Ch12 Test Review

#1. Find the lengths of the sides of the triangle PQR. Is it a right triangle? Is it an isosceles triangle? $P(3,-2,-3)$, $Q(7,0,1)$, $R(1,2,1)$.

$PQ=\sqrt{(7-3)^2+(0+2)^2+(1+3)^2}=6$

$PR=\sqrt{(1-3)^2+(2+2)^2+(1+3)^2}=6$

2 sides equal $\boxed{\text{Isosceles}}$

$QR=\sqrt{(7-1)^2+(0-2)^2+(1-1)^2}=\sqrt{40}$

Pythagorean theorem:

$6^2+6^2 \stackrel{?}{=} \sqrt{40}^2$

$72 \neq 40$

no, so $\boxed{\text{not a right triangle}}$

#2. Determine whether the points lie on a straight line.

(i) $A(2,4,2)$, $B(3,7,-2)$, $C(1,3,3)$.

$AB=\sqrt{(3-2)^2+(7-4)^2+(-2-2)^2}=\sqrt{26}$

$AC=\sqrt{(2-1)^2+(4-3)^2+(2-3)^2}=\sqrt{3}$

$BC=\sqrt{(3-1)^2+(7-3)^2+(-2-3)^2}=\sqrt{45}$

on a line if $\dfrac{\text{2 shorter dist}}{\text{add to exactly longest}}$

$\sqrt{26}+\sqrt{3} \stackrel{?}{=} \sqrt{45}$

$6.83 \neq 6.708$ $\boxed{\text{not on a line}}$

(ii) $D(0,-5,5)$, $E(1,-2,4)$, $F(3,4,2)$.

$DE=\sqrt{(1-0)^2+(-2+5)^2+(4-5)^2}=\sqrt{11}$

$DF=\sqrt{(3-0)^2+(4+5)^2+(2-5)^2}=\sqrt{99}$

$EF=\sqrt{(3-1)^2+(4+2)^2+(2-4)^2}=\sqrt{44}$

$\sqrt{11}+\sqrt{44} \stackrel{?}{=} \sqrt{99}$

$9.94987 = 9.94987$

$\boxed{\text{yes, on a line}}$

#3. For $\vec{a}=\langle 5,-12\rangle$, $\vec{b}=\langle -3,-6\rangle$, find:

(i) $\vec{a}+\vec{b}$ $=\langle 5,-12\rangle+\langle -3,-6\rangle$

$=\boxed{\langle 2,-18\rangle}$

(ii) $2\vec{a}+3\vec{b}$ $=2\langle 5,-12\rangle+3\langle -3,-6\rangle$

$=\langle 10,-24\rangle+\langle -9,-18\rangle$

$=\boxed{\langle 1,-42\rangle}$

(iii) $|\vec{a}-\vec{b}|$ $\vec{a}-\vec{b}=\langle 5,-12\rangle-\langle -3,-6\rangle$

$=\langle 8,-6\rangle$

$|\vec{a}-\vec{b}|=\sqrt{8^2+6^2}=\sqrt{100}=\boxed{10}$

#4. Find a unit vector that has the direction as $\langle -4,2,4\rangle$.

$|\langle -4,2,4\rangle|=\sqrt{4^2+2^2+4^2}=\sqrt{36}=6$

$\boxed{\frac{1}{6}\langle -4,2,4\rangle}$

or

$\boxed{\left\langle \frac{-4}{6},\frac{2}{6},\frac{4}{6}\right\rangle}$

or

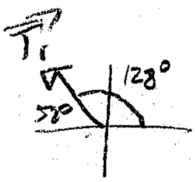
$\boxed{\left\langle -\frac{2}{3},\frac{1}{3},\frac{2}{3}\right\rangle}$

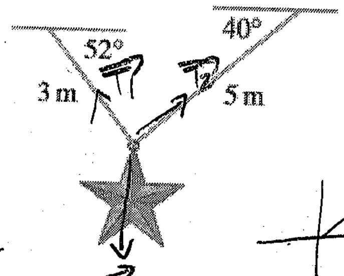
#5. Rope 3m and 5m in length are fastened to a holiday decoration that is suspended over a town square. The decoration has a mass of 5 kg. The ropes, fasted at different heights, make angles of 52° and 40° with the horizontal. Find the tension in each wire.

