)^2 - \frac{1}{2}(0)^2(1)^2$

$= 2 - 0$

$= \boxed{2}$

#6. Set up (but do not evaluate) the line integral.
(hint: use Green's Theorem)

(a) $\int_C \overset{P}{(xy)}\,dx + \overset{Q}{(x^2y^3)}\,dy$ along C which traces the path around a triangle from $(0,0)$ to $(1,0)$ to $(1,2)$ and back to $(0,0)$.

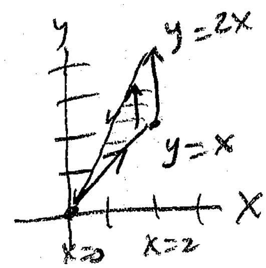


$\oint \vec{F}\cdot d\vec{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$

$\frac{\partial Q}{\partial x} = 2xy^3 \quad \frac{\partial P}{\partial y} = x$

$\boxed{\int_0^1 \int_0^{2x} (2xy^3 - x)\,dy\,dx}$

(b) $\int_C \overset{P}{(xy^2)}\,dx + \overset{Q}{(2x^2y)}\,dy$ along C which traces the path around a triangle from $(0,0)$ to $(2,2)$ to $(2,4)$ and back to $(0,0)$.



$\oint \vec{F}\cdot d\vec{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$

$\frac{\partial Q}{\partial x} = 4xy \quad \frac{\partial P}{\partial y} = 2xy$

$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 4xy - 2xy = 2xy$

$\boxed{\int_0^2 \int_x^{2x} 2xy\,dy\,dx}$

#7. Find the equation of the tangent plane to the parametric surface.

at $(2,3,0)$: $\begin{cases} u+v=2 \\ 3u^2=3 \to u=\pm 1 \\ u-v=0 \end{cases}$

$$\begin{array}{r} u+v=2 \\ +\;u-v=0 \\ \hline 2u=2 \end{array} \quad \underline{u=1} \quad \underline{v=1}$$

(a) $x=u+v,\ y=3u^2,\ z=u-v$ at $(2,3,0)$.

$\vec{r}=\langle u+v, 3u^2, u-v\rangle$

$\vec{r}_u=\langle 1, 6u, 1\rangle \quad \vec{r}_v=\langle 1,0,-1\rangle$

$$\vec{n}=\vec{r}_u\times\vec{r}_v=\begin{vmatrix} + & - & + \\ 1 & 6u & 1 \\ 1 & 0 & -1\end{vmatrix} = \langle -6u-0, -(-1-1), 0-6u\rangle = \langle -6u, 2, -6u\rangle$$

at $\overset{u\,v}{(1,1)}$, $\vec{n}=\langle -6(1), 2, -6(1)\rangle=\langle -6,2,-6\rangle$

$ax+by+cz=\vec{n}\cdot\vec{r_0}$

$-6x+2y-6z=\langle -6,2,-6\rangle\cdot\langle 2,3,0\rangle=(-6)(2)+(2)(3)+(-6)(0)=-6$

$$\boxed{\begin{array}{c} -6x+2y-6z=-6 \\ \text{or} \\ -3x+y-3z=-3 \\ \text{or} \\ 3x-y+3z=3 \end{array}}$$

(b) $x=u^2,\ y=v^2,\ z=uv$ at $u=1,\ v=1$.

at $u=1, v=1$: $\vec{r}(1,1)=\langle (1)^2,(1)^2,(1)(1)\rangle=\langle 1,1,1\rangle$ $(\vec{r_0})$

$\vec{r}=\langle u^2, v^2, uv\rangle$

$\vec{r}_u=\langle 2u, 0, v\rangle \quad \vec{r}_v=\langle 0, 2v, u\rangle$

$$\vec{n}=\vec{r}_u\times\vec{r}_v=\begin{vmatrix} + & - & + \\ 2u & 0 & v \\ 0 & 2v & u\end{vmatrix}=\langle 0-2v^2, -(2u^2-0), 4uv-0\rangle = \langle -2v^2, -2u^2, 4uv\rangle$$

