

Ch12 Test Review

#1. Find the lengths of the sides of the triangle PQR. Is it a right triangle? Is it an isosceles triangle? $P(3, -2, -3)$, $Q(7, 0, 1)$, $R(1, 2, 1)$.

$$PQ = \sqrt{(7-3)^2 + (0+2)^2 + (1+3)^2} = 6 \quad \text{2 sides equal}$$

$$PR = \sqrt{(1-3)^2 + (2+2)^2 + (1+3)^2} = 6 \quad \text{isosceles}$$

$$QR = \sqrt{(7-1)^2 + (0-2)^2 + (1-1)^2} = \sqrt{40}$$

Pythagorean theorem:

$$6^2 + 6^2 \stackrel{?}{=} \sqrt{40}^2$$

$$72 \neq 40$$

no, so not a right triangle

#2. Determine whether the points lie on a straight line.

(i) $A(2, 4, 2)$, $B(3, 7, -2)$, $C(1, 3, 3)$.

$$AB = \sqrt{(3-2)^2 + (7-4)^2 + (-2-2)^2} = \sqrt{26}$$

$$AC = \sqrt{(2-1)^2 + (4-3)^2 + (2-3)^2} = \sqrt{3}$$

$$BC = \sqrt{(3-1)^2 + (7-3)^2 + (-2-3)^2} = \sqrt{45}$$

on a line if $\frac{\text{2 shorter dist}}{\text{add to exactly longest}}$

$$\sqrt{26} + \sqrt{3} \stackrel{?}{=} \sqrt{45}$$

$$6.83 \neq 6.708 \quad \text{not on a line}$$

(ii) $D(0, -5, 5)$, $E(1, -2, 4)$, $F(3, 4, 2)$.

$$DE = \sqrt{(1-0)^2 + (-2+5)^2 + (4-5)^2} = \sqrt{11}$$

$$DF = \sqrt{(3-0)^2 + (4+5)^2 + (2-5)^2} = \sqrt{99}$$

$$EF = \sqrt{(3-1)^2 + (4+2)^2 + (2-4)^2} = \sqrt{44}$$

$$\sqrt{11} + \sqrt{44} \stackrel{?}{=} \sqrt{99}$$

$$9.94987 = 9.94987$$

yes, on a line

#3. For $\vec{a} = \langle 5, -12 \rangle$, $\vec{b} = \langle -3, -6 \rangle$, find:

(i) $\vec{a} + \vec{b} = \langle 5, -12 \rangle + \langle -3, -6 \rangle$
 $= \langle 2, -18 \rangle$

(ii) $2\vec{a} + 3\vec{b} = 2\langle 5, -12 \rangle + 3\langle -3, -6 \rangle$
 $= \langle 10, -24 \rangle + \langle -9, -18 \rangle$
 $= \langle 1, -42 \rangle$

(iii) $|\vec{a} - \vec{b}| \quad \vec{a} - \vec{b} = \langle 5, -12 \rangle - \langle -3, -6 \rangle$
 $= \langle 8, -6 \rangle$

$$|\vec{a} - \vec{b}| = \sqrt{8^2 + 6^2} = \sqrt{100} = 10$$

#4. Find a unit vector that has the direction as $\langle -4, 2, 4 \rangle$.

$$|\langle -4, 2, 4 \rangle| = \sqrt{4^2 + 2^2 + 4^2} = \sqrt{36} = 6$$

$$\frac{1}{6} \langle -4, 2, 4 \rangle$$

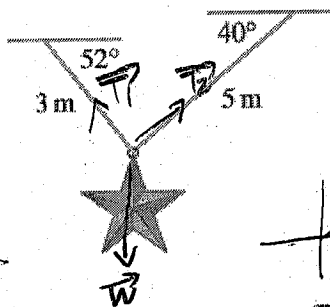
or

$$\left\langle -\frac{4}{6}, \frac{2}{6}, \frac{4}{6} \right\rangle$$

or

$$\left\langle -\frac{2}{3}, \frac{1}{3}, \frac{2}{3} \right\rangle$$

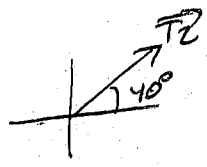
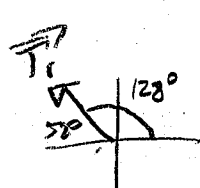
#5. Rope 3m and 5m in length are fastened to a holiday decoration that is suspended over a town square. The decoration has a mass of 5 kg. The ropes, fasted at different heights, make angles of 52° and 40° with the horizontal. Find the tension in each wire.