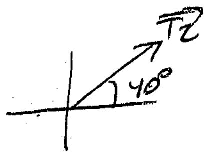
$W = mg$

$w = (5)(9.81)$

$w = 49\ N$







stationary, so $\vec{T_1} + \vec{T_2} + \vec{W} = \vec{0}$

$\vec{T_1} = \langle |T_1|\cos 128°, |T_1|\sin 128° \rangle$

$\vec{T_2} = \langle |T_2|\cos 40°, |T_2|\sin 40° \rangle$

$\vec{W} = \langle 0, -49 \rangle$

$\Sigma x = 0$; $|T_1|\cos 128° + |T_2|\cos 40° + 0 = 0$

$\Sigma y = 0$; $|T_1|\sin 128° + |T_2|\sin 40° - 49 = 0$

System:
$$\left[\begin{array}{cc|c} \cos 128° & \cos 40° & 0 \\ \sin 128° & \sin 40° & 49 \end{array}\right]$$

rref

$$\left[\begin{array}{cc|c} 1 & 0 & 37.5591 \\ 0 & 1 & 30.1858 \end{array}\right] \begin{array}{l} = |T_1| \\ = |T_2| \end{array}$$

$\vec{T_1} = \langle 37.5591\cos 128°, 37.5591\sin 128° \rangle$

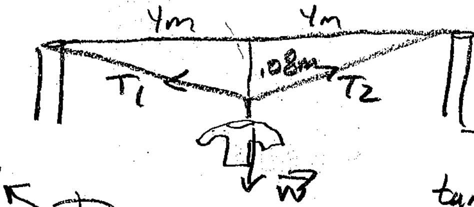
$= \langle -23.124, 29.597 \rangle$ Newtons (N)

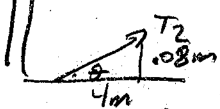
$\vec{T_2} = \langle 30.1858\cos 40°, 30.1858\sin 40° \rangle$

$= \langle 23.124, 19.403 \rangle$ N

#6. A clothesline is tied between two poles, 8m apart. The line is quite taut and has negligible sag. When a wet shirt with a mass ot 0.8 kg is hung at the middle of the line, the midpoint is pulled down 8 cm. Find the tension in each half of the clothesline.

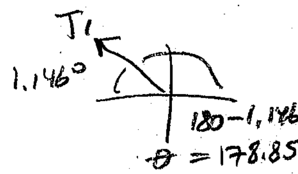
$= .08\,m$ (convert to standard units)





$\tan\theta = \frac{.08}{4}$

$\theta = \tan^{-1}\left(\frac{.08}{4}\right) = 1.1$



$\theta = 178.854°$

stationary, so $\vec{T_1} + \vec{T_2} + \vec{W} = 0$

$\vec{T_1} = \langle |T_1|\cos 178.854°, |T_1|\sin 178.854° \rangle$

$\vec{T_2} = \langle |T_2|\cos 1.146°, |T_2|\sin 1.146° \rangle$

$|\vec{W}| = mg = .08(9.81) = .7848$

$\vec{W} = \langle 0, -7.848 \rangle$ N

$\Sigma x = 0$: $|T_1|\cos 178.854° + |T_2|\cos 1.146° + 0 = 0$

$\Sigma y = 0$: $|T_1|\sin 178.854° + |T_2|\sin 1.146° - .7848 =$

System:
$$\left[\begin{array}{cc|c} \cos 178.854° & \cos 1.146° & 0 \\ \sin 178.854° & \sin 1.146° & 7.848 \end{array}\right]$$

rref

$$\left[\begin{array}{ccc} 1 & 0 & 196.1986 \\ 0 & 1 & 196.1986 \end{array}\right] \begin{array}{l} = |T_1| \\ = |T_2| \end{array}$$