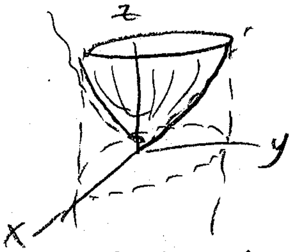
at $\overset{u\,v}{(1,1)}$, $\vec{n}=\langle -2(1)^2, -2(1)^2, 4(1)(1)\rangle=\langle -2,-2,4\rangle$

$ax+by+cz=\vec{n}\cdot\vec{r_0}$

$-2x-2y+4z=\langle -2,-2,4\rangle\cdot\langle 1,1,1\rangle=(-2)(1)+(-2)(1)+(4)(1)=0$

$$\boxed{\begin{array}{c} -2x-2y+4z=0 \\ \text{or} \\ x+y-2z=0 \end{array}}$$

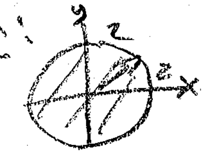
#8. Set up (but do not evaluate) the surface integral.



(a) $\int\limits_S (z)\,dS$ where S is the part of the paraboloid $z = x^2 + y^2$ that lies inside the cylinder $x^2 + y^2 = 4$.

Surface: $\vec{r}(x,y) = \langle x, y, x^2+y^2 \rangle$

Parameter domain:



Polar:
$r=0$ to $r=2$
$\theta=0$ to $\theta=2\pi$

$$\iint f(x,y,z)\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA$$

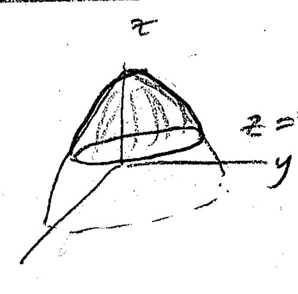
$$\frac{\partial z}{\partial x}=2x \quad \frac{\partial z}{\partial y}=2y$$

$$\int_0^{2\pi}\int_0^{2}(x^2+y^2)\sqrt{1+(2x)^2+(2y)^2}\,r\,dr\,d\theta$$

$\hookrightarrow$ to polar: $r^2\sqrt{1+4(x^2+y^2)} = r^2\sqrt{1+4r^2}$

$$\boxed{\int_0^{2\pi}\int_0^{2} r^2\sqrt{1+4r^2}\;r\,dr\,d\theta}$$

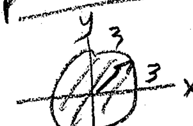
(b) $\int\limits_S (xz)\,dS$ where S is the part of the paraboloid $z = 16 - x^2 - y^2$ that lies above the plane $z = 7$.



intersection:
$\begin{cases} z = 16-x^2-y^2 \\ z=7 \end{cases}$
$7 = 16-x^2-y^2$
$x^2+y^2=9$

surface:
$\vec{r}(x,y) = \langle x, y, 16-x^2-y^2 \rangle$

parameter domain:



polar:
$r=0$ to $r=3$
$\theta=0$ to $\theta=2\pi$

$$\iint f(x,y,z)\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA$$

$$\frac{\partial z}{\partial x}=-2x \quad \frac{\partial z}{\partial y}=-2y$$

$$\int_0^{2\pi}\int_0^{3} x(16-x^2-y^2)\sqrt{1+(-2x)^2+(-2y)^2}\,r\,dr\,d\theta$$

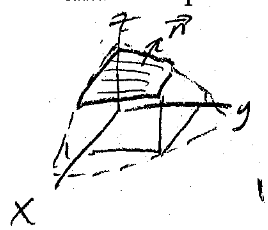
to polar:
$(r\cos\theta)(16-r^2)\sqrt{1+4r^2}$

$$\boxed{\int_0^{2\pi}\int_0^{3}(r\cos\theta)(16-r^2)\sqrt{1+4r^2}\;r\,dr\,d\theta}$$

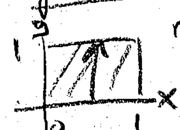
Ch16 Test Review (for test day 2)

#9. Set up (but do not evaluate) the integral to evaluate the flux of $\vec{F}$ over S