$$W = mg$$

$$W = (5)(9.81)$$

$$W = 49 \text{ N}$$



stationary, so $\vec{T}_1 + \vec{T}_2 + \vec{W} = \vec{0}$

$$\vec{T}_1 = \langle |T_1| \cos 128^\circ, |T_1| \sin 128^\circ \rangle$$

$$\vec{T}_2 = \langle |T_2| \cos 40^\circ, |T_2| \sin 40^\circ \rangle$$

$$\vec{W} = \langle 0, -49 \rangle$$

$$\sum x = 0: |T_1| \cos 128^\circ + |T_2| \cos 40^\circ + 0 = 0$$

$$\sum y = 0: |T_1| \sin 128^\circ + |T_2| \sin 40^\circ - 49 = 0$$

system:

$$\begin{bmatrix} \cos 128^\circ & \cos 40^\circ & 0 \\ \sin 128^\circ & \sin 40^\circ & 49 \end{bmatrix}$$

ref

$$\begin{bmatrix} 1 & 0 & 37.5591 \\ 0 & 1 & 30.1858 \end{bmatrix} = |T_1|$$

$$\begin{bmatrix} 1 & 0 & 37.5591 \\ 0 & 1 & 30.1858 \end{bmatrix} = |T_2|$$

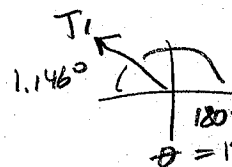
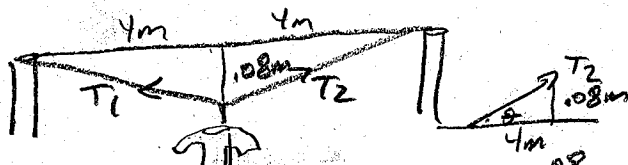
$$\vec{T}_1 = \langle 37.5591 \cos 128^\circ, 37.5591 \sin 128^\circ \rangle$$

$$= \langle -23.124, 29.597 \rangle \text{ Newtons (N)}$$

$$\vec{T}_2 = \langle 30.1858 \cos 40^\circ, 30.1858 \sin 40^\circ \rangle$$

$$= \langle 23.124, 19.403 \rangle \text{ N}$$

#6. A clothesline is tied between two poles, 8m apart. The line is quite taut and has negligible sag. When a wet shirt with a mass of 0.8 kg is hung at the middle of the line, the midpoint is pulled down 8 cm. Find the tension in each half of the clothesline.



$$\tan \theta = \frac{0.08}{4}$$

$$\theta = \tan^{-1} \left(\frac{0.08}{4} \right) = 1.146^\circ$$

stationary, so $\vec{T}_1 + \vec{T}_2 + \vec{W} = \vec{0}$

$$\vec{T}_1 = \langle |T_1| \cos 178.854^\circ, |T_1| \sin 178.854^\circ \rangle$$

$$\vec{T}_2 = \langle |T_2| \cos 1.146^\circ, |T_2| \sin 1.146^\circ \rangle$$

$$|\vec{W}| = mg = 0.8(9.81) = 7.848 \text{ N}$$

$$\vec{W} = \langle 0, -7.848 \rangle \text{ N}$$

$$\sum x = 0: |T_1| \cos 178.854^\circ + |T_2| \cos 1.146^\circ + 0 = 0$$

$$\sum y = 0: |T_1| \sin 178.854^\circ + |T_2| \sin 1.146^\circ - 7.848 = 0$$

system:

$$\begin{bmatrix} \cos 178.854^\circ & \cos 1.146^\circ & 0 \\ \sin 178.854^\circ & \sin 1.146^\circ & 7.848 \end{bmatrix}$$

ref

$$\begin{bmatrix} 1 & 0 & 196.1986 \\ 0 & 1 & 196.1986 \end{bmatrix} = |T_1|$$

$$\begin{bmatrix} 1 & 0 & 196.1986 \\ 0 & 1 & 196.1986 \end{bmatrix} = |T_2|$$

$$\vec{T}_1 = \langle 196.1986 \cos 178.854^\circ, 196.1986 \sin 178.854^\circ \rangle$$

$$= \langle -196.16, 3.92 \rangle \text{ N}$$

$$\vec{T}_2 = \langle 196.16, 3.92 \rangle \text{ N}$$

#7. Find the angle between the vectors (in exact and decimal form). $\vec{a} = \langle 4, 0, 2 \rangle$, $\vec{b} = \langle 2, -1, 0 \rangle$.