$\vec{T_1} = \langle 196.16\cos 178.854°, 196.61986\sin 178.854° \rangle$

$= \langle -196.16, 3.92 \rangle$ N

$\vec{T_2} = \langle 196.16, 3.92 \rangle$ N

#7. Find the angle between the vectors (in exact and decimal form). $\vec{a} = \langle 4,0,2\rangle$, $\vec{b} = \langle 2,-1,0\rangle$.

$\vec{a}\cdot\vec{b} = |a||b|\cos\theta$

$$\cos\theta = \frac{\langle 4,0,2\rangle\cdot\langle 2,-1,0\rangle}{\sqrt{4^2+0^2+2^2}\sqrt{2^2+1^2+0^2}}$$

$$= \frac{(4)(2)+(0)(-1)+(2)(0)}{\sqrt{20}\sqrt{5}}$$

$$= \frac{8}{\sqrt{100}} = \frac{8}{10} = \frac{4}{5}$$

$$\theta = \boxed{\cos^{-1}\left(\frac{4}{5}\right)} \approx \boxed{0.6435 \text{ radians} \approx 36.87^\circ}$$

#9. If $\vec{a} = \langle 1,2,1\rangle$ and $\vec{b} = \langle 0,1,3\rangle$, find $\vec{a} \times \vec{b}$ and $\vec{b} \times \vec{a}$.

$$\vec{a}\times\vec{b} = \begin{vmatrix} + & - & + \\ 1 & 2 & 1 \\ 0 & 1 & 3 \end{vmatrix}$$

$$= \langle 6-1, -(3-0), 1-0\rangle$$

$$= \boxed{\langle 5,-3,1\rangle}$$

$$\vec{b}\times\vec{a} = \begin{vmatrix} + & - & + \\ 0 & 1 & 3 \\ 1 & 2 & 1 \end{vmatrix}$$

$$= \langle 1-6, -(0-3), 0-1\rangle$$

$$= \boxed{\langle -5,3,-1\rangle}$$

(also, $\vec{b}\times\vec{a} = -(\vec{a}\times\vec{b})$)

#8. For what values of b are the vectors $\langle -6,b,2\rangle$ and $\langle b,b^2,b\rangle$ orthogonal? .

orthogonal ($\perp$) if dotproduct $=0$

$\langle -6,b,2\rangle\cdot\langle b,b^2,b\rangle = 0$

$-6b + b^3 + 2b = 0$

$b^3 - 4b = 0$

$b(b^2-4) = 0$

$b(b-2)(b+2) = 0$

$\boxed{b=0,\ b=2,\ b=-2}$

#10. Find two unit vectors orthogonal to both $\langle 1,-1,1\rangle$ and $\langle 0,4,4\rangle$.

(cross-product is $\perp$ to vectors)

$$\langle 1,-1,1\rangle\times\langle 0,4,4\rangle = \begin{vmatrix} + & - & + \\ 1 & -1 & 1 \\ 0 & 4 & 4 \end{vmatrix}$$

$$= \langle -4-4, -(4-0), 4-0\rangle$$

$= \langle -8,-4,4\rangle$ unit vector: $|\langle -8,-4,4\rangle|$

$= \sqrt{8^2+4^2+4^2} = \sqrt{96}$

$\boxed{\left\langle \frac{-8}{\sqrt{96}}, \frac{-4}{\sqrt{96}}, \frac{4}{\sqrt{96}}\right\rangle}$ $\frac{1}{\sqrt{96}}\langle -8,-4,4\rangle$

other vector is 180° apart

$$= \boxed{\left\langle \frac{8}{\sqrt{96}}, \frac{4}{\sqrt{96}}, \frac{-4}{\sqrt{96}}\right\rangle}$$

#11. Find a vector equation, parametric equations, and symmetric equations for the line through the points $(6,1,-3)$ and $(2,4,5)$.