(a) $\vec{F}(x,y,z) = \langle xy,\ yz,\ zx \rangle$
S is the part of the paraboloid $z = 4 - x^2 - y^2$ that lies above the square $0 \le x \le 1,\ 0 \le y \le 1$ and has upward orientation.



surface:
$\vec r(x,y) = \langle x, y, 4-x^2-y^2 \rangle$
parameter domain:



rectangular:
$y=0$ to $y=1$
$x=0$ to $x=1$

$\iint \vec F \cdot (\vec r_u \times \vec r_v)\, dA$

$\vec r_u = \vec r_x = \langle 1, 0, -2x \rangle$ $\quad \vec r_v = \vec r_y = \langle 0, 1, -2y \rangle$

$\vec n = \vec r_u \times \vec r_v = \begin{vmatrix} + & - & + \\ 1 & 0 & -2x \\ 0 & 1 & -2y \end{vmatrix}$

$= \langle 0+2x, -(-2y-0), 1-0 \rangle = \langle 2x, 2y, 1 \rangle$

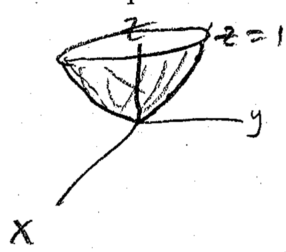
$\vec F(\vec r) = \langle xy, y(4-x^2-y^2), (4-x^2-y^2)x \rangle$

$\vec F \cdot \vec n = \langle xy, y(4-x^2-y^2), (4-x^2-y^2)x \rangle \cdot \langle 2x, 2y, 1 \rangle$

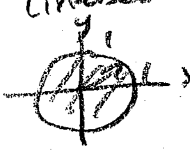
$= 2x^2y + 2y^2(4-x^2-y^2) + (4-x^2-y^2)x$

$$\boxed{\int_0^1 \int_0^1 \left(2x^2y + 2y^2(4-x^2-y^2) + (4-x^2-y^2)x\right) dy\, dx}$$

(b) $\vec{F}(x,y,z) = \langle x^2,\ xy,\ z \rangle$
S is the part of the paraboloid $z = x^2 + y^2$ that lies below the plane $z = 1$ with upward orientation.



surface:
$\vec r(x,y) = \langle x, y, x^2+y^2 \rangle$
parameter domain:
(intersection $z = x^2+y^2 = 1$)



polar:
$r=0$ to $r=1$
$\theta=0$ to $\theta=2\pi$

$\iint \vec F \cdot (\vec r_u \times \vec r_v)\, dA$

$\vec r_u = \vec r_x = \langle 1, 0, 2x \rangle$ $\quad \vec r_v = \vec r_y = \langle 0, 1, 2y \rangle$

$\vec n = \vec r_u \times \vec r_v = \begin{vmatrix} + & - & + \\ 1 & 0 & 2x \\ 0 & 1 & 2y \end{vmatrix}$

$= \langle 0-2x, -(2y-0), 1-0 \rangle$ $\quad \langle -2x, -2y, 1 \rangle$

$\vec F(\vec r) = \langle x^2, xy, x^2+y^2 \rangle$

$\vec F \cdot \vec n = \langle x^2, xy, x^2+y^2 \rangle \cdot \langle -2x, -2y, 1 \rangle$

$= -2x^3 - 2xy^2 + x^2 + y^2$

to polar...

$-2x(x^2+y^2) + (x^2+y^2) = -2xr^2 + r^2$

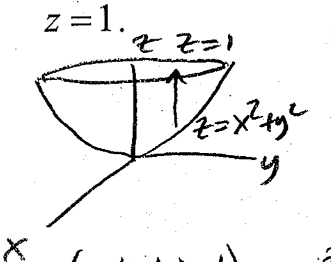
$= r^2(-2x+1) = r^2(-2r\cos\theta + 1)$

$$\boxed{\int_0^{2\pi} \int_0^1 r^2(-2r\cos\theta + 1)\, r\, dr\, d\theta}$$

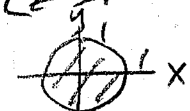
#10. Using the Divergence Theorem, write the **triple-integral** (but do not evaluate) which is equivalent to the surface integral $\iint_S \vec{F} \cdot d\vec{S}$ which $= \iiint \operatorname{div} \vec{F}\, dV$

calculates the flux of $\vec{F}$ across S.