$$\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos \theta$$

$$\cos \theta = \frac{\langle 4, 0, 2 \rangle \cdot \langle 2, -1, 0 \rangle}{\sqrt{4^2 + 0^2 + 2^2} \sqrt{2^2 + (-1)^2 + 0^2}}$$

$$= \frac{(4)(2) + (0)(-1) + (2)(0)}{\sqrt{20} \sqrt{5}}$$

$$= \frac{8}{\sqrt{100}} = \frac{8}{10} = \frac{4}{5}$$

$$\theta = \cos^{-1}\left(\frac{4}{5}\right) \approx \begin{cases} 0.6435 \text{ radians} \\ 36.87^\circ \end{cases}$$

#9. If $\vec{a} = \langle 1, 2, 1 \rangle$ and $\vec{b} = \langle 0, 1, 3 \rangle$, find $\vec{a} \times \vec{b}$ and $\vec{b} \times \vec{a}$.

$$\vec{a} \times \vec{b} = \begin{vmatrix} + & - & + \\ 1 & 2 & 1 \\ 0 & 1 & 3 \end{vmatrix}$$

$$= \langle 6-1, -(3-0), 1-0 \rangle$$

$$= \langle 5, -3, 1 \rangle$$

$$\vec{b} \times \vec{a} = \begin{vmatrix} + & - & + \\ 0 & 1 & 3 \\ 1 & 2 & 1 \end{vmatrix}$$

$$= \langle 1-6, -(0-3), 0-1 \rangle$$

$$= \langle -5, 3, -1 \rangle$$

(also, $\vec{b} \times \vec{a} = -(\vec{a} \times \vec{b})$)

#8. For what values of b are the vectors $\langle -6, b, 2 \rangle$ and $\langle b, b^2, b \rangle$ orthogonal?

orthogonal (\perp) if dot product = 0

$$\langle -6, b, 2 \rangle \cdot \langle b, b^2, b \rangle = 0$$

$$-6b + b^3 + 2b = 0$$

$$b^3 - 4b = 0$$

$$b(b^2 - 4) = 0$$

$$b(b-2)(b+2) = 0$$

$$\boxed{b=0, b=2, b=-2}$$

#10. Find two unit vectors orthogonal to both $\langle 1, -1, 1 \rangle$ and $\langle 0, 4, 4 \rangle$.

(cross-product is \perp to vectors)

$$\langle 1, -1, 1 \rangle \times \langle 0, 4, 4 \rangle = \begin{vmatrix} + & - & + \\ 1 & -1 & 1 \\ 0 & 4 & 4 \end{vmatrix}$$

$$= \langle -4-4, -(4-0), 4-0 \rangle$$

$$= \langle -8, -4, 4 \rangle$$

also $\langle 8, 4, -4 \rangle$

so $-\langle -8, -4, 4 \rangle$

$$= \langle 8, 4, -4 \rangle$$

or any scale multiple

e.g. $\langle 2, 1, -1 \rangle$

$\langle 4, 2, -2 \rangle$

⋮

#11. Find a vector equation, parametric equations, and symmetric equations for the line through the points (6,1,-3) and (2,4,5).

$$\vec{r}_0 = \langle 6, 1, -3 \rangle \text{ (can choose either point)}$$

$$\vec{v} = \langle 2-6, 4-1, 5+3 \rangle = \langle -4, 3, 8 \rangle$$

vector equation: $\vec{r} = \vec{r}_0 + t\vec{v}$

$$\vec{r} = \langle 6, 1, -3 \rangle + t\langle -4, 3, 8 \rangle$$

parametric equations: $\vec{r} = \langle 6-4t, 1+3t, -3+8t \rangle$

$$\begin{cases} x = 6-4t \\ y = 1+3t \\ z = -3+8t \end{cases} \quad \begin{matrix} -\infty \leq t \leq \infty \\ \text{include range} \\ \text{for parameter} \end{matrix}$$

symmetric (solve for t):

$$\begin{matrix} x = 6-4t & y = 1+3t & z = -3+8t \\ 4t = 6-x & 3t = y-1 & 8t = z+3 \end{matrix}$$

$$t = \frac{6-x}{4} = \frac{y-1}{3} = \frac{z+3}{8}$$

#12. Find a vector equation for the line of intersection of the planes $x+y+z=1$ and $x+z=0$.