$\vec{r_0} = \langle 6,1,-3\rangle$ (can choose either point)

$\vec{v} = \langle 2-6, 4-1, 5+3\rangle = \langle -4, 3, 8\rangle$

vector equation: $\vec{r} = \vec{r_0} + t\vec{v}$

$\boxed{\vec{r} = \langle 6,1,-3\rangle + t\langle -4,3,8\rangle}$

parametric equations: $\vec{r} = \langle 6-4t, 1+3t, -3+8t\rangle$

$$\boxed{\begin{aligned} x &= 6-4t \\ y &= 1+3t \quad -\infty \le t \le \infty \\ z &= -3+8t \end{aligned}}$$

include range for parameter

symmetric equations (solve for t):

$x = 6-4t$ $\quad y = 1+3t$ $\quad z = -3+8t$

$4t = 6-x$ $\quad 3t = y-1$ $\quad 8t = z+3$

$t = \boxed{\dfrac{6-x}{4} = \dfrac{y-1}{3} = \dfrac{z+3}{8}}$

#12. Find a vector equation for the line of <u>intersection</u> of the planes $x+y+z=1$ and $x+z=0$.

intersection = system solution

$\begin{cases} x+y+z=1 \\ x \quad +z=0 \end{cases}$ $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{array}\right]$

rref $\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{array}\right]$ $x+z=0,\ x=-z$; $y=1$

general solution: $(-z, 1, z)$

everything in terms of z, so make z the parameter t $(z=t)$

$x = -t$

$y = 1$

$z = t$

as a vector equation:

$$\boxed{\begin{aligned} \vec{r} &= \langle -t, 1, t\rangle \\ \infty &\le t \le \infty \end{aligned}}$$

#13. Find an equation of the plane through the point $(-2,8,10)$ and perpendicular to the line $x = 1+t,\ y = 2t,\ z = 4-3t$.

line: $\vec{r} = \vec{r_0} + t\vec{v} = \langle 1,0,4\rangle + t\underbrace{\langle 1,2,-3\rangle}_{\vec{v}}$

so $\vec{v} = \langle 1,2,-3\rangle$

if new plane is $\perp$ to line, then $\vec{v}$ is the normal, $\vec{n}$, for the plane: $\vec{n} = \langle 1,2,-3\rangle$

for the plane, $\vec{r_0} = \langle -2,8,10\rangle$

$ax+by+cz = \vec{n}\cdot\vec{r_0}$

$1x+2y-3z = \langle 1,2,-3\rangle \cdot \langle -2,8,10\rangle$

$= (1)(-2)+(2)(8)+(-3)(10)$

$= -2+16-30$

$\boxed{x+2y-3z=-16}$

#14. Find an equation of the plane that contains the line $x = 3+2t,\ y = t,\ z = 8-t$ and is parallel to the plane $2x+4y+8z=17$.

$\vec{n}$ for given plane is $\vec{n} = \langle 2,4,8\rangle$

is also normal for new plane (parallel to given plane)

for $\vec{r_0}$ on plane, choose any t to plug into parametric equations:

choose $t=0$ $\quad x = 3+2(0) = 3$

$y = (0) = 0$

$z = 8-(0) = 8$

so $\vec{r_0} = \langle 3,0,8\rangle$

now, $ax+by+cz = \vec{n}\cdot\vec{r_0}$

$2x+4y+8z = \langle 2,4,8\rangle \cdot \langle 3,0,8\rangle$

$= (2)(3)+(4)(0)+(8)(8)$

$= 6+0+64$

$\boxed{2x+4y+8z = 70}$

or

$\boxed{x+2y+4z = 35}$

#15. Draw at least two traces for each coordinate plane for $4x^2 - 16y^2 + z^2 = 16$. What kind of solid is this? What is its main axis? Sketch the solid in $\mathbb{R}^3$.