(a) $\vec{F} = \langle x^2, xy, z \rangle$ and S is the part of the paraboloid $z = x^2 + y^2$ that lies below the plane $z = 1$.



intersection: $\begin{cases} z = x^2+y^2 \\ z = 1 \end{cases}$ $x^2+y^2=1$



(cylindrical)

$z = x^2+y^2$ to $z = 1$

$r = 0$ to $r = 1$

$\theta = 0$ to $\theta = 2\pi$

$\operatorname{div} \vec{F} = \frac{\partial}{\partial x}[x^2] + \frac{\partial}{\partial y}(xy) + \frac{\partial}{\partial z}[z]$

$= 2x + x + 1 = 3x + 1$

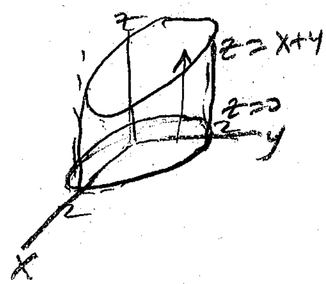
to cylindrical $= 3(r\cos\theta) + 1$

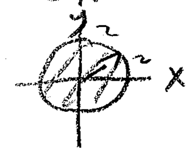
$$\int_0^{2\pi}\int_0^{1}\int_{x^2+y^2}^{1} (3r\cos\theta + 1)\, r\, dz\, dr\, d\theta$$

$$\boxed{\int_0^{2\pi}\int_0^{1}\int_{r^2}^{1} (3r\cos\theta + 1)\, r\, dz\, dr\, d\theta}$$

(b) $\vec{F} = \langle x^2, -x^2yz, 2xy^2 \rangle$ and S is the cylinder $x^2 + y^2 = 4$ between the planes $z = x + 4$ and $z = 0$.

(cylindrical)





$z = 0$ to $z = x + 4$

$r = 0$ to $r = 2$

$\theta = 0$ to $\theta = 2\pi$

$\operatorname{div} \vec{F} = \frac{\partial}{\partial x}[x^2] + \frac{\partial}{\partial y}[-x^2yz] + \frac{\partial}{\partial z}[2xy^2]$

$= 2x + (-x^2z) + 0$

$= 2x - x^2z$ to cylindrical

$= 2(r\cos\theta) - (r\cos\theta)^2 z$

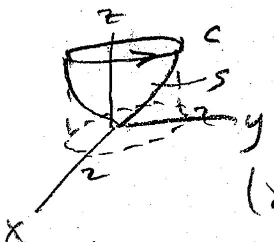
$$\int_0^{2\pi}\int_0^{2}\int_0^{x+4} (2r\cos\theta - (r\cos\theta)^2 z)\, r\, dz\, dr\, d\theta$$

$$\boxed{\int_0^{2\pi}\int_0^{2}\int_0^{r\cos\theta + 4} (2r\cos\theta - (r\cos\theta)^2 z)\, r\, dz\, dr\, d\theta}$$

#11. Using Stokes' Theorem, write the **single-integral** (but do not evaluate) which calculates

$$\iint_S \left(curl\, \vec{F} \right) \bullet d\vec{S} = \int \vec{F} \cdot d\vec{r} = \underline{\int \vec{F}(\vec{r}) \cdot \vec{r}\,'\, dt}$$

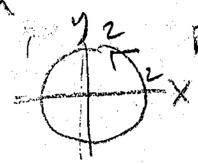
(a) $\vec{F} = \langle x^2z^2,\ y^2z^2,\ xyz \rangle$ and S is the part of the paraboloid $z = x^2 + y^2$ that inside the cylinder $x^2 + y^2 = 4$ and is oriented upward.