intersection = system solution:

$$\begin{cases} x+y+z=1 \\ x+z=0 \end{cases} \quad \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{array} \right]$$

rref $\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{array} \right]$ $x+z=0, x=-z$
 $y=1$

general solution: $(-z, 1, z)$
 everything in terms of z , so make z the parameter t ($z=t$)

$$\begin{matrix} x = -t \\ y = 1 \\ z = t \end{matrix}$$

as a vector equation:

$$\vec{r} = \langle -t, 1, t \rangle \quad -\infty \leq t \leq \infty$$

#13. Find an equation of the plane through the point $(-2, 8, 10)$ and perpendicular to the line

$$x=1+t, y=2t, z=4-3t.$$

line: $\vec{r} = \vec{r}_0 + t\vec{v} = \langle 1, 0, 4 \rangle + t\langle 1, 2, -3 \rangle$

so $\vec{v} = \langle 1, 2, -3 \rangle$

if new plane is \perp to line, then \vec{v} is the normal, \vec{n} , for the plane: $\vec{n} = \langle 1, 2, -3 \rangle$

for the plane, $\vec{r}_0 = \langle -2, 8, 10 \rangle$

$$ax+by+cz = \vec{n} \cdot \vec{r}_0$$

$$\begin{aligned} 1x+2y-3z &= \langle 1, 2, -3 \rangle \cdot \langle -2, 8, 10 \rangle \\ &= (1)(-2) + (2)(8) + (-3)(10) \\ &= -2 + 16 - 30 \end{aligned}$$

$$x+2y-3z = -16$$

#14. Find an equation of the plane that contains the line $x=3+2t, y=t, z=8-t$ and is parallel to the plane $2x+4y+8z=17$.

\vec{n} for given plane is $\vec{n} = \langle 2, 4, 8 \rangle$

is also normal for new plane (parallel to given plane)

for \vec{r}_0 on plane, choose any t to plug into parametric equations:

$$\begin{aligned} \text{choose } t=0 & \quad x=3+2(0)=3 \\ & \quad y=(0)=0 \\ & \quad z=8-(0)=8 \end{aligned}$$

so $\vec{r}_0 = \langle 3, 0, 8 \rangle$

now, $ax+by+cz = \vec{n} \cdot \vec{r}_0$

$$\begin{aligned} 2x+4y+8z &= \langle 2, 4, 8 \rangle \cdot \langle 3, 0, 8 \rangle \\ &= (2)(3) + (4)(0) + (8)(8) \\ &= 6 + 0 + 64 \end{aligned}$$

$$2x+4y+8z = 70$$

$$x+2y+4z = 35$$

#15. Draw at least two traces for each coordinate plane for $4x^2 - 16y^2 + z^2 = 16$. What kind of solid is this? What is its main axis? Sketch the solid in \mathbb{R}^3 .

xy (select z)

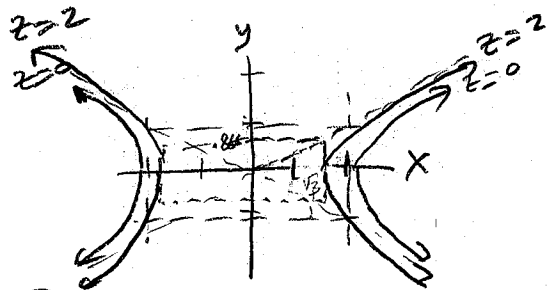
$$z=0: 4x^2 - 16y^2 = 16$$

$$\frac{x^2}{4} - \frac{y^2}{1} = 1$$

$$z=2: 4x^2 - 16y^2 + z^2 = 16$$

$$4x^2 - 16y^2 = 12$$

$$\frac{x^2}{3} - \frac{y^2}{\frac{3}{4}} = 1$$



xz (select y)

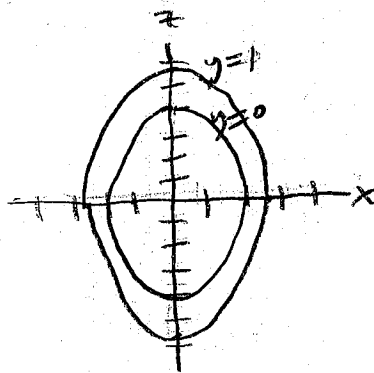
$$y=0: 4x^2 + z^2 = 16$$

$$\frac{x^2}{4} + \frac{z^2}{16} = 1$$

$$y=1: 4x^2 - 16 + z^2 = 16$$

$$4x^2 + z^2 = 32$$

$$\frac{x^2}{8} + \frac{z^2}{32} = 1$$



yz (select x)