xy (select z)

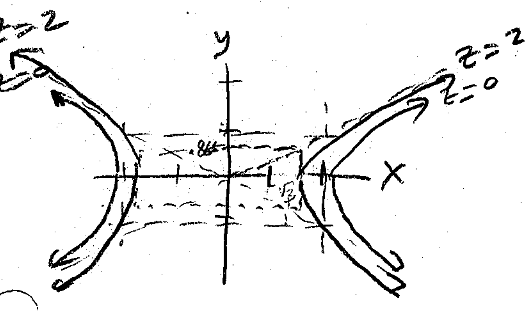
$z=0$: $4x^2 - 16y^2 = 16$

$$\frac{x^2}{4} - \frac{y^2}{1} = 1$$

$z=2$: $4x^2 - 16y^2 + z^2 = 16$

$4x^2 - 16y^2 = 12$

$$\frac{x^2}{3} - \frac{y^2}{\frac{3}{4}} = 1$$



xz (select y)

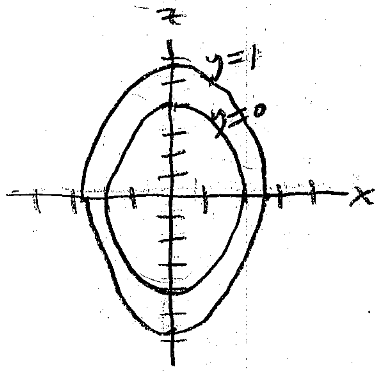
$y=0$: $4x^2 + z^2 = 16$

$$\frac{x^2}{4} + \frac{z^2}{16} = 1$$

$y=1$: $4x^2 - 16 + z^2 = 16$

$4x^2 + z^2 = 32$

$$\frac{x^2}{8} + \frac{z^2}{32} = 1$$



yz (select x)

$x=0$: $-16y^2 + z^2 = 16$

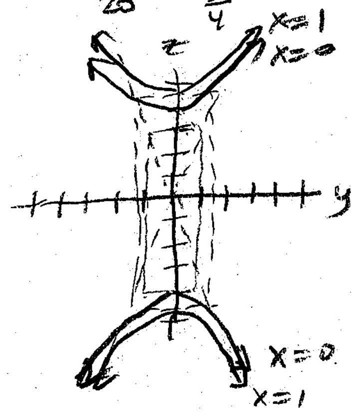
$z^2 - 16y^2 = 16$

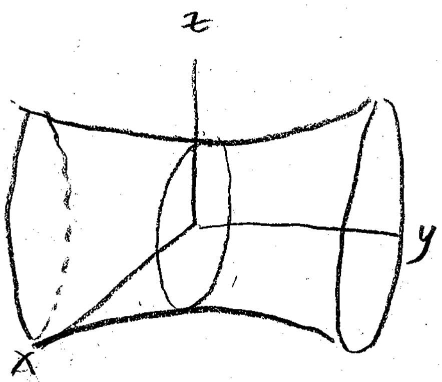
$$\frac{z^2}{16} - \frac{y^2}{1} = 1$$

$x=1$: $-4 - 16y^2 + z^2 = 16$

$z^2 - 16y^2 = 20$

$$\frac{z^2}{20} - \frac{y^2}{\frac{5}{4}} = 1$$



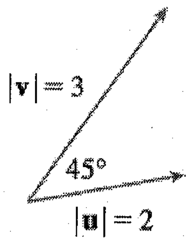


$4x^2 - 16y^2 + z^2 = 16$

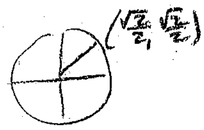
$$\frac{x^2}{4} - \frac{y^2}{1} + \frac{z^2}{16} = 1$$

$\boxed{\text{hyperboloid of one sheet with axis in y-axis direction}}$

#16. If $\vec{u}$ and $\vec{v}$ are the vectors shown in the figure:



(i) Find $\vec{u} \bullet \vec{v}$.