Intersection:

$$\begin{cases} z = x^2 + y^2 \\ x^2 + y^2 = 4 \end{cases}$$

$(x^2 + y^2 \text{ w/ } z = 4)$



$\vec{r}(t) = \langle 2\cos t, 2\sin t, 4 \rangle$
$(0 \le t \le 2\pi)$

$\vec{r}\,' = \langle -2\sin t, 2\cos t, 0 \rangle$

$\vec{F}(\vec{r}) = \langle (2\cos t)^2(4)^2, (2\sin t)^2(4)^2, (2\cos t)(2\sin t)(4) \rangle$

$= \langle 64\cos^2 t, 64\sin^2 t, 16\cos t \sin t \rangle$

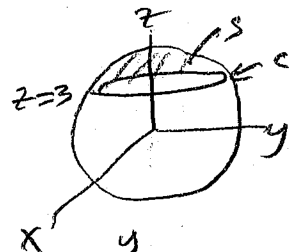
$\vec{F} \cdot \vec{r}\,' =$

$\langle 64\cos^2 t, 64\sin^2 t, 16\cos t \sin t \rangle \cdot \langle -2\sin t, 2\cos t, 0 \rangle$

$= -128\cos^2 t \sin t + 128\sin^2 t \cos t$

$$\boxed{\int_0^{2\pi} (-128\cos^2 t \sin t + 128\sin^2 t \cos t)\, dt}$$

(b) $\vec{F} = \langle 2x^2,\ -2y^2,\ x^2z \rangle$ and S is the part of the sphere $x^2 + y^2 + z^2 = 13$ that lies above the plane $z = 3$ and is oriented upward.

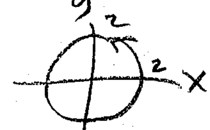


Intersection

$$\begin{cases} x^2 + y^2 + z^2 = 13 \\ z = 3 \end{cases}$$

$x^2 + y^2 + (3)^2 = 13$

$x^2 + y^2 = 4$



$\vec{r}(t) = \langle 2\cos t, 2\sin t, 3 \rangle$
$(0 \le t \le 2\pi)$

$\vec{r}\,' = \langle -2\sin t, 2\cos t, 0 \rangle$

$\vec{F}(\vec{r}) = \langle 2(2\cos t)^2, -2(2\sin t)^2, (2\cos t)^2(3) \rangle$

$= \langle 8\cos^2 t, -8\sin^2 t, 12\cos^2 t \rangle$

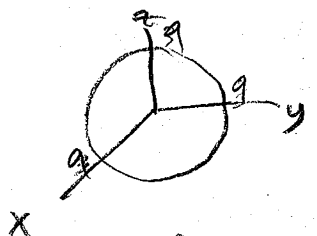
$\vec{F} \cdot \vec{r}\,' =$

$\langle 8\cos^2 t, -8\sin^2 t, 12\cos^2 t \rangle \cdot \langle -2\sin t, 2\cos t, 0 \rangle$

$= -16\cos^2 t \sin t - 16\sin^2 t \cos t$

$$\boxed{\int_0^{2\pi} (-16\cos^2 t \sin t - 16\sin^2 t \cos t)\, dt}$$

#12. Using the Divergence Theorem, write (but do not evaluate) the **triple-integral** which is equivalent to the surface integral $\iint_S \vec{F} \bullet \vec{dS}$ which $= \underline{\iiint \operatorname{div} \vec{F}\, dV}$ calculates the flux of $\vec{F}$ across S if $\vec{F} = \langle 3x^2,\ xyz,\ z^3 \rangle$ and S is the surface of a sphere of radius 9 centered at the origin.



$$\operatorname{div}\vec{F} = \frac{\partial}{\partial x}[3x^2] + \frac{\partial}{\partial y}[xyz] + \frac{\partial}{\partial z}[z^3]$$

$$= 6x + xz + 3z^2$$

→ to spherical...

$$= 6(\rho\sin\phi\cos\theta) + (\rho\sin\phi\cos\theta)(\rho\cos\phi) + 3(\rho\cos\phi)^2$$

(spherical)