$$x=0: -16y^2 + z^2 = 16$$

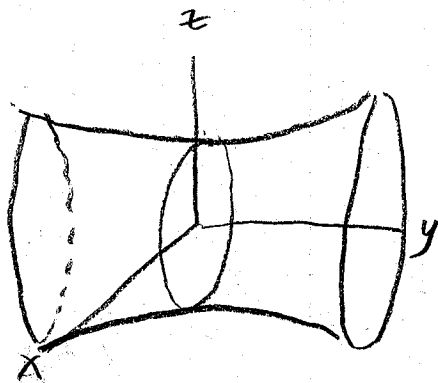
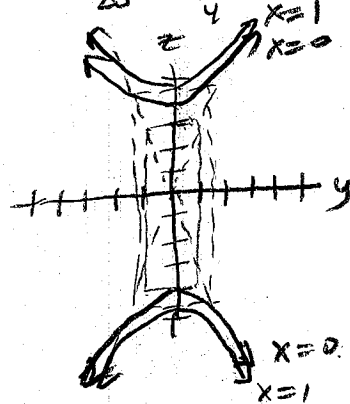
$$z^2 - 16y^2 = 16$$

$$\frac{z^2}{16} - \frac{y^2}{1} = 1$$

$$x=1: -4 - 16y^2 + z^2 = 16$$

$$z^2 - 16y^2 = 20$$

$$\frac{z^2}{20} - \frac{y^2}{\frac{5}{4}} = 1$$

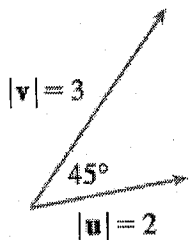


$$4x^2 - 16y^2 + z^2 = 16$$

$$\frac{x^2}{4} - \frac{y^2}{1} + \frac{z^2}{16} = 1$$

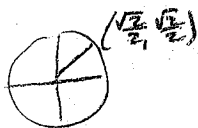
hyperboloid of one sheet
with axis in y-axis direction

#16. If \vec{u} and \vec{v} are the vectors shown in the figure:



(i) Find $\vec{u} \cdot \vec{v}$.

$$\begin{aligned} \vec{u} \cdot \vec{v} &= |\vec{u}| |\vec{v}| \cos \theta \\ &= (2)(3) \cos 45^\circ \\ &= 6 \left(\frac{\sqrt{2}}{2} \right) = \boxed{3\sqrt{2}} \end{aligned}$$



(ii) Find $|\vec{u} \times \vec{v}|$.

$$\begin{aligned} |\vec{u} \times \vec{v}| &= |\vec{u}| |\vec{v}| \sin \theta \\ &= (2)(3) \sin 45^\circ \\ &= 6 \left(\frac{\sqrt{2}}{2} \right) = \boxed{3\sqrt{2}} \end{aligned}$$

(iii) Is $\vec{u} \times \vec{v}$ directed into the page or out of the page?

by right-hand rule,
 $\vec{u} \times \vec{v}$ is out of the page

#17. For $\vec{a} = \vec{i} + \vec{j} - 2\vec{k}$, and $\vec{b} = 3\vec{i} - 2\vec{j} + \vec{k}$
 Find each of the following:

(i) $2\vec{a} + 3\vec{b}$

$$\begin{aligned} &2\langle 1, 1, -2 \rangle + 3\langle 3, -2, 1 \rangle \\ &\langle 2, 2, -4 \rangle + \langle 9, -6, 3 \rangle \\ &\boxed{\langle 11, -4, -1 \rangle} \end{aligned}$$

(ii) $|\vec{b}| = \sqrt{3^2 + 2^2 + 1^2} = \boxed{\sqrt{14}}$

(iii) $\vec{a} \cdot \vec{b} = \langle 1, 1, -2 \rangle \cdot \langle 3, -2, 1 \rangle$

$$\begin{aligned} &(1)(3) + (1)(-2) + (-2)(1) \\ &3 - 2 - 2 \\ &\boxed{-1} \end{aligned}$$

(iv) $\vec{a} \times \vec{b} = \begin{vmatrix} + & - & + \\ 1 & 1 & -2 \\ 3 & -2 & 1 \end{vmatrix}$

$$\begin{aligned} &= \langle (1)(1) - (-2)(-2), -[(1)(1) - (-2)(3)], (1)(-2) - (1)(3) \rangle \\ &= \langle 1 - 4, -(1 + 6), -2 - 3 \rangle \\ &= \boxed{\langle -3, -7, -5 \rangle} \end{aligned}$$

but I'd like to see this step on the test to show work (plus the determinant matrix)

don't need to show this much detail