$\vec{u} \cdot \vec{v} = |u||v|\cos\theta$

$= (3)(2)\cos 45°$

$= 6\left(\frac{\sqrt{2}}{2}\right) = \boxed{3\sqrt{2}}$

(ii) Find $\left|\vec{u} \times \vec{v}\right|$.

$|\vec{u} \times \vec{v}| = |u||v|\sin\theta$

$= (3)(2)\sin 45°$

$= 6\left(\frac{\sqrt{2}}{2}\right) = \boxed{3\sqrt{2}}$

(iii) Is $\vec{u} \times \vec{v}$ directed into the page or out of the page?

by right-hand rule,

$\vec{u} \times \vec{v}$ is $\boxed{\text{out of the page}}$

$\langle 1, 1, -2 \rangle$ $\langle 3, -2, 1 \rangle$

#17. For $\vec{a} = \vec{i} + \vec{j} - 2\vec{k}$, *and* $\vec{b} = 3\vec{i} - 2\vec{j} + \vec{k}$

Find each of the following:

(i) $2\vec{a} + 3\vec{b}$ $2\langle 1,1,-2\rangle + 3\langle 3,-2,1\rangle$

$\langle 2,2,-4\rangle + \langle 9,-6,3\rangle$

$\boxed{\langle 11,-4,-1\rangle}$

(ii) $\left|\vec{b}\right| = \sqrt{3^2+2^2+1^2} = \boxed{\sqrt{14}}$

(iii) $\vec{a} \bullet \vec{b} = \langle 1,1,-2\rangle \cdot \langle 3,-2,1\rangle$

$(1)(3) + (1)(-2) + (-2)(1)$

$3 - 2 - 2$

$\boxed{-1}$

(iv) $\vec{a} \times \vec{b} = \begin{vmatrix} + & - & + \\ 1 & 1 & -2 \\ 3 & -2 & 1 \end{vmatrix}$

$= \langle (1)(1) - (-2)(-2), -[(1)(1) - (-2)(3)], (1)(-2) - (1)(3) \rangle$

$= \langle 1 - 4, -(1+6), -2-3 \rangle$

$= \boxed{\langle -3, -7, -5 \rangle}$

don't need to show this much detail

but I'd like to see this step on the test to show work (plus the determinant matrix) :)

Find the scalar and vector projection of $\langle 5,\ 2,\ 7\rangle$ onto $\langle 4, 6, 1\rangle$.

$\vec{a} = \langle 5, 2, 7\rangle$

$\vec{b} = \langle 4, 6, 1\rangle \quad |\vec{b}| = \sqrt{4^2+6^2+1^2} = \sqrt{53} \quad \vec{u}_b = \frac{1}{\sqrt{53}}\langle 4, 6, 1\rangle$

Scalar projection of $\vec{a}$ onto $\vec{b}$

$$= \vec{a} \cdot \vec{u}_b$$

$$= \langle 5, 2, 7\rangle \cdot \left(\frac{1}{\sqrt{53}}\langle 4, 6, 1\rangle\right)$$

$$= \frac{1}{\sqrt{53}}\left((5)(4)+(2)(6)+(7)(1)\right)$$

$$= \frac{1}{\sqrt{53}}(39)$$

$$= \boxed{\frac{39}{\sqrt{53}}}$$

vector projection of $\vec{a}$ onto $\vec{b}$

$$= \frac{39}{\sqrt{53}}\vec{u}_b$$

$$= \frac{39}{\sqrt{53}}\left(\frac{1}{\sqrt{53}}\langle 4, 6, 1\rangle\right)$$

$$= \boxed{\frac{39}{53}\langle 4, 6, 1\rangle}$$