$\rho = 0$ to $\rho = 3$

$\phi = 0$ to $\phi = \pi$

$\theta = 0$ to $\theta = 2\pi$

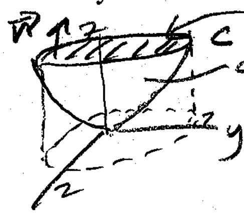
$$\boxed{\int_0^{2\pi}\int_0^{\pi}\int_0^{9} \left[6(\rho\sin\phi\cos\theta) + (\rho\sin\phi\cos\theta)(\rho\cos\phi) + 3(\rho\cos\phi)^2\right]\rho^2\sin\phi\, d\rho\, d\phi\, d\theta}$$

#13. Using Stokes' Theorem, write the **double-integral** (but do not evaluate) which is equivalent to the line integral $\int_C \vec{F} \cdot d\vec{r}$ which sums the contributions of the given field along the given path. $= \iint (\text{curl}\vec{F}) \cdot \vec{n}\, dA$

Hints: You can assume the path is oriented such that the normal vector to the enclosed surface is in the 'positive' direction.

Remember: you can choose any surface that is bounded by the curve of the given surface – you don't have to use the original surface.

(a) $\vec{F} = \langle x^2z^2,\ x^2,\ xyz \rangle$ along the path which is the intersection of the paraboloid $z = x^2 + y^2$ and the cylinder $x^2 + y^2 = 4$.

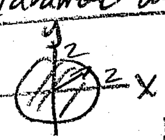


use this S plane $z=4$

$\vec{n} = \langle 0,0,1 \rangle$

surface: $\vec{r}(x,y) = \langle x, y, 4 \rangle$

parameter domain



polar: $r=0$ to $r=2$, $\theta=0$ to $\theta=2\pi$

$$\text{curl}\vec{F} = \begin{vmatrix} + & - & + \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2z^2 & x^2 & xyz \end{vmatrix}$$

$= \langle xz - 0, -(yz - 2x^2z), 2x - 0 \rangle$

$= \langle xz, 2x^2z - yz, 2x \rangle$

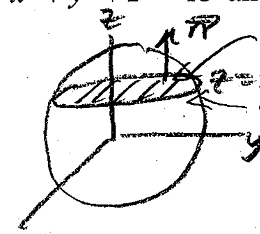
$\text{curl}\vec{F} \cdot \vec{n} = \langle xz, 2x^2z - yz, 2x \rangle \cdot \langle 0,0,1 \rangle$

$= 2x$

to polar $\Rightarrow 2r\cos\theta$

$$\boxed{\int_0^{2\pi}\int_0^2 (2r\cos\theta)\, r\,dr\,d\theta}$$

(b) $\vec{F} = \langle x + y^2,\ y + z^2,\ z + x^2 \rangle$ along the path which is the intersection of the sphere $x^2 + y^2 + z^2 = 13$ and the plane $z = 3$.



use this S surface:

$\vec{r}(x,y) = \langle x, y, 3 \rangle$

$\vec{n} = \langle 0, 0, 1 \rangle$

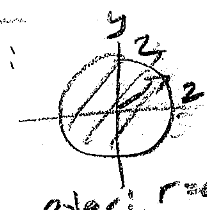
parameter domain

set by intersection:

$\begin{cases} x^2+y^2+z^2 = 13 \\ z = 3 \end{cases}$

$x^2+y^2+(3)^2 = 13$

$x^2+y^2 = 4$



polar: $r=0$ to $r=2$, $\theta=0$ to $\theta=2\pi$

$$\text{curl}\vec{F} = \begin{vmatrix} + & - & + \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x+y^2 & y+z^2 & z+x^2 \end{vmatrix}$$

$= \langle 0 - 2z, -(2x - 0), 0 - 2y \rangle$

$= \langle -2z, -2x, -2y \rangle$

$\text{curl}\vec{F} \cdot \vec{n} = \langle -2z, -2x, -2y \rangle \cdot \langle 0,0,1 \rangle$

$= -2y$ to polar...

$= -2r\sin\theta$

$$\boxed{\int_0^{2\pi}\int_0^2 (-2r\sin\theta)\, r\,dr\,d\theta}$$